7.1 + 7.2 Systems of First Order Linear Eqs

solve more than one equation at a time

$$\begin{cases} x_1'(t) = 2x_1(t) + 5x_2(t) \\ x_2'(t) = -x_1(t) + x_2(t) \end{cases} \Rightarrow \begin{array}{l} x_1(t) = ? \\ x_2(t) = ? \end{array}$$

(this is a <u>coupled</u> system — they depend on each other)

the system above is <u>homogeneous</u> because there are <u>no other functions of t</u> or <u>other constants</u> on the right side

this system is <u>nonhomogeneous</u>

$$\begin{cases} x_1'(t) = 2x_1(t) + 5x_2(t) + \underline{\cos(t)} \\ x_2'(t) = -x_1(t) + x_2(t) + \underline{5} \end{cases}$$

an nth-order differential equation can be written as a system of n first-order eqs.

example $y'' + \frac{1}{2}y' + 2y = \sin t$

this is a 2nd-order eq.

→ equivalent to sys of Two 1st-order eqs.

let $x_1 = y$ $\quad x_2 = y'$

write first-order eqs for these

$$\boxed{\begin{cases} x_1' = x_2 \\ x_2' = -2x_1 - \frac{1}{2}x_2 + \sin t \end{cases}}$$

$$\underset{x_2'}{\underset{\uparrow}{y''}} = -\frac{1}{2}\underset{x_2}{\underset{\uparrow}{y'}} - 2\underset{x_1}{\underset{\uparrow}{y}} + \sin t$$

<u>example</u> (Project 1) $u'' + ku + \epsilon u^3 = 0$

in Matlab, we represented it as a system in function file

let $x_1 = u$

$x_2 = u'$

$$\begin{cases} x_1' = x_2 \\ x_2' = -kx_1 - \epsilon (x_1)^3 \end{cases}$$

nonlinear sys.

example $y^{(4)} - y = 0$

4th-order $\longrightarrow$ 4 first-order eqs.

let $x_1 = y$ $\Rightarrow$ lower order deriv. (0^{th} order)

$x_2 = y'$ step up in deriv.

$x_3 = y''$

$x_4 = y'''$ until the $(n-1)^{th}$ deriv.

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ \boxed{x_4' = x_1} \end{cases}$$

$x_4' = (y''')' = y^{(4)}$

from DE, $y^{(4)} = y$

x_1

$\rightarrow$ DE

turn sys back to nth-order DE, then solve.

example

$$x_1' = -2x_1 + x_2 \quad \text{— } (1)$$

$$x_2' = x_1 - 2x_2 \quad \text{— } (2)$$

what is the equivalent 2nd-order DE?

solve either for one variable

from (1)

$$x_2 = x_1' + 2x_1$$

$$x_2' = x_1'' + 2x_1'$$

solve for the one that appears least often

sub these into (2)

$$x_1'' + 2x_1' = x_1 - 2(x_1' + 2x_1)$$

no x_2 any more

$$x_1'' + 4x_1' + 3x_1 = 0$$

$$\boxed{x_1(t) = C_1 e^{-t} + C_2 e^{-3t}}$$

to find x_2, use (1)

$$x_2 = x_1' + 2x_1$$

$$= -c_1 e^{-t} - 3c_2 e^{-3t} + 2\left(c_1 e^{-t} + c_2 e^{-3t}\right)$$

$$\boxed{x_2 = c_1 e^{-t} - c_2 e^{-3t}}$$

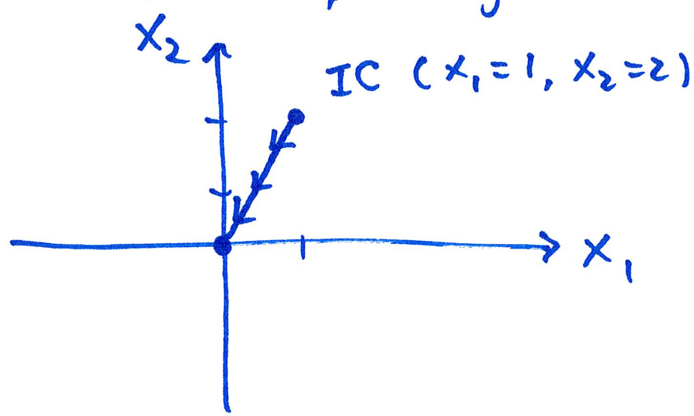
$$\boxed{x_1 = c_1 e^{-t} + c_2 e^{-3t}}$$

IC's: $\quad x_1(0) = 1, \quad x_2(0) = 2$

then $\quad x_1 = \frac{3}{2} e^{-t} - \frac{1}{2} e^{-3t}$

$$x_2 = \frac{3}{2} e^{-t} + \frac{1}{2} e^{-3t}$$

if we graph x_1 vs x_2, we get a phase plot.



solving systems by transforming into higher-order eqs is not hard but can be cumbersome.

→ borrow concepts from linear algebra to speed things up.

Review of Matrices

$$\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}$$ book notation

$$A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

we can add/subtract matrices if they are of the <u>same size</u>

$$A + B = \begin{bmatrix} 4 & 3 \\ 4 & 5 \end{bmatrix} \qquad A - B = \begin{bmatrix} -2 & 5 \\ -8 & 1 \end{bmatrix}$$

$$3A = \begin{bmatrix} 3 & 12 \\ -6 & 9 \end{bmatrix} \qquad 2A + 3B = \begin{bmatrix} 11 & 5 \\ 14 & 12 \end{bmatrix}$$

multiply matrices ~~iff~~ is only possible if inner dimensions match

$$A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\underset{\text{rows}}{2} \times \underset{\text{columns}}{2} \qquad 2 \times 2 \qquad 2 \times 1$$

$$\underset{2\times \boxed{2}}{A}\quad \underset{\boxed{2}\times 1}{C}$$

must match

$$\underset{2\times \boxed{1}}{C}\quad \underset{\boxed{2}\times 2}{A} \rightarrow$$ is meaningless (operation is NOT allowed)

don't match

$$AC = \underset{2\times 2}{\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 3 \\ -1 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 1\cdot 3 + 4\cdot -1 \\ -2\cdot 3 + 3\cdot -1 \end{bmatrix}} = \begin{bmatrix} -1 \\ -9 \end{bmatrix}$$

size of AC

$$AB = \underset{2\times 2}{\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} 27 & 7 \\ 12 & 8 \end{bmatrix}}$$

$AB = BA$? NO. In general $AB \neq BA$.

we can express systems of DEs as matrix eqs.

<u>example</u>

$$x_1' = -2x_1 + x_2$$

$$x_2' = x_1 - 2x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_2 \\ x_1 - 2x_2 \end{bmatrix}$$

let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

$$\vec{x}' = A\vec{x}$$

<u>example</u> $y^{(4)} - y = 0$ write as matrix eq.

$x_1 = y$ $\quad$ $x_1' = x_2$

$x_2 = y'$ $\quad$ $x_2' = x_3$

$x_3 = y''$ $\quad$ $x_3' = x_4$

$x_4 = y'''$ $\quad$ $x_4' = x_1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0x_1 + x_2 + 0x_3 + 0x_4 \\ 0x_1 + 0x_2 + x_3 + 0x_4 \\ 0x_1 + 0x_2 + 0x_3 + x_4 \\ x_1 + 0x_2 + 0x_3 + 0x_4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$