28. $u'' + 2u = 0 \qquad u(0) = 0, \quad u'(0) = 2$

$$u = \sqrt{2}\sin\sqrt{2}t \qquad u' = 2\cos\sqrt{2}t$$

$$\text{period} = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$$

9. DE: $u'' + 20u' + 196u = 0 \qquad u(0) = 2 \quad u'(0) = 0$

$\vdots$

$$u = e^{-10t}\left[2\cos(\underbrace{4\sqrt{6}}_{}t) + \frac{5}{\sqrt{6}}\sin(4\sqrt{6}t)\right]$$

$$\omega = 4\sqrt{6}$$

$$u = \frac{7}{\sqrt{6}}e^{-10t}\cos\left(4\sqrt{6}t - \tan^{-1}\left(\frac{5}{2\sqrt{6}}\right)\right)$$

$|y| < 0.05$

$-0.05 = u$

$$T_d = \frac{2\pi}{4\sqrt{6}}$$

undamped: $u'' + 196u = 0$

$\omega_0 = 14 \qquad T = \frac{2\pi}{14}$

$$\frac{T_d}{T} = 1.43$$

3.8 Forced Vibrations

$$mu'' + \gamma u' + ku = f(t)$$

$f(t)$ is a periodic function
"forcing function"
"disturbing function"

focus on $\gamma = 0$ cases

example

$$u'' + 16u = 2\cos 2t \qquad u(0) = u'(0) = 0$$

$$u = C_1 \cos 4t + C_2 \sin 4t + Y$$

undetermined coeff is good.

$$Y = A\cos 2t + B\sin 2t$$

$$\vdots$$

$$A = \frac{1}{6} \qquad B = 0$$

then w/ ICs, $C_1 = -\frac{1}{6}$, $C_2 = 0$

$$u(t) = \frac{1}{6}\cos 2t - \frac{1}{6}\cos 4t$$

same amplitude, different frequencies

transform w/ identities and a "trick"

$$\cos At - \cos Bt$$

$$= \cos\left(\frac{At-Bt}{2} + \frac{At+Bt}{2}\right)$$

$$- \cos\left(\frac{At-Bt}{2} - \frac{At+Bt}{2}\right)$$

$$u = \frac{1}{6}\cos 2t - \frac{1}{6}\cos 4t = \frac{1}{6}\left(\cos 2t - \cos 4t\right)$$

$$= \frac{1}{6}\left[\cos(-t+3t) - \cos(-t-3t)\right]$$

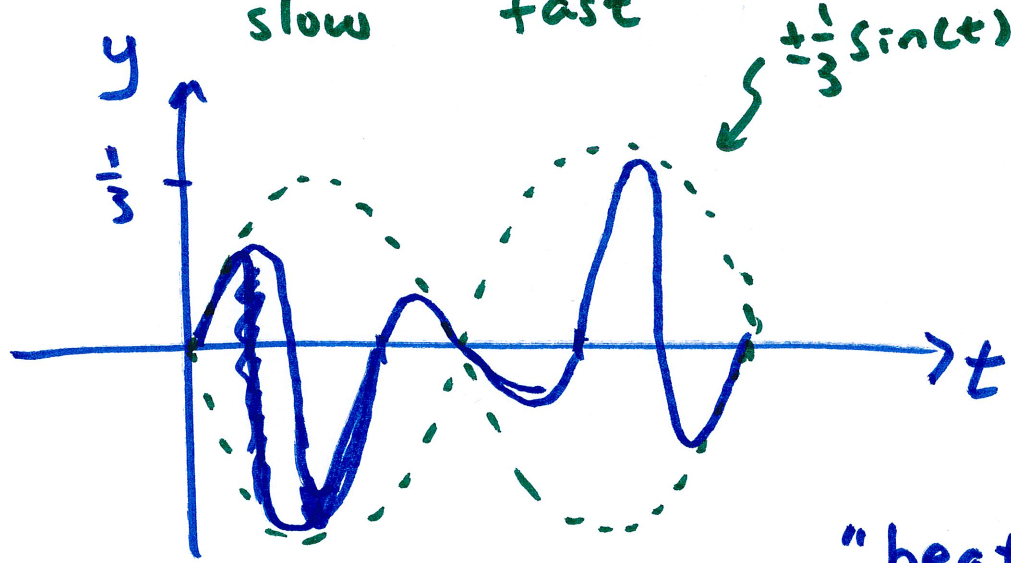
then use $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$$= \frac{1}{6}\left[\cancel{\cos(-t)\cos(3t)} - \sin(-t)\sin(3t)\right.$$

$$\left. - \cancel{\cos(-t)\cos(3t)} + \sin(-t)\sin(3t)\right]$$

$$= \frac{1}{6}\left[-2\underbrace{\sin(-t)}_{-\sin(t)}\sin(3t)\right] = \frac{1}{3}\sin(t)\sin(3t)$$

$$u = \frac{1}{3}\underbrace{\sin(t)}_{\text{slow}}\ \underbrace{\sin(3t)}_{\text{fast}}$$



"beat"

two waves w/ nearly same frequencies

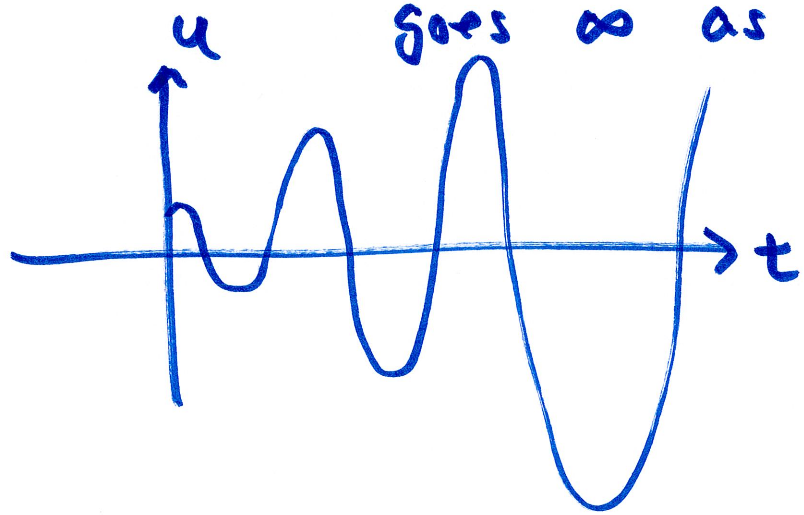
example $\quad u'' + 16u = 2\cos 4t \qquad u(0) = \frac{1}{3} \quad u'(0) = 0$

$$u = c_1 \cos 4t + c_2 \sin 4t + A\,t \cos 4t + B\,t \sin 4t$$

$$\vdots$$

$$\boxed{u = \frac{1}{3}\cos 4t + 4t \sin 4t}$$

goes ∞ as $t \to \infty$



"resonance"

applied force has same or nearly the same frequency as fundamental freq.

Computer Project 1. Nonlinear Springs

Goal: Investigate the behavior of nonlinear springs.

Tools needed: `ode45`, `plot`

Description: For certain (nonlinear) spring-mass systems, the spring force is not given by Hooke's Law but instead satisfies

$$F_{\text{spring}} = ku + \epsilon u^3,$$

where $k > 0$ is the spring constant and ϵ is small but may be positive or negative and represents the "strength" of the spring ($\epsilon = 0$ gives Hooke's Law). The spring is called a *hard spring* if $\epsilon > 0$ and a *soft spring* if $\epsilon < 0$.

Questions: Suppose a nonlinear spring-mass system satisfies the initial value problem

$$\begin{cases} u'' + u + \epsilon u^3 = 0 \\ u(0) = 0, \quad u'(0) = 1 \end{cases}$$

Use `ode45` and `plot` to answer the following:

1. Let $\epsilon = 0.0, 0.2, 0.4, 0.6, 0.8, 1.0$ and plot the solutions of the above initial value problem for $0 \le t \le 20$. Estimate the amplitude of the spring. Experiment with various $\epsilon > 0$. What appears to happen to the amplitude as $\epsilon \to \infty$? Let μ^+ denote the first time the mass reaches equilibrium after $t = 0$. Estimate μ^+ when $\epsilon = 0.0, 0.2, 0.4, 0.6, 0.8, 1.0$. What appears to happen to μ^+ as $\epsilon \to \infty$?

2. Let $\epsilon = -0.1, -0.2, -0.3, -0.4$ and plot the solutions of the above initial value problem for $0 \le t \le 20$. Estimate the amplitude of the spring. Experiment with various $\epsilon < 0$. What appears to happen to the amplitude as $\epsilon \to -\infty$? Let μ^- denote the first time the mass reaches equilibrium after $t = 0$. Estimate μ^- when $\epsilon = -0.1, -0.2, -0.3, -0.4$. What appears to happen to μ^- as $\epsilon \to -\infty$?

3. Given that a certain nonlinear hard spring satisfies the initial value problem

$$\begin{cases} u'' + \frac{1}{5}u' + \left(u + \frac{1}{5}u^3\right) = \cos \omega t \\ u(0) = 0, \quad u'(0) = 0 \end{cases}$$

plot the solution $u(t)$ over the interval $0 \le t \le 60$ for $\omega = 0.5, 0.7, 1.0, 1.3, 2.0$. Continue to experiment with various values of ω, where $0.5 \le \omega \le 2.0$, to find a value ω^* for which $|u(t)|$ is largest over the interval $40 \le t \le 60$.

ode45 Differential Equation Solver

This routine uses a variable step Runge-Kutta Method to solve differential equations numerically. The syntax for `ode45` for first order differential equations and that for second order differential equations are basically the same. However, the `.m` files are quite different.

I. First Order Equations $\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$

A. Create a `.m` file for $f(t,y)$ (see the tutorial on numerical methods and `.m` files on how to do this). Save file as, for example, `yp.m`.

B. *Basic syntax for* `ode45`. At a MATLAB prompt type:

```
[t,y]=ode45('yp',[t0,tf],y0);
```

(your version of `ode45` may not require brackets around `t0,tf`)

$$\begin{cases} \texttt{yp} = \text{the } \texttt{.m} \text{ file of the function } f(t,y) \text{ saved as } \texttt{yp.m} \\ \texttt{t0,tf} = \text{initial and terminal values of } t \\ \texttt{y0} = \text{initial value of } y \text{ at } \texttt{t0} \end{cases}$$

C. For example, to numerically solve $\begin{cases} t^2 y' = y + 3t \\ y(1) = -2 \end{cases}$ over $1 \le t \le 4$:

* Create and save the file `yp.m` for the function $\frac{1}{t^2}(y + 3t)$.
* At a MATLAB prompt type:

```
[t,y]=ode45('yp',[1,4],-2);
```

(your version of `ode45` may not require brackets around `1,4`)

* To print results type: `[t,y]`
* To plot results type: `plot(t,y)`
* To plot results type with a '+' symbol: `plot(t,y,'+')`

II. Second Order Equations $\begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ y(t_0) = y_0,\ y'(t_0) = y_1 \end{cases}$

A. First convert 2nd order equation to an equivalent system of 1st order equations. Let $x_1 = y$, $x_2 = y'$:

$$\begin{cases} x_1' = x_2, \\ x_2' = -q(t)x_1 - p(t)x_2 + g(t) \\ x_1(t_0) = y_0,\ x_2(t_0) = y_1 \end{cases}$$

B. Create and save a `.m` file which will return a *vector-valued* function. This is a little tricky so here is a specific example.

* Suppose the system is as below and $0 \le t \le 4$

$$\begin{cases} x_1' = x_2, \\ x_2' = -t\,x_1 - e^t x_2 + 3\sin 2t \\ x_1(0) = 2,\ x_2(0) = 8 \end{cases}$$

* Create the following function file and save it as `F.m`:

```
function xp=F(t,x)
xp=zeros(2,1); % since output must be a column vector
xp(1)=x(2);
xp(2)=-t*x(1)+exp(t)*x(2)+3*sin(2*t);
```

(Don't forget the ";" after each line.)

C. *Basic syntax for* `ode45`. At a MATLAB prompt, type:

```
[t,x]=ode45('F',[t0,tf],[x10,x20]);
```

$$\begin{cases} \texttt{F} = \text{the } \texttt{.m} \text{ file of the vector-function saved as above} \\ \texttt{t0,tf} = \text{initial and terminal values of } t \\ \texttt{x10} = \text{initial value of } x_1 \text{ at } \texttt{t0}\text{: } x10 = x_1(t_0) \\ \texttt{x20} = \text{initial value of } x_2 \text{ at } \texttt{t0}\text{: } x20 = x_2(t_0) \end{cases}$$

(The example above becomes: `[t,x]=ode45('F',[0,4],[2,8]);`)

* Since $x_1(t) = y$, to print out the values of the solution y for $t_0 \le t \le t_f$, at a MATLAB prompt type: `[t,x(:,1)]`
* To plot the solution on a graph t vs y, type: `plot(t,x(:,1))` (since the vector **x** has 1st component $x_1 = y$ and 2nd component $x_2 = y'$.)
* To plot x_1 vs x_2 (phase plane) type: `plot(x(:,1),x(:,2))`