6. π^{-1+2i} $\qquad e^{\ln x} = x \qquad e^{a+ib} = e^{a}(\cos b + i \sin b)$

$$= \left(e^{\ln \pi}\right)^{-1+2i}$$

$$= e^{-\ln \pi} \, e^{i(2\ln \pi)} = e^{\ln \pi^{-1}}\left(\cos 2\ln \pi + i \sin 2\ln \pi\right)$$

$$= \frac{1}{\pi}\left(\cos 2\ln \pi + i \sin 2\ln \pi\right)$$

23. $3u'' - u' + 2u = 0 \qquad u(0) = 2,\; u'(0) = 0$

$3r^2 - r + 2 = 0$

$$r = \frac{1 \pm \sqrt{1 - 4(3)(2)}}{2(3)} = \frac{1 \pm i\sqrt{23}}{6}$$

3.4 Repeated Roots; Reduction of Order

$y'' + 6y' + 9y = 0$

$r^2 + 6r + 9 = 0 \qquad (r+3)(r+3) = 0$

$r_1 = -3, \; r_2 = -3$

first solution: $y_1 = e^{-3t}$

2nd solution: $y_2 = ?$ can't use $y_2 = y_1$ b/c Wronskian $= 0$ for all t

to find y_2, use method of <u>Reduction of Order</u>

assume $y_2 = v(t)\, y_1$ $v(t)$: function of t

$= v e^{-3t}$ plug y_2 into DE

product rule

$y_2' = -3ve^{-3t} + v'e^{-3t}$

$y_2'' = 9ve^{-3t} - 6v'e^{-3t} + v''e^{-3t}$

Sub into $y'' + 6y' + 9y = 0$

$$\cancel{9v}e^{-3t} - \cancel{6v'}e^{-3t} + v''e^{-3t} - \cancel{18ve^{-3t}} + \cancel{6v'e^{-3t}} + \cancel{9ve^{-3t}} = 0$$

$v''e^{-3t} = 0$ $\qquad e^{-3t} \neq 0$ so

$v'' = 0$ $\qquad v' = c_1$ $\qquad v = c_1 t + c_2$

2nd solution: $y_2 = v y_1 = (c_1 t + c_2) e^{-3t}$

$$= \underbrace{c_1 t e^{-3t}}_{\text{for simplicity, choose } c_1 = 1} + \underbrace{c_2 e^{-3t}}_{\text{repeat of } y_1 \text{ don't include in } y_2 \text{ choose } c_2 = 0}$$

$y_2 = t y_1$

$$\boxed{y_2 = t e^{-3t}}$$

for 2nd order linear homogeneous, constant coefficient DE, V is always t

$$y'' + a y' + b y = 0$$ ← any of this form w/ repeated roots.

example

$$9y'' - 12y' + 4y = 0 \qquad y(0) = 2, \quad y'(0) = -1$$

$$9r^2 - 12r + 4 = 0$$

$$(3r - 2)(3r - 2) = 0 \qquad r = 2/3, \; 2/3$$

$$y_1 = e^{2/3 t} \qquad y_2 = t e^{2/3 t}$$

because DE is linear, homogeneous constant coeff

$$y = c_1 e^{2/3 t} + c_2 t e^{2/3 t}$$

$$y' = \frac{2}{3} c_1 e^{2/3 t} + c_2 \left(\frac{2}{3} e^{2/3 t} \cdot t + e^{2/3 t} \right)$$

$2 = c_1$

$-1 = \frac{2}{3}(2) + c_2$ $\qquad c_2 = -7/3$

$$\boxed{y(t) = 2e^{2/3 t} - \frac{7}{3} t e^{2/3 t}}$$

if t is small, first term (y_1) dominates

Solution looks like y_1 first.

if t is large, 2nd term (y_2) is dominant

Solution looks like y_2 later

example $t^2y'' + 2ty' - 2y = 0$ linear, homogeneous

$\boxed{\text{but not constant coeff.}}$

$y_1 = t$, find y_2 using Reduction of Order.

$V(t) \neq t$ in general!

$$y_2 = v(t)\, y_1 = vt$$

$$y_2' = v + v't$$

$$y_2'' = v' + v' + v''t = 2v' + v''t$$

$$t^2(2v' + v''t) + 2t(v + v't) - 2vt = 0$$

$$2t^2v' + v''t^3 + \cancel{2tv} + 2v't^2 - \cancel{2vt} = 0$$

$$t^3 v'' + 4t^2 v' = 0$$

$$\rightarrow \quad t^3 (v')' + 4t^2 (v') = 0$$

1st order in v'
linear
separable

$$(v')' = -\frac{4t^2}{t^3}(v')$$

$$\frac{d(v')}{dt} = -\frac{4}{t}(v')$$

$$\frac{1}{(v')}\, d(v') = -\frac{4}{t}\, dt$$

$$\ln|v'| = -4\ln t + k_1$$

$$= \ln t^{-4} + k_1$$

$$v' = k_1 t^{-4}$$

$$v = \frac{k_1}{-3} t^{-3} + k_2 = k_1 t^{-3} + k_2$$

$$y_1 = t \qquad y_2 = vt$$

$$= (k_1 t^{-3} + k_2)t$$

$$= k_1 t^{-2} + k_2 t$$

already in y_1, don't want so choose $k_2 = 0$

not in y_1, choose $k_1 = 1$ for simplicity

$$\boxed{y_2 = t^{-2}}$$