24. $y'' + 4y = \begin{cases} 1 & 0 \le t < \pi \\ 0 & \pi \le t < \infty \end{cases}$ $\quad y(0)=1, \quad y'(0)=0$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\pi e^{-st}\, dt = -\frac{1}{s} e^{-st} \Big|_0^\pi$$

$$= -\frac{1}{s} e^{-s\pi} + \frac{1}{s}$$

$$\mathcal{L}\{y'' + 4y\} = s^2 Y - s y(0) \cancel{- y'(0)} + 4Y$$

$$s^2 Y + 4Y = s + \frac{1 - e^{-\pi s}}{s}$$

$$(s^2 + 4) Y = s + \frac{1 - e^{-\pi s}}{s}$$

$$Y = \frac{s}{s^2 + 4} + \frac{1 - e^{-\pi s}}{s(s^2 + 4)}$$

21. $y'' - 2y' + 2y = \cos t \qquad y(0) = 1 \qquad y'(0) = 0$

$$s^2Y - sy(0) - \cancel{y'(0)} - 2sY + 2y(0) + 2Y = \frac{s}{s^2+1}$$

$$(s^2 - 2s + 2)Y = \frac{s}{s^2+1} + s - 2 \qquad \frac{s-2}{(s-1)^2+1}$$

$$Y = \frac{s}{(s^2+1)(s^2-2s+2)} + \boxed{\frac{s-2}{s^2-2s+2}} \qquad \frac{s-1}{(s-1)^2+1} - \frac{1}{(s-1)^2+1}$$

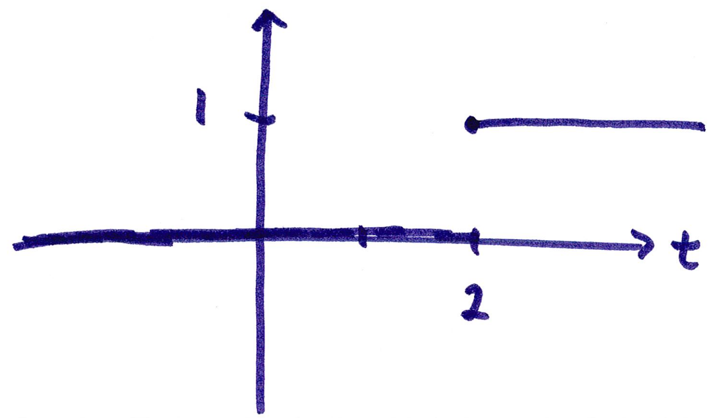
$$\frac{s}{(s^2+1)(s^2-2s+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-2s+2}$$

Solve for A, B, C, D

6.3 Step Functions

define a unit step function $u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$

example: $u_2(t) = \begin{cases} 0 & t < 2 \\ 1 & t \geq 2 \end{cases}$



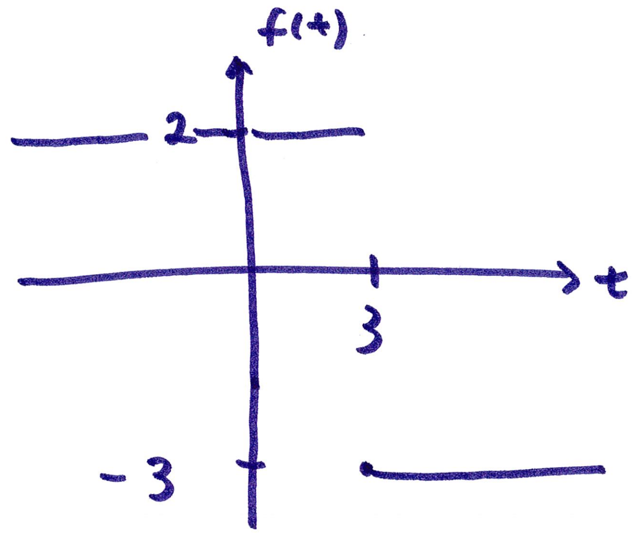
$$\mathcal{L}\{u_c(t)\} = \int_c^\infty e^{-st}\,dt = \lim_{A\to\infty} \int_c^A e^{-st}\,dt$$

$$= \lim_{A\to\infty} -\frac{1}{s} e^{-st}\Big|_c^A = \lim_{A\to\infty} \cancel{-\frac{1}{s}e^{-sA}} + \frac{1}{s}e^{-cs}$$

$$\boxed{\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}}$$

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

<u>example</u> Find LT of $f(t) = \begin{cases} 2 & -\infty < t < 3 \\ -3 & 3 \leq t < \infty \end{cases}$



$$f(t) = 2 - 5u_3(t)$$

$$\boxed{F(s) = \frac{2}{s} - \frac{5}{s}e^{-3s}}$$

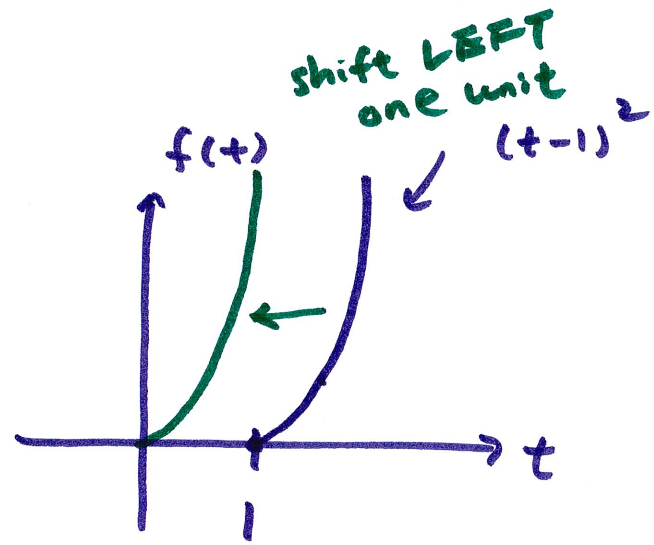
generalize: $u_c(t) \cdot g(t) = \begin{cases} 0 & t < c \\ g(t) & t \geq c \end{cases}$

$$\mathcal{L}\{u_c(t)\cdot f(t-c)\} = e^{-cs}\underbrace{\mathcal{L}\{f(t)\}}_{\text{NOT LT of this}} = e^{-cs}F(s)$$

<u>example</u>

$$f(t) = u_1(t)\,(t-1)^2$$

$$= \begin{cases} 0 & t<1 \\ (t-1)^2 & t\geq 1 \end{cases}$$



$F(s) = e^{-s}\mathcal{L}\{$ what $(t-1)^2$ would look like if it is shifted to start at $t=0\}$

shift $(t-1)^2$ LEFT one unit

replace t with $t+1$ — transform THIS

so $(t-1)^2$ becomes $\boxed{t^2}$ after shift — NOT this

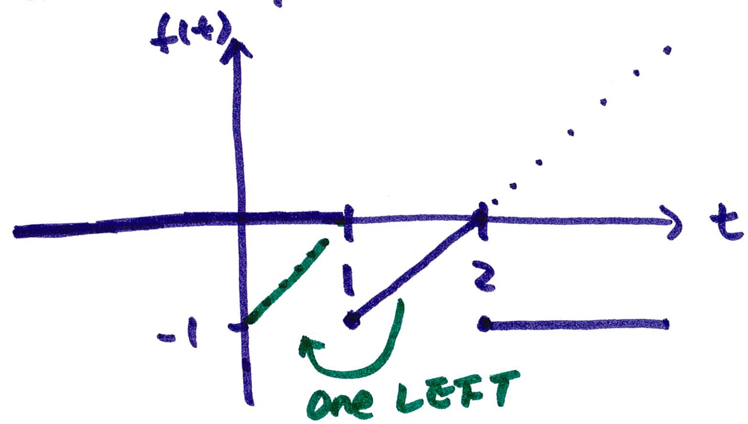
$$f(t) = u_1(t)\,(t-1)^2$$

$$F(s) = e^{-s}\,\mathcal{L}\{t^2\} = e^{-s}\,\frac{2!}{s^{2+1}}$$

$$= \boxed{e^{-s}\,\frac{2}{s^3}}$$

<u>example</u>

$$f(t) = u_1(t)\,(t-2) - u_2(t)\,(t-1)$$



$t=2:$ $\quad t-2-(t-1)$

$\quad -1$

$t \geq 2:$ $\quad (t-2)-(t-1)$

$\quad -1$ for all t

$$F(s) = e^{-s}\mathcal{L}\{t-1\} - e^{-2s}\mathcal{L}\{t+1\}$$

$t-1$ shift LEFT 2

$$= \boxed{e^{-s}\left(\frac{1}{s^2} - \frac{1}{s}\right) - e^{-2s}\left(\frac{1}{s^2} + \frac{1}{s}\right)}$$

Inverse LT $\qquad \mathcal{L}\{u_c(t)\, f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$

<u>example</u>

$$\underbrace{\mathcal{L}^{-1}\left\{\frac{2(s-1)}{s^2-2s+2}\boxed{e^{-2s}}\right\}}$$

discontinuity at $t=2$

e^{-2s}

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2(s-1)}{s^2-2s+2}\right\} = 2e^{t}\cos t$$

shift RIGHT two units

$$= \boxed{2u_2(t)\, e^{t-2}\cos(t-2)}$$

Reverse ALL steps we took to do LT