Existence & Uniqueness (continued)

linear: $y' + p(t)y = g(t) \quad y(t_0) = y_0$

unique solution on interval where $p(t)$ and $g(t)$ are continuous and containing t_0

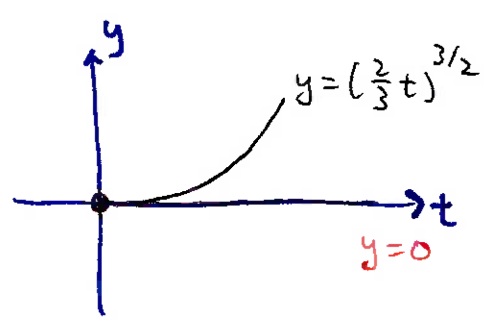
nonlinear: $y' = f(t, y) \quad y(t_0) = y_0$

last time: $y' = y^{1/3} \qquad y(0) = 0$

$$\vdots$$

$$y = \left(\tfrac{2}{3}t\right)^{3/2}$$

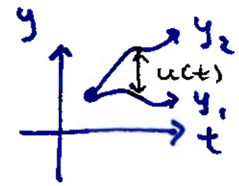
but $y = 0$ for all t is also a solution



Solution is NOT unique out of $y(0) = 0$

what conditions do nonlinear eqs. have to satisfy to guarantee a unique solution?

for linear, we defined $u(t) = y_2 - y_1$



we showed that $u(t) = 0$ (unique solution) as long as $p(t)$ and $g(t)$ are continuous.

let's do that for nonlinear $y' = f(t, y)$ $\quad y(t_0) = y_0$

let y_1 and y_2 be two solutions

define $u(t) = y_2 - y_1$

$y_1' = f(t, y_1)$ because y_1 is a solution

$y_2' = f(t, y_2)$

subtract: $\quad y_2' - y_1' = f(t, y_2) - f(t, y_1)$

we will use the Mean-Value Theorem on the right

MVT: $\dfrac{f(b)-f(a)}{b-a} = f'(c) \qquad a<c<b$

right side of $y_2' - y_1' = \underbrace{f(t,y_2) - f(t,y_1)}_{\text{using MVT}}$

$$\frac{f(t,y_2)-f(t,y_1)}{y_2-y_1} = \boxed{\frac{\partial f}{\partial y}(t, y^*)}$$

depends explicitly on t only

$y_1 < y^* < y_2$

$y_2' - y_1' = f(t,y_2) - f(t,y_1)$ becomes

$$u'(t) = \frac{\partial f}{\partial y}(t,y^*) \cdot u(t)$$

$y_2 - y_1$

$$u'(t) - \frac{\partial f}{\partial y}(t) \cdot u(t) = 0$$

is linear! $\quad y' + p(t)\,y = g(t)$

$-\frac{\partial f}{\partial y}$ is acting like $p(t)$

$$\mu(t) = e^{-\int \frac{\partial f}{\partial y}\, dt}$$ integrating factor $u(t_0) = 0$

$\vdots$

$$u(t) = u(t_0)\, e^{\int \frac{\partial f}{\partial y}\, dt}$$ $\rightarrow$ will be 0 for ALL t as long as $\frac{\partial f}{\partial y}$ is continuous

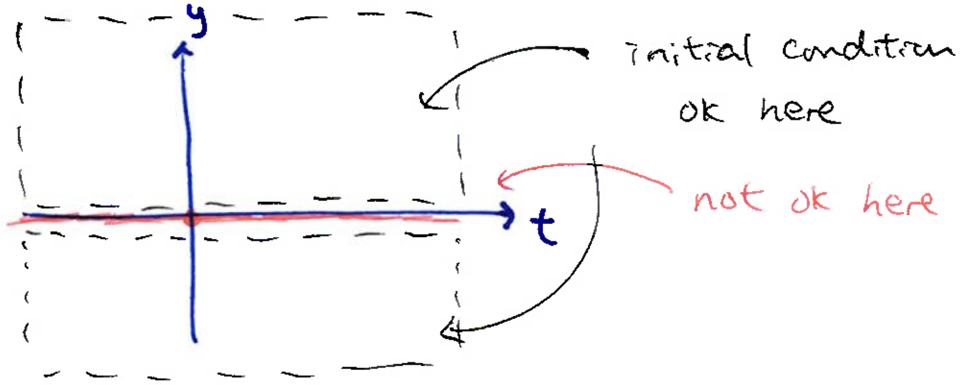
For nonlinear $y' = f(t,y)$ $y(t_0) = y_0$

a unique solution is guaranteed on some interval wherever both $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous and containing $y(t_0) = y_0$

let's revisit $y' = y^{1/3}$ $y(0) = 0$

$f(t,y) = y^{1/3}$ continuous for all (t,y)

$\frac{\partial f}{\partial y} = \frac{1}{3} y^{-2/3} = \frac{1}{3y^{2/3}}$ not continuous for $y = 0$



initial condition $y(0)=0$ is NOT at a location where BOTH f and $\frac{\partial f}{\partial y}$ are continuous

$\rightarrow$ solution MAY NOT be unique

Autonomous Diff. Eqs.

1st-order eqs. of the form $y' = f(y)$

$\uparrow$ no t

rate of change of y depends on its own size

in a lot of applications

$\frac{dy}{dt} = ky$ exponential growth $(k>0)$

" decay $(k<0)$

as population model, it does not model environmental limitations

an improved model is

$$\frac{dy}{dt} = \underbrace{(r - ay)}_{\text{no longer just constant } K}(y) \qquad r, a \text{ constants}$$

growth rate is NOT constant

note $y' = 0$ when $y = 0$

$y' = 0$ when $y = \frac{r}{a}$ "carrying capacity" (environmental limit)

rewrite: $\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$ logistic growth model

r → "intrinsic growth rate"

$K = \frac{r}{a}$

we want to know : as $t \to \infty$ will $y \to 0$ or $y \to \frac{r}{a}$

does the initial condition matter ?

for example, let's look at

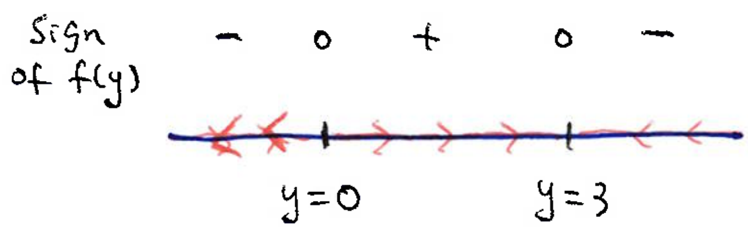
$$\frac{dy}{dt} = 3\left(1 - \frac{y}{3}\right)y = f(y)$$

$f(y) = 0 \rightarrow$ critical values/points

(y values where $y' = 0$)

here, the critical values are $y = 0$, $y = 3$

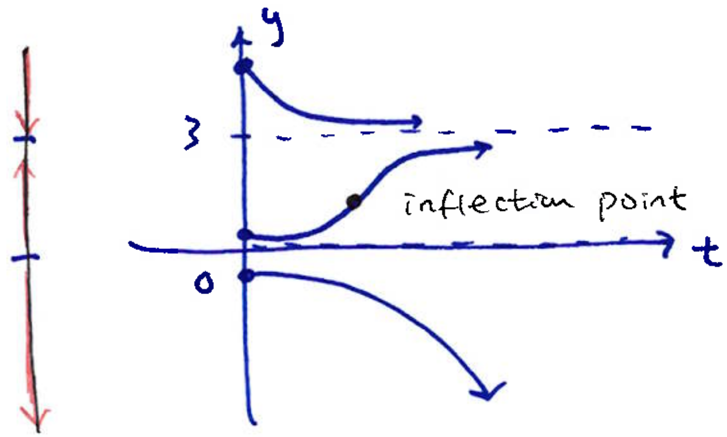
let's draw a <u>phase line</u>



$f(y) = y'$

$f(y) > 0 \rightarrow y' > 0$ increasing (to right)

$f(y) < 0 \rightarrow y' < 0$ decreasing (to left) sketch some solutions



$y' = 0 \rightarrow$ equilibrium solutions