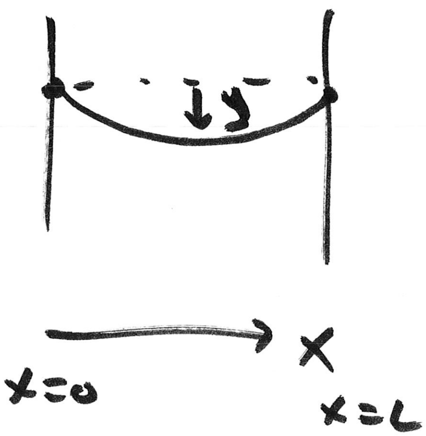
K. $y^{(4)} = -P \qquad P > 0 \qquad L > 0$

$y(0) = y(L) = 0$

$y'(0) = y'(L) = 0$



a). $L = 4,\ P = 24$

$y^{(4)} = 0 \rightarrow r = 0, 0, 0, 0$

$y = c_1 + c_2 t + c_3 t^2 + c_4 t^3 + Y \qquad Y = At^4$

$Y' = 4At^3 \quad Y'' = 12At^2 \quad Y''' = 24At \quad Y^{(4)} = 24A$

$24A = -P \qquad A = -\frac{P}{24} = -\frac{24}{24} = -1$

$y = c_1 + c_2 t + c_3 t^2 + c_4 t^3 \cancel{+} - t^4$

$0 = c_1 + 0 \rightarrow c_1 = 0$

$0 = c_1 + 4c_2 + 16c_3 + 64c_4 - 256 \quad \ldots$

b). max displacement $\rightarrow$ max $y \rightarrow y' = 0$

6.1 The Laplace Transform

the Laplace transform of $f(t)$ is defined as

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt}$$

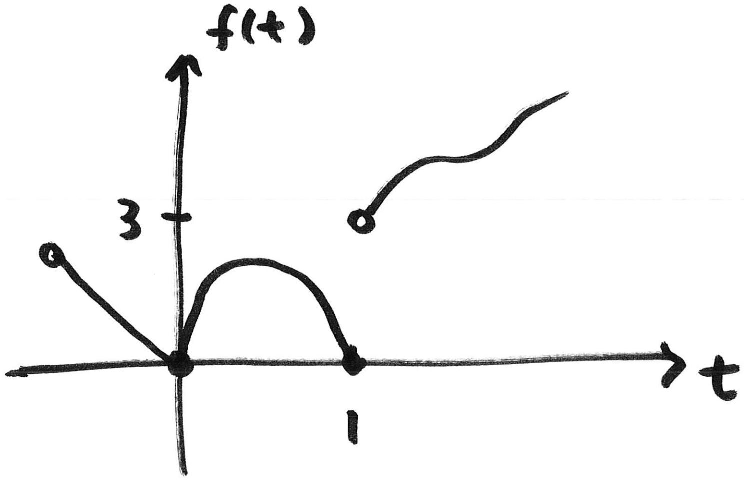
named after Pierre-Simon de Laplace (1749-1827)

application to DEs is due to Oliver Heaviside (1850-1925)

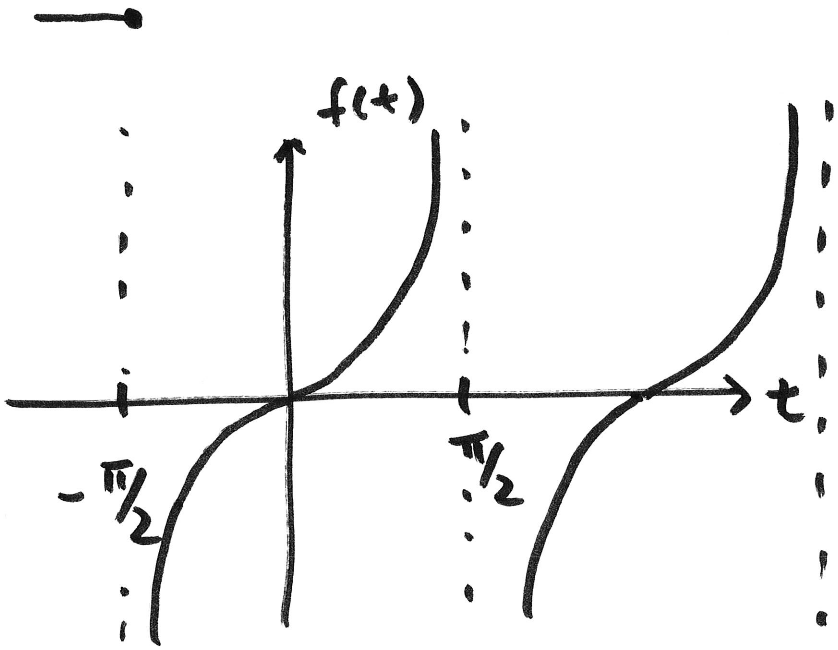
$f(t)$ must be at least piecewise continuous

→ finite number of discontinuities

→ the limit of $f(t)$ at each end of an subinterval must be finite.



piecewise continuous



NOT piecewise continuous because it has infinite limit at at least one end of a subinterval

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

<u>example</u> $f(t) = 1$

$$F(s) = \int_0^\infty e^{-st}\,dt$$

$$= \lim_{A\to\infty} \int_0^A e^{-st}\,dt$$

$$= \lim_{A\to\infty} -\frac{1}{s} e^{-st} \Big|_0^A$$

$$= \lim_{A\to\infty} \left(-\frac{1}{s} e^{-sA} + \frac{1}{s}\right)$$

integral converges if $s > 0$
(otherwise first term blows up)

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0}$$

LIATE

example $f(t) = t$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} \cdot t \, dt$$

$$= \lim_{A\to\infty} \int_0^A t e^{-st} \, dt$$

$u = t \quad dv = e^{-st} dt$

$u = dt \quad v = -\frac{1}{s} e^{-st}$

$$= \lim_{A\to\infty} \left(-\frac{t}{s} e^{-st} \Big|_0^A + \frac{1}{s} \int_0^A e^{-st} \, dt \right)$$

$$= \lim_{A\to\infty} \left(-\frac{A}{s} e^{-sA} - \frac{1}{s^2} e^{-st} \Big|_0^A \right)$$

$$= \lim_{A\to\infty} \left(-\frac{A}{s} e^{-sA} - \frac{1}{s^2} e^{-sA} + \frac{1}{s^2} \right)$$

$s > 0$

$$= \boxed{\frac{1}{s^2}, \quad s > 0}$$

<u>example</u> $\mathcal{L}\{1+t\}$

$$= \int_0^\infty (1+t)e^{-st}\,dt$$

$$= \int_0^\infty 1\cdot e^{-st}\,dt + \int_0^\infty t\cdot e^{-st}\,dt$$

$$= \mathcal{L}\{1\} + \mathcal{L}\{t\}$$

so, $\boxed{\mathcal{L}\{f(t) \pm g(t)\} = \mathcal{L}\{f(t)\} \pm \mathcal{L}\{g(t)\}}$

also, $\boxed{\mathcal{L}\{c\cdot f(t)\} = c\cdot \mathcal{L}\{f(t)\}}$

constant

LT is <u>linear</u>

<u>example</u> $\mathcal{L}\{\sin bt\}$ b is a constant

$$\sin bt = \frac{e^{ibt} - e^{-ibt}}{2i} = \frac{1}{2i}\left(e^{ibt} - e^{-ibt}\right)$$

$$\mathcal{L}\left\{\frac{1}{2i}\left(e^{ibt} - e^{-ibt}\right)\right\} = \frac{1}{2i}\left(\mathcal{L}\{e^{ibt}\} - \mathcal{L}\{e^{-ibt}\}\right)$$

$$= \frac{1}{2i}\int_0^\infty e^{-st}\left(e^{ibt} - e^{-ibt}\right)dt = \frac{1}{2i}\lim_{A\to\infty}\int_0^A e^{-st}\left(e^{ibt} - e^{-ibt}\right)dt$$

$$= \frac{1}{2i}\lim_{A\to\infty}\int_0^A e^{(ib-s)t} - e^{-(ib+s)t}\,dt$$

$$= \frac{1}{2i}\lim_{A\to\infty}\left[\frac{1}{ib-s}e^{(ib-s)t}\Big|_0^A + \frac{1}{ib+s}e^{-(ib+s)t}\Big|_0^A\right]$$

$e^{ibA}e^{-sA}$ $\qquad$ $e^{-ibA}e^{-sA}$

$$= \frac{1}{2i}\lim_{A\to\infty}\left[\underbrace{\frac{1}{ib-s}e^{(ib-s)A}}_{\text{to } 0} - \frac{1}{ib-s} + \underbrace{\frac{1}{ib+s}e^{-(ib+s)A}}_{\text{to } 0} - \frac{1}{ib+s}\right]$$

recall $e^{ibt} = \cos bt + i\sin bt$ magnitude is bounded

$$= \frac{1}{2i}\left(\frac{1}{s-ib} - \frac{1}{s+ib}\right) = \frac{1}{2i}\,\frac{2ib}{s^2+b^2} = \boxed{\frac{b}{s^2+b^2}}$$

$\boxed{s > 0}$

2nd way: $\displaystyle\int_0^\infty e^{-st}\sin bt\,dt$ $\qquad$ integrate by parts <u>twice</u>

7/18

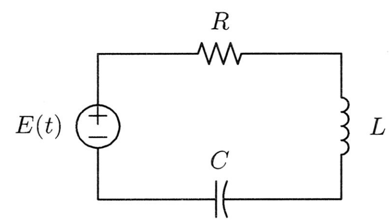
Computer Project 2. RLC Circuits

Goal: Investigate the charge on a capacitor in an RLC circuit with varying voltage.

Tools needed: `ode45`, `plot`

Description: If $Q(t) =$ charge on a capacitor at time t in an RLC circuit (with R, L and C being the resistance, inductance and capacitance, respectively) and $E(t) =$ applied voltage, then Kirchhoff's Laws give the following 2nd order differential equation for $Q(t)$:

$$L\,Q''(t) + R\,Q'(t) + \frac{1}{C}\,Q(t) = E(t) \qquad (*)$$



Questions: Assume $L = 1$, $C = 1/5$, $R = 4$ and $E(t) = 10\cos\omega t$.

1. Use `ode45` (and plot routines) to plot the solution of $(*)$ with $Q(0) = 0$ and $Q'(0) = 0$ over the interval $0 \le t \le 80$ for $\omega = 0, 0.5, 1, 2, 4, 8, 16$.

2. Let $A(\omega) =$ maximum of $|Q(t)|$ over the interval $30 \le t \le 80$ (this approximates the amplitude of the steady-stat solution). Experiment with various values of ω and discuss what appears to happen to $A(\omega)$ as $\omega \to \infty$ and as $\omega \to 0$. Also, interpret your findings in terms of an equivalent spring-mass system.

Remark: There is an analogy between spring-mass system and RLC circuits given by:

Spring-mass system	RLC circuit
$m\,u'' + c\,u' + k\,u = F(t)$	$L\,Q'' + R\,Q' + \frac{1}{C}\,Q = E(t)$
$u =$ Displacement	$Q =$ Charge
$u' =$ Velocity	$Q' = I =$ Current
$m =$ Mass	$L =$ Inductance
$c =$ Damping constant	$R =$ Resistance
$k =$ Spring constant	$1/C = (\text{Capacitance})^{-1}$
$F(t) =$ External force	$E(t) =$ Voltage