6.2 Solutions of Initial-Value Problems

use LT to solve $y'' + p(t)y' + g(t)y = g(t)$

$y(0) = y_0, \quad y'(0) = y_0'$

definition of LT: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

we often look up a Table for commonly used LT
(Table 6.2.1 p. 321)

Inverse LT: $\mathcal{L}^{-1}\{F(s)\} = f(t)$

formula is complicated and is not practical. Use Table.

<u>example</u> $F(s) = \dfrac{2s-3}{s^2-4}$ find $f(t)$

note denominator: $s^2 - a^2$ a: constant

closest match: #7 $\mathcal{L}\{\sinh at\} = \dfrac{a}{s^2-a^2}$

#8 $\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2}$

$$F(s) = \frac{2s}{s^2-4} - \frac{3}{s^2-4}$$ a is 2

$$= 2 \cdot \frac{s}{s^2-4} - \frac{3}{2}\,\frac{2}{s^2-4}$$

$$f(t) = \mathcal{L}^{-1}\{F\} = \boxed{2\cosh 2t - \frac{3}{2}\sinh 2t}$$

example $F(s) = \dfrac{3s}{s^2 - s - 6}$

$$\boxed{\#2 \ \mathcal{L}\{e^{at}\} = \frac{1}{s-a}}$$

$$= \frac{3s}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

$$3s = \frac{A}{\cancel{s-3}}\cancel{(s-3)}(s+2) + \frac{B}{\cancel{s+2}}(s-3)\cancel{(s+2)}$$

$$3s = A(s+2) + B(s-3)$$

let $s=3$ $\quad 9 = 5A \quad A = \frac{9}{5}$

let $s=-2$ $\quad -6 = -5B \quad B = \frac{6}{5}$

or $\quad 3s = (A+B)s + (2A-3B)$

then solve $\quad A+B=3$

$\quad 2A-3B=0$

$$F(s) = \frac{9}{5}\frac{1}{s-3} + \frac{6}{5}\frac{1}{s+2}$$

$$\boxed{f(t) = \frac{9}{5}e^{3t} + \frac{6}{5}e^{-2t}}$$

if y is not specified, then $\boxed{\mathcal{L}\{y\} = Y}$

$\mathcal{L}\{y'\} = ?$

use definition: $\mathcal{L}\{y'\} = \int_0^\infty e^{-st} y' \, dt$

$u = e^{-st}$ $\quad dv = y' \, dt$

$du = -se^{-st} \, dt$ $\quad v = y$

$$\mathcal{L}\{y'\} = \lim_{A\to\infty} \left(ye^{-st}\Big|_0^A + s\int_0^A e^{-st} y \, dt \right)$$

$$= \lim_{A\to\infty} \left[\overset{0}{\cancel{\underset{y(A)}{y}e^{-sA}}} - y(0) \right] + s \underbrace{\int_0^\infty e^{-st} y \, dt}_{\mathcal{L}\{y\} = Y}$$

$$\boxed{\mathcal{L}\{y'\} = sY - y(0)}$$

$$\mathcal{L}\{y''\} = \mathcal{L}\{(y')'\} = s\mathcal{L}\{y'\} - y'(0)$$

$$= s\,(sY - y(0)) - y'(0)$$

$$\boxed{\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)}$$

similarly,

$$\mathcal{L}\{y'''\} = s^3Y - s^2y(0) - sy'(0) - y''(0)$$

$$\mathcal{L}\{y^{(4)}\} = s^4Y - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)$$

example Solve $y''-2y'+2y=0$ $\quad y(0)=0,\ y'(0)=1$

LT both sides

$$\mathcal{L}\{y''\}-2\mathcal{L}\{y'\}+2\mathcal{L}\{y\}=\mathcal{L}\{0\}$$

$$s^2Y-\cancel{sy(0)}-y'(0)-2(sY-\cancel{y(0)})+2Y=0$$

solve for Y

$$s^2Y-2sY+2Y=y'(0)$$

$$\underbrace{(s^2-2s+2)}_{\text{characteristic polynomial}}Y=1$$

$$Y=\frac{1}{s^2-2s+2}$$

find $\mathcal{L}^{-1}\{Y\}=y$

denominator not factorable

$\to$ complete the square.

$$Y = \frac{1}{s^2 - 2s + \underbrace{\left(\frac{-2}{2}\right)^2}_{\frac{1}{2} \text{ of coeff. of } s} + 2 - \underbrace{\left(\frac{-2}{2}\right)^2}} = \frac{1}{s^2 - 2s + 1 \; + 1}$$

$$Y = \frac{1}{(s-1)^2 + 1^2}$$

so $\boxed{y(t) = e^t \sin t}$

$$\boxed{\#9 \quad \mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}}$$

<u>example</u> : $y''' + y' = 1 \qquad y(0) = y'(0) = y''(0) = 0$

$$\mathcal{L}\{y'''\} + \mathcal{L}\{y'\} = \mathcal{L}\{1\}$$

$$s^3 Y - s^2 \cancel{y(0)} - s\cancel{y'(0)} - \cancel{y''(0)} + sY - \cancel{y(0)} = \frac{1}{s}$$

$$(s^3 + s)Y = \frac{1}{s}$$

$$Y = \frac{1}{s(s^3+s)} = \frac{1}{s^2(s^2+1)} = \frac{1}{\underbrace{s \cdot s}_{\text{repeated linear}} \cdot (s^2+1)}$$

$$\frac{1}{s \cdot s \cdot (s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$1 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$

$$1 = \cancel{As^3}\ (A+C)s^3 + (B+D)s^2 + As + B$$

$$B = 1, \quad A = 0, \quad \begin{array}{c} B + D = 0 \\ D = -1 \end{array}, \quad \begin{array}{c} A + C = 0 \\ C = 0 \end{array}$$

$$Y = \frac{1}{s^2} + \frac{-1}{s^2+1} = At$$

$$\boxed{y = t - \sin t}$$

$$y''' + y' = 1$$

$$r^3 + r = 0 \qquad r = 0, \pm i$$

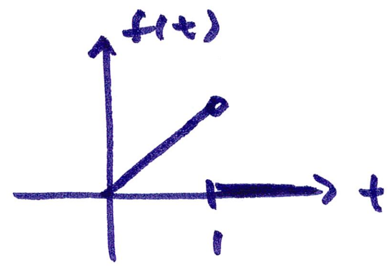
$$y = C_1 + C_2 \cos t + C_3 \sin t + At$$

LT doesn't care about duplication.

LT can handle discontinuous forcing functions
(just needs to be piecewise continuous)

<u>example</u>

$$y'' + y = \begin{cases} t & 0 \le t < 1 \\ 0 & 1 \le t < \infty \end{cases} \qquad y(0) = y'(0) = 0$$



$$s^2 Y - \cancel{s y(0)} - \cancel{y'(0)} + Y = \mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}\, t\,dt + \cancel{\int_1^\infty e^{-st}\, 0\,dt}$$

$$= 1 - \frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s}$$

$$(s^2 + 1)\,Y = 1 - \frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s}$$

$$Y = \frac{1 - \frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s}}{s^2 + 1}$$

$\mathcal{L}^{-1}\{Y\}$ discussed in 6.3