17. $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

$$\begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-1-\lambda)+8=0$$

$$\lambda^2-2\lambda+5=0$$

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

$\underline{\lambda = 1+2i}$

$$(A-\lambda I)\vec{x} = \vec{0}$$

$$\left[\begin{array}{cc|c} 2-2i & -2 & 0 \\ 4 & -2-2i & 0 \end{array}\right]$$

multiply row 1 by $2+2i$

$$\left[\begin{array}{cc|c} 8 & -4-4i & 0 \\ 4 & -2-2i & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & -1-i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = r$

$2x_1 = (1+i)x_2$

choose $x_2 = 1$ or $1-i$

$$\vec{x} = \begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

7.5 Homogeneous Systems with Constant Coefficients

solve $\vec{x}' = A\vec{x}$ where A is a constant matrix

if this is scalar eq. $y' = ay$, then solution is in the form of $y = e^{rt}$ (because exponential is related to its own derivative)

$\vec{x}' = A\vec{x}$

exponential is expected, but has to be a <u>vector</u>

use form $\vec{x} = \vec{v}e^{rt}$

some constant vector

plug into DE $\vec{x}' = \vec{v}re^{rt}$

$\vec{v}r\cancel{e^{rt}} = A\vec{v}\cancel{e^{rt}}$ divide by e^{rt} (because $e^{rt} \neq 0$)

$$\boxed{\begin{gathered} A\vec{v} = r\vec{v} \\ (A - rI)\vec{v} = \vec{0} \\ \det(A - rI) = 0 \end{gathered}}$$

this is the same as eq. we solve to find eigenvalues and eigenvectors

so, r must be eigenvalue and $\vec{v}$ must be an eigenvector there are n linearly independent solutions $\cancel{\vec{X}^{(n)} = \vec{V}_n e}$

$$\vec{X}^{(n)} = \vec{V}_n e^{\lambda_n t}$$ because there are n eigenvalue / eigenvector pairs

general solution: $\vec{X}(t) = C_1 \vec{V}_1 e^{\lambda_1 t} + C_2 \vec{V}_2 e^{\lambda_2 t} + \cdots + C_n \vec{V}_n e^{\lambda_n t}$

<u>example</u> $y'' + 5y' + 6y = 0$

characteristic eq. $r^2 + 5r + 6 = 0 \rightarrow r = -2, \ r = -3$

so $y = C_1 e^{-2t} + C_2 e^{-3t}$

$y' = C_1 \cdot -2e^{-2t} + C_2 \cdot -3e^{-3t}$

change to system: $x_1 = y \qquad x_1' = x_2$

$x_2 = y' \qquad x_2' = -6x_1 - 5x_2$

$$\vec{X}' = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \vec{X}$$

eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ -6 & -5-\lambda \end{vmatrix} = 0$

$$(-\lambda)(-5-\lambda)+6=0$$

$$\lambda^2+5\lambda+6=0$$ same as characteristic eq.

$$\lambda=-2, \quad \lambda=-3$$

find eigenvectors

$\lambda_1=-2$ $\quad (A-\lambda I)\vec{v}=\vec{0} \qquad \det(A-\lambda I)=0$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -6 & -3 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$V_2=r \qquad 2V_1+V_2=0 \qquad V_1=-\frac{1}{2}V_2=-\frac{1}{2}r$

choose $r=-2$

$$\vec{V}_1=\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$ first element is often made to be 1 by choosing r

$\lambda_2=-3$ $\quad \vec{V}_2=\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

general solution:

$$\vec{X} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-3t}$$

recall $\vec{X} = \begin{bmatrix} y \\ y' \end{bmatrix}$

1st row: $y = c_1 e^{-2t} + c_2 e^{-3t}$

2nd row: $y' = c_1 \cdot -2e^{-2t} + c_2 \cdot -3e^{-3t}$

if we collect the solutions and put them into a matrix, this matrix is called the <u>fundamental matrix</u>

$$\Phi(t) = \begin{bmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{bmatrix}$$

$$\det\left[\Phi(t)\right] = \text{Wronskian}$$

example

$$\vec{X}' = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix} \vec{X} \qquad \vec{X}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} 3-\lambda & -2 \\ 4 & -3-\lambda \end{vmatrix} = 0 \qquad (3-\lambda)(-3-\lambda)+8=0$$

$$\lambda^2 - 1 = 0 \qquad \lambda = \pm 1$$

$\lambda_1 = 1$, $\quad (A-\lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 4 & -4 & 0 \end{array}\right]$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$(A-\lambda I)$

$\lambda_2 = -1$, $$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

general solution: $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$

$$\vec{x}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + 2c_2 \end{bmatrix} \qquad 3 = -c_2 \qquad c_2 = -3$$

$$c_1 = 6$$

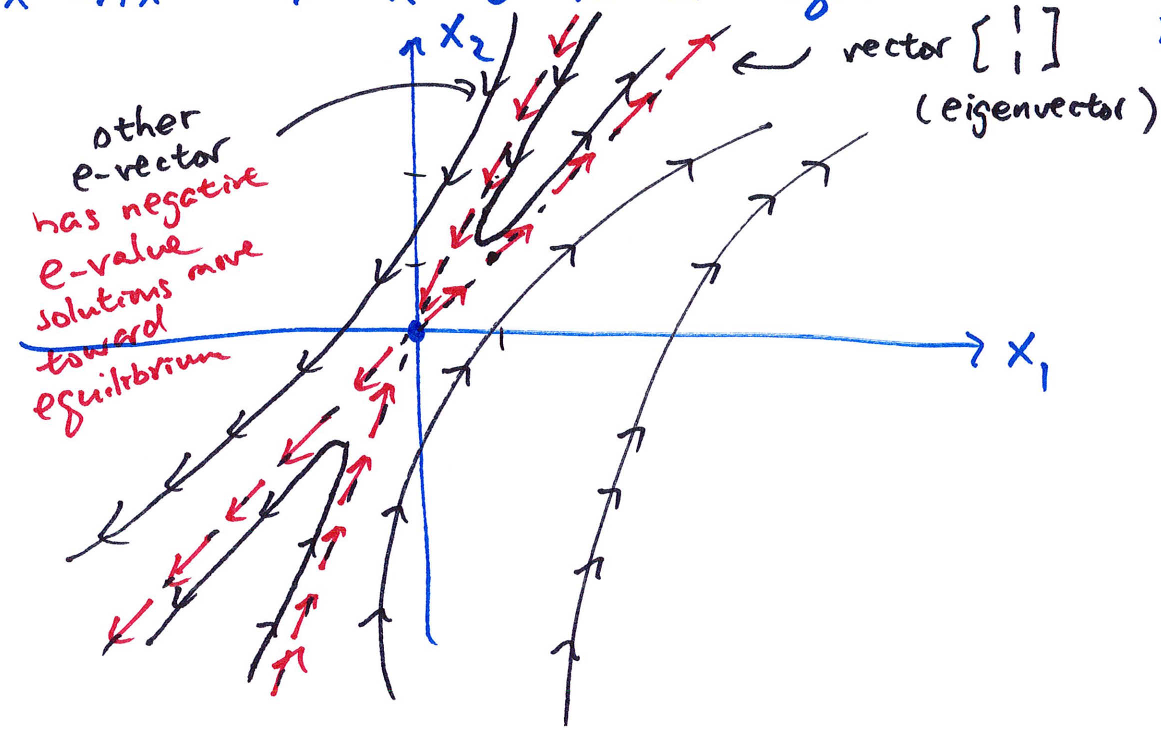
$$\vec{x}(t) = \begin{bmatrix} 6 \\ 6 \end{bmatrix} e^{t} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} e^{-t} = \begin{bmatrix} 6e^{t} - 3e^{-t} \\ 6e^{t} - 6e^{-t} \end{bmatrix}$$

phase plot $\quad x_1$ vs x_2

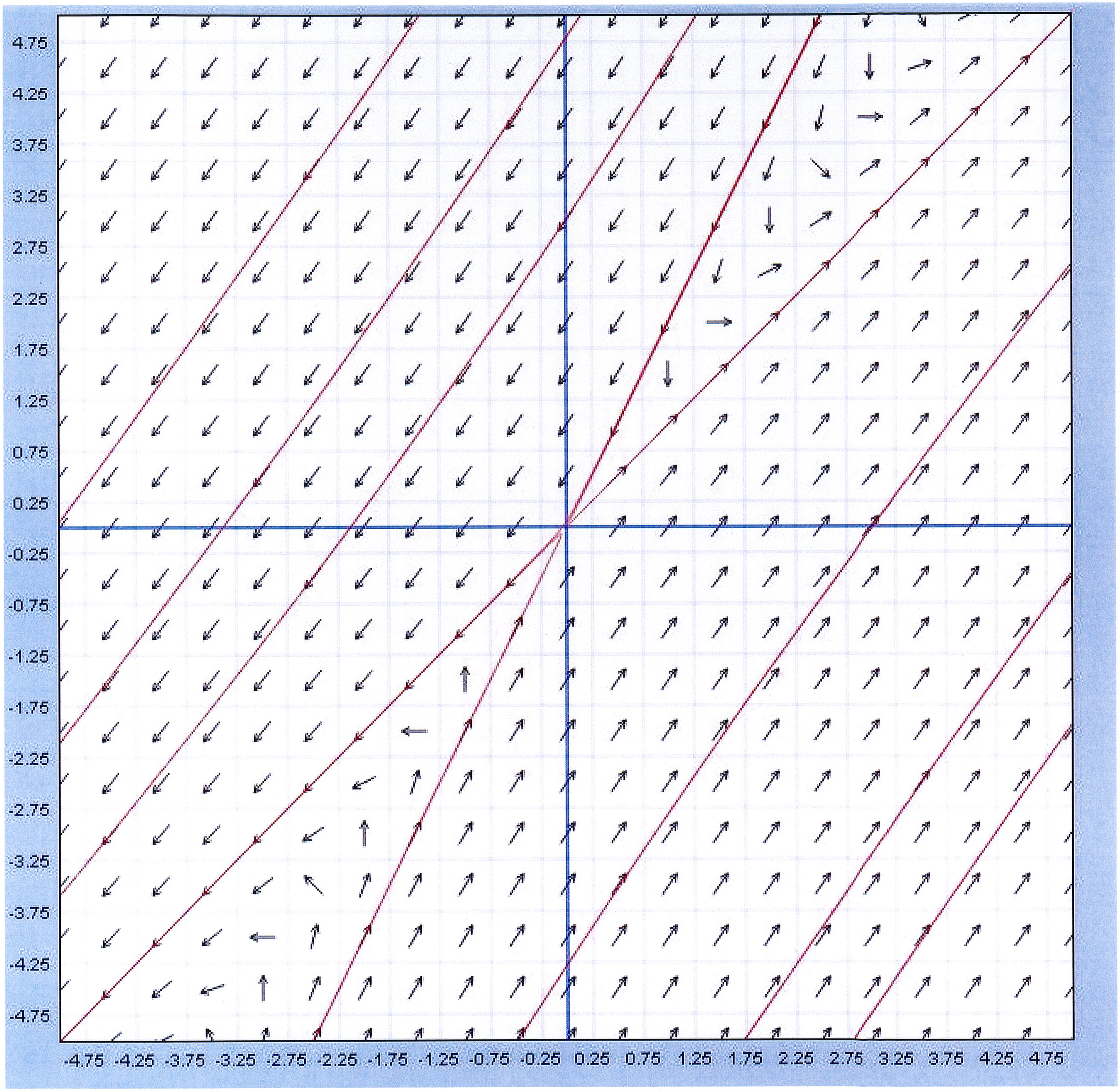
$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$$

$\vec{x}\,' = A\vec{x} \rightarrow \vec{x} = \vec{0}$ is an equilibrium

$$\vec{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$$



→ positive eigenvalue ($\lambda = 1$)
solutions move AWAY from equilibrium


4.75
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this kind of equilibrium is called a <u>saddle point</u>

(stable in some directions, unstable in others)

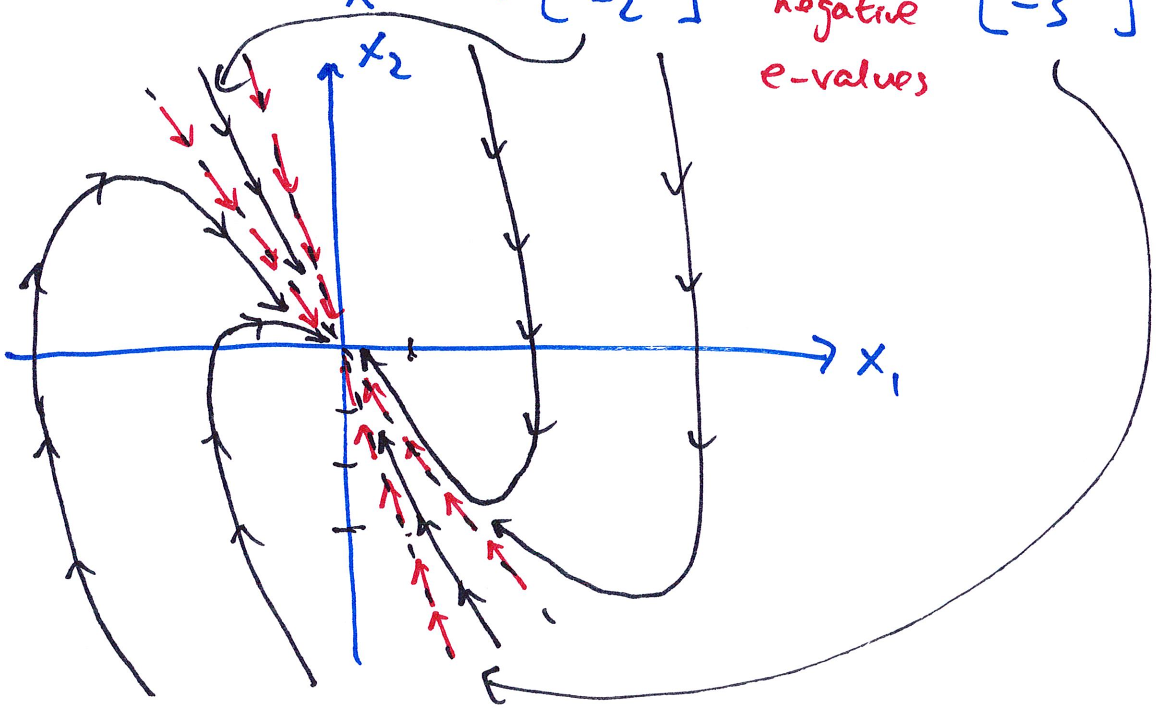
→ eigenvalues are <u>real</u> and of <u>opposite</u> <u>signs</u>

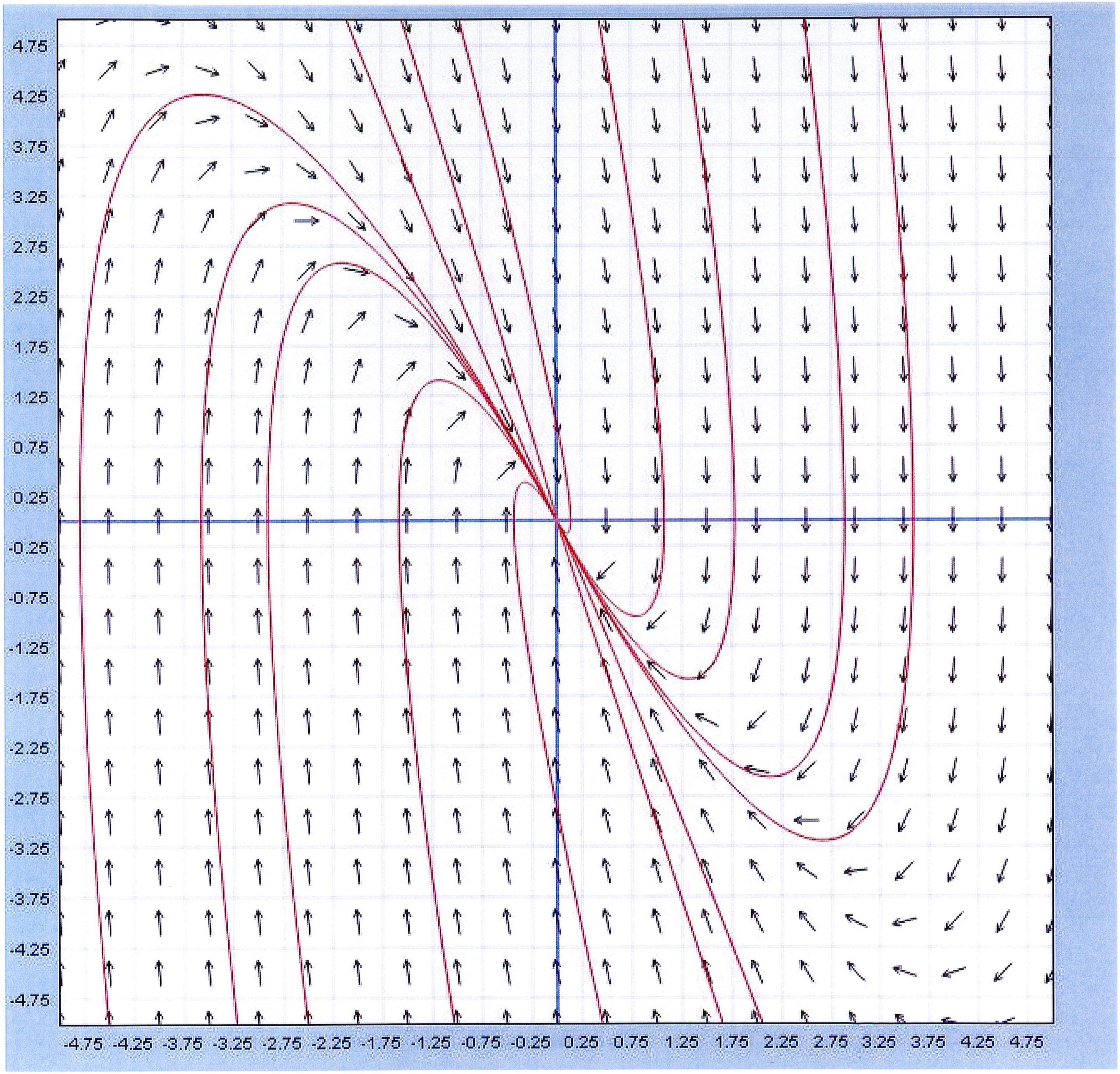
<u>example</u>

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \vec{x} \quad \text{has e-values } -2, -3$$

solution: $\vec{x} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-3t}$

negative e-values

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this equilibrium is an asymptotically stable node

→ two real eigenvalue and both are negative

if both positive and real → unstable node

(opposite of the above)