# 1st-Order Eqs, Existence and Uniqueness of Solutions

given $\frac{dy}{dt} = f(t,y) \quad y(t_0) = y_0$

Questions: 1. Is there a solution? (Existence)

2. If so, is that solution unique? (Uniqueness)

let's start with linear eqs.

$$y' + p(t)y = g(t) \quad y(t_0) = y_0$$

we know that if an integrating factor exists, we can (in principle) solve it

$$\mu(t) = e^{\int p(t)\,dt}$$

1st requirement: this MUST exist

so $\int p(t)\,dt$ must exist

→ $p(t)$ MUST be continuous at least on some interval of $t$

the eq can be written as

$$\frac{d}{dt}\left[\mu(t)\, y\right] = \mu(t)\, g(t)$$

then

$$\mu(t)\, y = \underbrace{\int \mu(t)\, g(t)}_{} + c$$

2nd requirement : $\int \mu(t) g(t)\, dt$ MUST exist

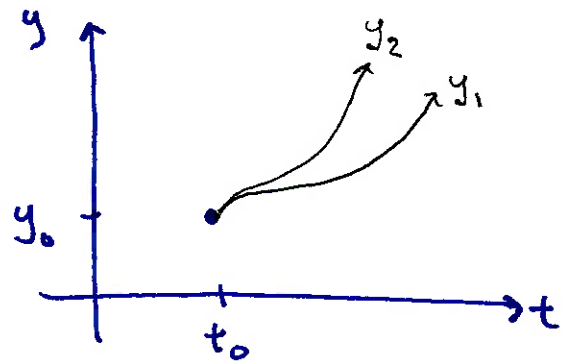
so $g(t)$ must be continuous

AND $p(t)$ and $g(t)$ are continuous on some common interval

so, for $y' + p(t)y = g(t)$, $y(t_0) = y_0$, ~~the~~ a solution exists on some interval of $t$ on which BOTH $p(t)$ and $g(t)$ are continuous AND containing the initial $t_0$

now, is the solution unique?

for example, can this happen?




eg: $y' + p(t)\,y = g(t) \qquad y(t_0) = y_0$

suppose that two solutions exist: $y_1$, $y_2$

let's define $u(t) = y_2(t) - y_1(t)$ — difference between them as a function of $t$

if $y_1$ and $y_2$ are solutions, then

$$y_1' + p(t)\,y_1 = g(t)$$

$$y_2' + p(t)\,y_2 = g(t)$$

Subtract: $(y_2' - y_1') + (p(t)\,y_2 - p(t)\,y_1) = 0$

$$\underbrace{(y_2' - y_1')}_{u'} + p(t)\underbrace{(y_2 - y_1)}_{u} = 0$$

$$\boxed{u' + p(t)\,u = 0 \qquad u(t_0) = 0}$$

we can solve this for $u$

linear and/or separable

$\vdots$

$$u(t) = C\,e^{-\int p(t)\,dt}$$

$$u(t) = u(t_0)\,e^{-\int p(t)\,dt}$$

$$= 0 \qquad \text{because } u(t_0) = 0$$

this means $y_2 - y_1 = 0$ so $y_2 = y_1 \rightarrow$ <u>one</u> solution only

For linear eq. $y' + p(t)y = g(t)$, $y(t_0) = y_0$

~~the~~ a solution exists and is unique on some interval of $t$

on which BOTH $p(t)$ and $g(t)$ are continuous and containing $t_0$

<u>example</u> $\ln(t)\, y' + y = \cot(t)$ $\quad y(2) = 3$

$$y' + \underbrace{\frac{1}{\ln(t)}}_{p(t)} y = \underbrace{\frac{\cot(t)}{\ln(t)}}_{g(t)}$$

$p(t)$ continuous on : $(0, 1), (1, \infty)$

$g(t)$ " " : $(0, 1), (1, \pi), (\pi, 2\pi), (2\pi, 3\pi), \ldots$

the interval where BOTH are continuous AND containing

$t_0 = 2$ is $(1, \pi)$

so, solution exists and is unique on $1 < t < \pi$

another nice property of linear eqs. is that the general solution

Contains ALL possible solutions

$y' = y$ $\qquad\qquad$ $y = 0$ is a solution

Solve as separable

$$\frac{dy}{dt} = y$$

$$\frac{1}{y}\,dy = dt \qquad (y \neq 0)$$

$$\vdots$$

$y = Ce^t$ $\qquad\qquad$ Note $y=0$ <u>is</u> included ($C=0$)

linear eqs. ALWAYS behave like this

not always true for nonlinear eqs.

for nonlinear eqs, $y' = f(t, y)$ $y(t_0) = y_0$

existance: $f(t, y)$ is continuous on some interval containing $t_0$

(for example, think of how you solve $y' = y^3$)

uniqueness is harder

consider $y' = y^{1/3}$

easy to solve as a separable

$$y^{-1/3}\, dy = dt$$

$$\int y^{-1/3}\, dy = \int dt$$

$$\frac{3}{2} y^{2/3} = t + C$$

$$y^{2/3} = \frac{2}{3} t + C$$

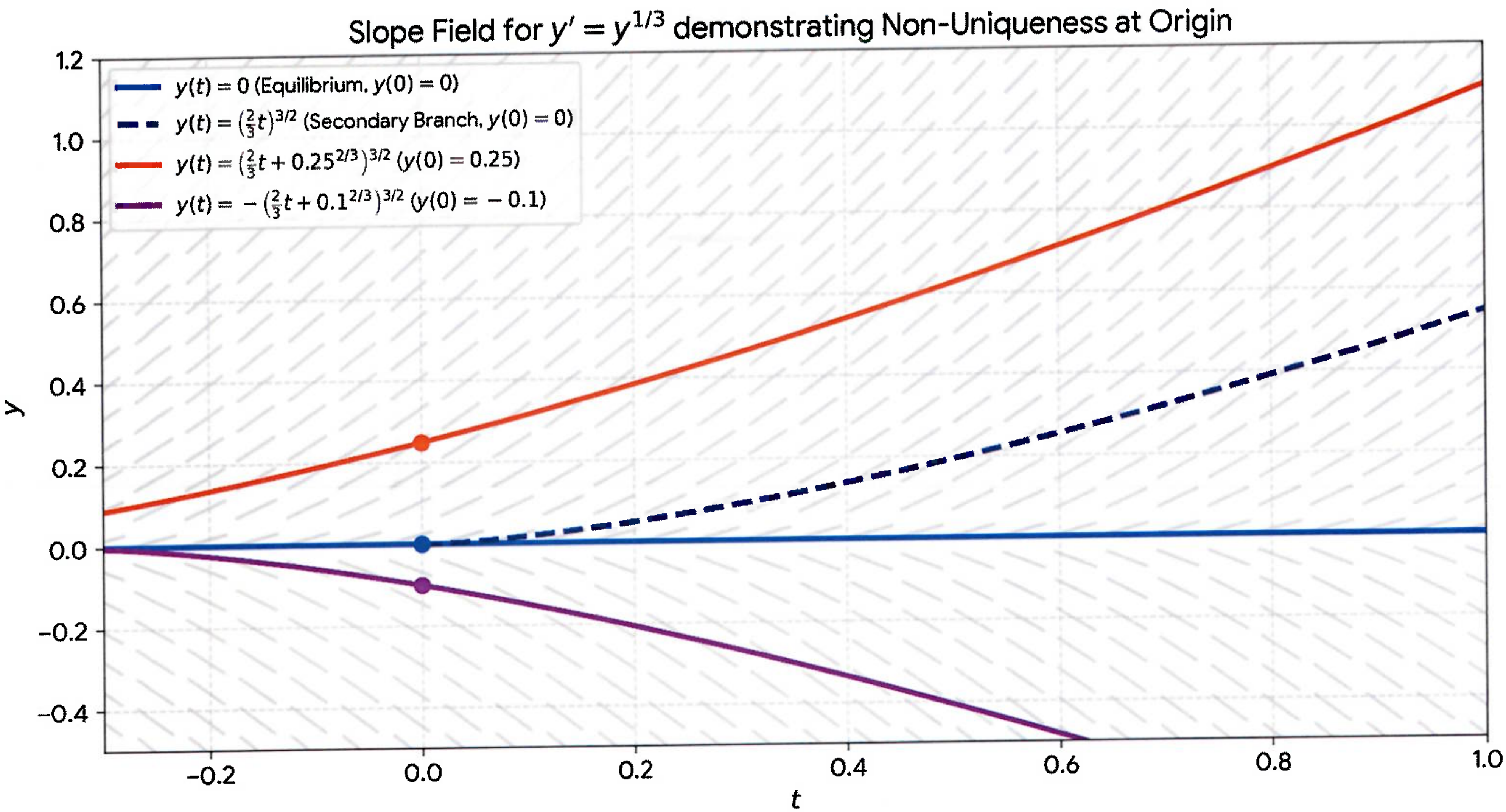
$$y = \left(\frac{2}{3}t + c\right)^{3/2}$$

note $y = 0$ <u>is</u> a solution to $y' = y^{1/3}$

but in the solution above, $y = \left(\frac{2}{3}t + c\right)^{3/2}$, if $y(0) = 0$

then we get $y = \left(\frac{2}{3}t\right)^{3/2}$

two solutions out of $y(0) = 0$?



Slope Field for $y' = y^{1/3}$ demonstrating Non-Uniqueness at Origin
$y(t) = 0$ (Equilibrium, $y(0) = 0$)
$y(t) = (\frac{2}{3}t)^{3/2}$ (Secondary Branch, $y(0) = 0$)
$y(t) = (\frac{2}{3}t + 0.25^{2/3})^{3/2}$ ($y(0) = 0.25$)
$y(t) = -(\frac{2}{3}t + 0.1^{2/3})^{3/2}$ ($y(0) = -0.1$)
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
y
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
t

what happens here is that for tiny change in $y$,

the slope can change dramatically near $(0,0)$

to track the change of slope of $y' = f(t,y)$ with respect to $y$, we look at $\frac{\partial f}{\partial y}$

for example, $y' = y^{1/3}$

$$\frac{\partial f}{\partial y} = \frac{1}{3} y^{-2/3} = \frac{1}{3y^{2/3}}$$

if $y$ is small, small changes in $y$ can make $\frac{\partial f}{\partial y}$ big

(continue next time)