4.1 + 4.2 Higher-Order Linear Equations

recall $y' + p(t)y = g(t)$, $y(t_0) = y_0$ has unique solution on interval where $p(t), g(t)$ are continuous and containing t_0.

$y'' + p(t)y' + q(t)y = g(t)$, $y(t_0) = y_0$, $y'(t_0) = y_0'$ has unique solution on interval where $p(t)$, $q(t)$, $g(t)$ are continuous and containing $t = t_0$.

for n^{th}-order linear eq.

$$y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + \cdots + p_n(t)y = g(t)$$

$$y(t_0) = y_0,\ y'(t_0) = y_0',\ y''(t_0) = y_0'',\ \ldots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}$$

has unique solution on interval where $p_1, p_2, \ldots, p_n$ and $g(t)$ are continuous, containing $t = t_0$.

example $y^{(5)} + \frac{t^2}{t-8} y''' + \frac{\sin t}{t-8} y'' + \frac{\ln t}{t-8} y = \frac{\sqrt{t}}{t-8}$

$t \neq 8$, $t > 0$

intervals where there are unique solutions:

$(0, 8)$ or $(8, \infty)$

need to to pick.

Solution of constant-coeff eq.

example $2y''' - 4y'' - 2y' + 4y = 0$

characteristic eq: $2r^3 - 4r^2 - 2r + 4 = 0$

$$r^3 - 2r^2 - r + 2 = 0$$

$$r^2(r-2) - (r-2) = 0$$

$$(r-2)(r^2-1) = 0 \qquad r = -1, 1, 2$$

$$\boxed{y = c_1 e^{-t} + c_2 e^{t} + c_3 e^{2t}}$$

example $y^{(4)} + 4y''' + 4y'' = 0$

$$r^4 + 4r^3 + 4r^2 = 0$$

$$r^2(r^2 + 4r + 4) = 0$$

$$r^2(r+2)^2 = 0 \qquad r = 0, 0, -2, -2$$

$$\boxed{y = c_1 + c_2 t + c_3 e^{-2t} + c_4 t e^{-2t}}$$

example

$$y''' - 5y'' + 3y' + y = 0$$

$$r^3 - 5r^2 + 3r + 1 = 0$$

factor by grouping is harder.

by inspection, $r = 1$ is a root

$$(r-1)(ar^2 + br + c) = 0$$

$$ar^3+br^2+cr-ar^2-br-c =$$
$$r^3-5r^2+3r+1$$

$a=1$, $\quad b-a=-5$, $\quad c-b=3$, $\quad -c=1$

$b=-4 \qquad c=-1$

$$(r-1)(\underbrace{r^2-4r-1}_{})=0$$

$$\downarrow$$

$r=1$

$$r=\frac{4\pm\sqrt{16+4}}{2}=\frac{4\pm2\sqrt5}{2}$$
$$=2\pm\sqrt5$$

$$\boxed{y=c_1e^t+c_2e^{(2+\sqrt5)t}+c_3e^{(2-\sqrt5)t}}$$

example An 8th-order DE has the following roots to the characteristic eq.

$r = 2,\ 3,\ 3,\ 3,\ 2+3i,\ 2-3i,\ 2+3i,\ 2-3i$

what is the general solution?

$$\boxed{\begin{aligned} y = {} & c_1 e^{2t} + c_2 e^{3t} + c_3 t e^{3t} + c_4 t^2 e^{3t} \\ & + c_5 e^{2t}\cos 3t + c_6 e^{2t}\sin 3t \\ & + c_7 t e^{2t}\cos 3t + c_8 t e^{2t}\sin 3t \end{aligned}}$$

<u>example</u> $y''' + y' = 0$ $\quad y(0)=0,\ y'(0)=1,\ y''(0)=2$

$$r^3 + r = 0$$

$$r(r^2+1)=0 \qquad r = 0,\ i,\ -i$$

$$y = C_1 + C_2 \cos t + C_3 \sin t$$

$$0 = C_1 + C_2$$

$$\boxed{y = 2 - 2\cos t + \sin t}$$

$$y' = -C_2 \sin t + C_3 \cos t$$

$$1 = C_3$$

$$y'' = -C_2 \cos t - C_3 \sin t$$

$$2 = -C_2 \qquad C_1 = 2$$

in general, solve system of n equations for n unknowns for an n^{th} order eq.