3.2 Solutions of Linear Homogeneous Eqs

If $y'' + p(t)y' + q(t)y = 0$, we know there are two solutions $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ where r_1 and r_2 are roots of the characteristic eq.

The linear combination $y = c_1 y_1 + c_2 y_2$ is also a solution, called the general Solution.

Why?

If y_1 and y_2 are solutions, then

$$y_1'' + p(t)y_1' + q(t)y_1 = 0$$

$$y_2'' + p(t)y_2' + q(t)y_2 = 0$$

If $y = c_1 y_1 + c_2 y_2$ is a solution for any c_1, c_2, then it also satisfies the DE.

$$y'' + p(t)\,y' + q(t)\,y = 0 \qquad \begin{aligned} y' &= c_1 y_1' + c_2 y_2' \\ y'' &= c_1 y_1'' + c_2 y_2'' \end{aligned}$$

$$c_1 y_1'' + c_2 y_2'' + p(t)\left[c_1 y_1' + c_2 y_2'\right] + q(t)\left[c_1 y_1 + c_2 y_2\right] = 0$$

$$c_1 \underbrace{\left[y_1'' + p(t)\,y_1' + q(t)\,y_1\right]}_{\substack{\text{zero, because} \\ \text{this is the DE} \\ \text{and } y_1 \text{ is solution}}} + c_2 \underbrace{\left[y_2'' + p(t)\,y_2' + q(t)\,y_2\right]}_{\text{zero, same reason}} = 0$$

→ this proves that $y = c_1 y_1 + c_2 y_2$ <u>is</u> a solution for ANY c_1, c_2.

BUT, we only care about c_1 and c_2 that satisfy the IC's : $y(t_0) = y_0$, $y'(t_0) = y_0'$

can we always find c_1 and c_2 for ANY IC's?

given $y'' + p(t)y' + g(t)y = 0$ has solutions y_1, y_2 and general solution $y = c_1 y_1 + c_2 y_2$ and IC's: $y(t_0) = y_0$, $y'(t_0) = y_0'$

to find c_1, c_2, we solve

$$y_0 = c_1 y_1(t_0) + c_2 y_2(t_0)$$

$$y_0' = c_1 y_1'(t_0) + c_2 y_2'(t_0)$$

rewrite: $$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

the inverse exists if and only if

$$\det\left(\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \right) \neq 0$$

this determinant is called the <u>Wronskian</u>

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

c_1 and c_2 ~~also~~ always exist for any IC's

if $W(y_1, y_2)(t_0) \neq 0$

<u>example</u> $y_1 = \cos t \quad y_2 = \sin t$

$$W(y_1, y_2) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = (\cos t)^2 - (-\sin t)(\sin t)$$

$$= \cos^2 t + \sin^2 t = 1$$

never zero for ANY t_0

<u>example</u> If the Wronskian ~~is~~ of f and g is $3e^{4t}$ and $f(t) = \cancel{e^{2t}}\ e^{2t}$ what is $g(t)$?

$$W = 3e^{4t} = \begin{vmatrix} e^{2t} & g \\ 2e^{2t} & g' \end{vmatrix}$$

$$3e^{4t} = e^{2t} g' - 2e^{2t} g$$ 1st order linear

$$e^{2t} g' - 2e^{2t} g = 3e^{4t}$$

$$g' - 2g = 3e^{2t} \qquad \mu = e^{-2t}$$

$$e^{-2t} g' - 2e^{-2t} g = 3$$

$$\frac{d}{dt}\left(e^{-2t} g\right) = 3$$

$$e^{-2t} y = 3t + \cancel{k}C$$

$$y = 3te^{2t} + Ce^{2t} = (3t+c)e^{2t}$$

placement of f and g is irrelevant

$$\begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} g & f \\ g' & f' \end{vmatrix} \quad \text{are}$$

both ok.

If $W(y_1, y_2)(t_0) \neq 0$, then y_1 and y_2 are the fundamental solutions and they form the fundamental set of solutions

Existence and Uniqueness of Solutions

The initial-value problem $y'' + p(t)y' + q(t)y = g(t)$

(ANY Linear but not necessarily homogeneous)

$y(t_0) = y_0$, $y'(t_0) = y_0'$ has a unique solution throughout an interval where p, q, g are continuous and containing $t = t_0$.

example $\quad 2t^2 y'' + 3t y' - y = 0 \qquad y(1) = 3$

$$y'' + \underbrace{\frac{3}{2t}}_{p} y' - \underbrace{\frac{1}{2t^2}}_{q} y = \underset{\overset{\uparrow}{g}}{0}$$

p, q, g continuous on $(-\infty, 0)$ and $(0, \infty)$

initial t is 1, so interval is $(0, \infty)$

Quick review of some complex numbers (for 3.3)

$i^2 = -1$ $\quad$ $i = \pm\sqrt{-1}$ $\quad$ imaginary number

complex number $\quad$ $a + bi$ $\quad$ a, b are constants

$$3 + 2i$$

real part $\quad$ imaginary part

complex exponential : $e^{it} = \cos(t) + i\sin(t)$

Euler's Identity

$$e^{(3+2i)t} = e^{3t}e^{i(2t)} = e^{3t}(\cos 2t + i \sin 2t)$$