10. $y'' - 2y' + y = \dfrac{e^t}{1+t^2}$ $\quad y_1 = e^t \quad y_2 = te^t$

$$y = u_1 e^t + u_2 te^t$$

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = g(t)$$

$$u_1' e^t + u_2' te^t = 0$$
$$u_1' e^t + u_2'(te^t + e^t) = \frac{e^t}{1+t^2}$$

$$u_2' e^t = \frac{e^t}{1+t^2} \qquad u_2' = \frac{1}{1+t^2} \qquad u_2 = \tan^{-1}(t) + C_2$$

$$u_1' = -u_2' t$$
$$= -\frac{t}{1+t^2} \qquad u_1 = -\frac{1}{2}\ln|1+t^2| + C_1$$

J. $t^2y'' - 4ty' + 4y = -2t^2$ $\quad y_1 = t \quad y_2 = t^4$

$y(1) = 2 \quad y'(1) = 0$

$$y = u_1 y_1 + u_2 y_2$$

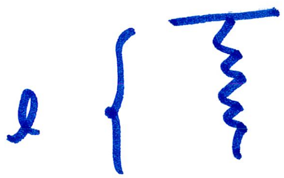
$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = g(t)$$

$$u_1' t + u_2' t^4 = 0$$
$$u_1' + 4u_2' t^3 = -2$$
$$u_1' t + 4u_2' t^4 = -2t$$

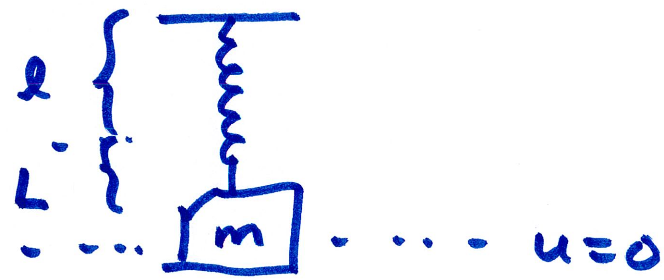
$$3u_2' t^4 = -2t$$
$$u_2' = -\frac{2}{3} t^{-3} \qquad u_2 = +\frac{1}{3} t^{-2} + C_2$$
$$u_1' = -u_2' t^3$$
$$= - -\frac{2}{3} t^{-3} t^3 = \frac{2}{3} \qquad u_1 = \frac{2}{3} t + C_1$$

3.7 Mechanical Vibrations

spring with natural length of ℓ, spring constant K



attach object mass m



↓ down positive

at equilibrium $(u=0)$

force due to spring $F_s = -KL$

and is equal to weight : mg

so $mg - KL = 0$

if $u > 0$, $F_s < 0$
$u < 0$, $F_s > 0$ $\Big\}$ $F_s = -K(L+u)$

now attach a damper that exerts a force proportional to speed in the oppositive direction of velocity. (like drag)

$$F_d = -\gamma u'$$

damping constant

Newton's 2nd Law: $F = ma$

$$mu'' = \underbrace{-k(L+u)}_{\text{spring}} + \underbrace{mg}_{\text{gravity}} - \underbrace{\gamma u'}_{\text{damper}}$$

$$= -kL - ku + mg - \gamma u'$$

$mg - Lk = 0$

$$\boxed{mu'' + \gamma u' + ku = 0}$$

Equation of Motion

$m, \gamma, k > 0$ constants

Linear, constant coeff, homogeneous

roots of char. eq.

$$mr^2+\gamma r+k=0$$

$$r=\frac{-\gamma\pm\sqrt{\gamma^2-4km}}{2m}$$

$$=-\frac{1}{2}\frac{\gamma}{m}\pm\frac{\sqrt{\gamma^2-4km}}{2m}$$

if $\gamma=0$ (undamped free vibration)

$$r=\pm\frac{\sqrt{-4km}}{2m}=\pm i\sqrt{\frac{k}{m}}$$

$$u=c_1\cos\sqrt{\frac{k}{m}}\,t+c_2\sin\sqrt{\frac{k}{m}}\,t$$

$\sqrt{\frac{k}{m}}$: circular frequency (ω_0)

$\frac{2\pi}{\sqrt{\frac{k}{m}}}$: period (T)

if $\gamma^2 > 4km$ (overdamped → no oscillation because roots are real) and distinct

$$u = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

need $\lim_{t\to\infty} u = 0$ from physical intuition

$\gamma, m, k > 0$ so $\gamma^2 > \gamma^2 - 4km$

$$r = -\frac{1}{2}\frac{\gamma}{m} \pm \frac{\sqrt{\gamma^2 - 4km}}{2m}$$ → less than γ

if $\gamma^2 = 4km$ (critically damped → no oscillation) repeated roots

$$u = c_1 e^{rt} + c_2 t e^{rt} \qquad r = -\frac{\gamma}{2m} < 0$$

crit. damped → return to $u=0$ in shortest possible time if $\gamma^2 < 4km$ (underdamped)

has oscillations

$$u = e^{-\gamma/2m\, t}\left(c_1 \cos \frac{\sqrt{\gamma^2 - 4km}}{2m} t + c_2 \sin \frac{\sqrt{\gamma^2 - 4km}}{2m} t \right)$$

<u>example</u> mass weighing 2 lb stretches spring 6 in. no damping.

$$mu'' + \cancel{\gamma u'} + ku = 0$$

deviation from natural

K: Hooke's Law $F = KX$

6 in

$$2\text{ lb} = K \cdot \left(\tfrac{1}{2}\text{ ft}\right)$$

$$K = 4 \text{ lb/ft}$$

m: $\quad 2\text{ lb} = m\,(32\text{ ft/s}^2)$

$$m = \frac{1}{16}\ \underbrace{\text{lb}\cdot\text{s}^2/\text{ft}}_{\text{slug}}$$

eg: $\quad mu'' + ku = 0$

$$\frac{1}{16}u'' + 4u = 0$$

$$u'' + 64u = 0$$

$$u = c_1 \cos 8t + c_2 \sin 8t$$

IC: $\quad \underbrace{u(0) = 3\text{ in}}_{\substack{\text{pull down}\\ \text{3 in from}\\ \text{equilibrium}}} \qquad \underbrace{u'(0) = 3\text{ in/s}}_{\substack{\text{downward initial}\\ \text{velocity}}}$

$\vdots$

$$c_1 = \frac{1}{4} \qquad c_2 = \frac{1}{32}$$

$u = \frac{1}{4}\cos 8t + \frac{1}{32}\sin 8t$

freq.

phase angle

rewrite as $u = R\cos(w_0 t - \delta)$

amplitude

$u = \underbrace{R\cos\delta}\cos w_0 t + \underbrace{R\sin\delta}\sin w_0 t$

$R\cos\delta = \frac{1}{4}$

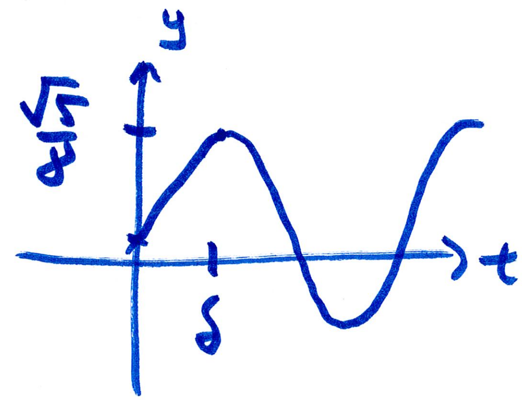
$R\sin\delta = \frac{1}{32}$

$\tan\delta = \frac{1}{8}$ $\quad \delta = \tan^{-1}\left(\frac{1}{8}\right)$

$R^2 = \left(\frac{1}{4}\right)^2 + \left(\frac{1}{32}\right)^2 = \frac{\sqrt{5}}{8}$

$u = \frac{\sqrt{5}}{8}\cos\left(8t - \cancel{\frac{\sqrt{5}}{8}}\right)$

$\tan^{-1}\left(\frac{1}{8}\right)$



example $\cancel{4}u'' + \cancel{4}u' + 37u = 0 \qquad u(0)=1,\ u'(0)=0$

$$u'' + u' + 9.25u = 0$$

$$r = -\frac{1}{2} \pm 3i$$

underdamped

$\vdots$

$$u(t) = e^{-\frac{1}{2}t}\underbrace{\left(\cos 3t + \frac{1}{6}\sin 3t\right)}_{R\cos(\omega t - \delta)}$$

see last example

$$= e^{-\frac{1}{2}t}\,\frac{\sqrt{37}}{6}\left(\cos\left(3t - \tan^{-1}\left(\frac{1}{6}\right)\right)\right)$$

↳ damped frequency

"quasi-frequency"

quasi-period: $\frac{2\pi}{3}$ &

$$= \boxed{\frac{\sqrt{37}}{6} e^{-\frac{1}{2}t}} \left(\cos 3t - \tan^{-1}\left(\frac{1}{6}\right) \right)$$