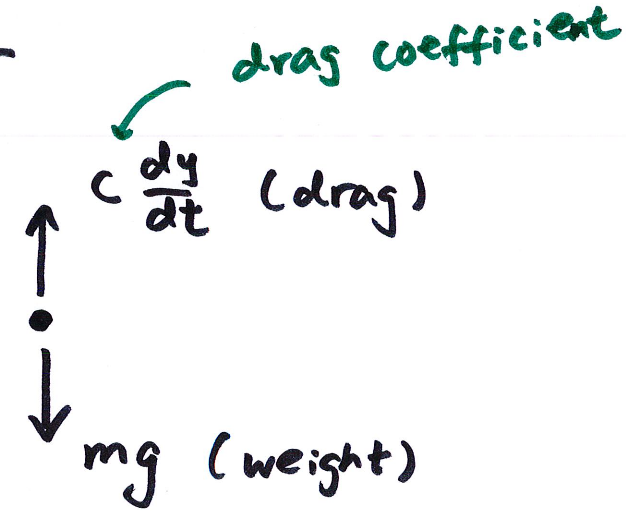
1.2 Solutions of Some DEs

object in free fall



$v(t) \to$ velocity $\left(\frac{dy}{dt}\right)$

DE: $$m\frac{dv}{dt} = mg - cv$$

let $m = 10$ kg $\quad c = 2$ kg/s

$$10\frac{dv}{dt} = 98 - 2v$$

$$\frac{dv}{dt} = 9.8 - 0.2v \qquad \Rightarrow \qquad v(t) = ?$$

assume: $v(0) = 0$

$$\frac{dv}{dt} = 0.2(49-v) = -0.2(v-49)$$

$$\frac{1}{v-49}\frac{dv}{dt} = -0.2$$

recall from calculus, $\frac{d}{dt}\ln|v(t)-49| = \frac{1}{v(t)-49}\frac{dv}{dt}$

$$\frac{d}{dt}\ln|v-49| = -0.2$$

integrate with respect to t

$$\ln|v-49| = -0.2t + A$$

constant

$$e^{\ln|v-49|} = e^{-0.2t+A}$$

also constant $= C$

$$v - 49 = e^{-0.2t} e^{A} = Ce^{-0.2t}$$

$$v = 49 + Ce^{-0.2t}$$

find C : $v(0) = 0$

$0 = 49 + C$ so $C = -49$

so $\boxed{v(t) = 49 - 49e^{-0.2t}}$

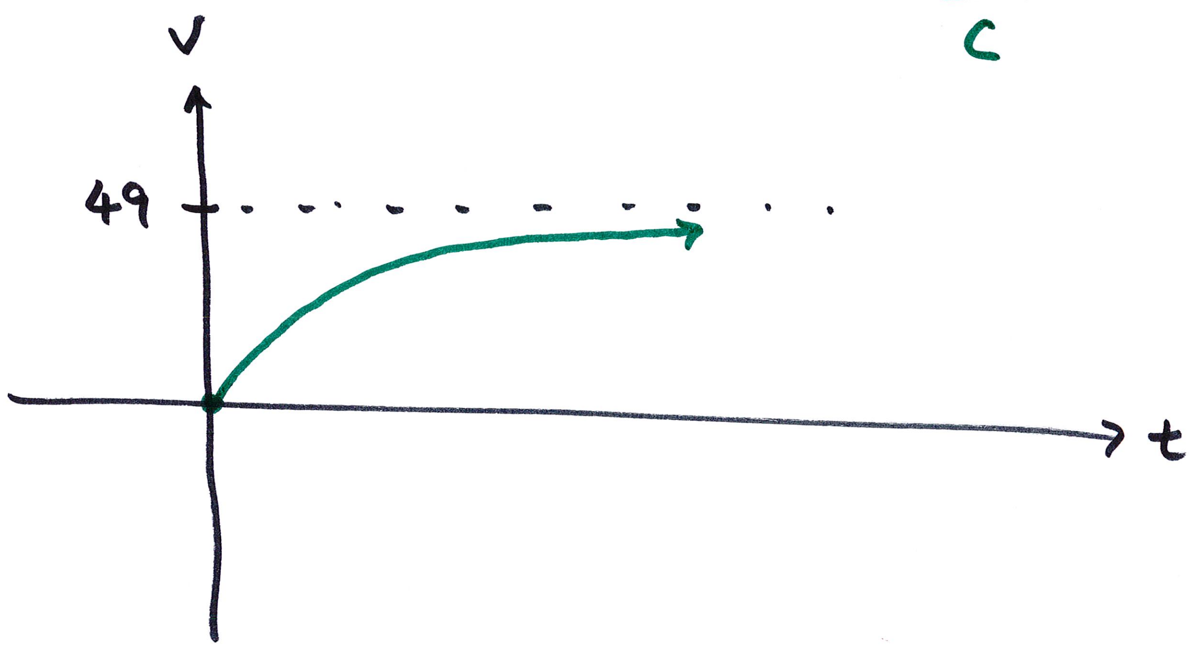
note: as $t \to \infty$ $v \to 49$ (because $\lim_{t\to\infty} e^{-t} = 0$)

$\downarrow$

terminal velocity

(weight balances drag)

$= \frac{mg}{c}$

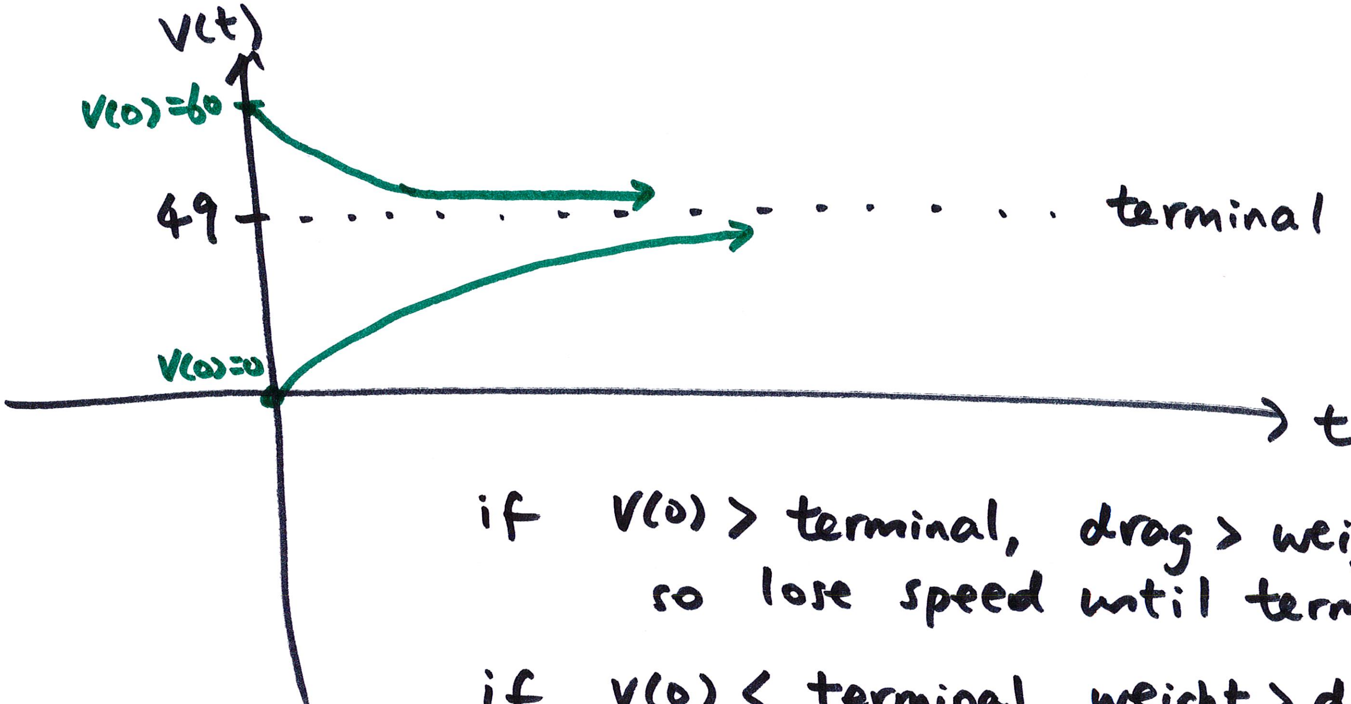


now, assume $v(0) = v_0$

$$V(t) = 49 + Ce^{-0.2t} \qquad V(0) = V_0$$

$$V_0 = 49 + C \quad \text{so} \quad C = V_0 - 49$$

$$\boxed{V(t) = 49 + (V_0 - 49)e^{-0.2t}}$$



if $V(0) >$ terminal, drag > weight initially, so lose speed until terminal

if $V(0) <$ terminal, weight > drag initially, so gain speed until terminal

if $V(0) =$ terminal, then drag = weight stays at terminal we can use the same steps to solve all equations of the form $\dfrac{dy}{dt} = ay + b$

a, b constants

1.3 Classifications of DEs

ordinary DE (ODE) : eqs with ordinary derivatives
(no partial derivs)

partial DE (PDE): eqs with at least one partial deriv.

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad \text{(heat equation)}$$

order of DE : order is the order of the highest derivative in eq.

$$y'' + y' + y = 0 \qquad \text{ODE of 2nd order}$$

$$\alpha^2 u_{xx} = u_t \qquad \text{PDE of 1st order}$$

linear vs nonlinear:

a DE is linear if it is a linear function of all its derivs of dependent variable

→ not multiplied together
not raised to any power other than 1
not part of transcendental functions
(e.g. exponential e^y
sine, cosine, etc)

examples:

$y'' + y' + y = 0$ linear "easy" to solve

$y'' + (y')^2 + y = 0$ nonlinear "hard to solve

$y'' + y'y + \sin(y) = 0$ nonlinear

$$y'' + y = 0$$ linear, 2nd order ODE

$$t y'' + y = 0$$ " " "

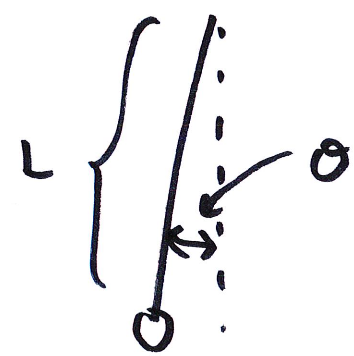
↑ indepent variable

$$t^2 y'' + t y' = \underbrace{\sin(t)}$$ linear 2nd order ODE

transcental ~~at~~ but w/ indep. var.

swinging pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = 0$$



θ: dependent variable

nonlinear

Solution: any function that satisfies the DE

example: $\frac{dy}{dt} = y$ function = its own deriv.

$y = e^t$ $y' = e^t$

→ $\boxed{y = Ce^t}$

solution of DE

many DEs will have $y = e^{rt}$ as a solution

r = constant

example: $y'' + 2y' - 3y = 0$

for what values of r is $y = e^{rt}$ a solution?

plug $y = e^{rt}$ into $y'' + 2y' - 3y = 0$

$$y' = re^{rt}$$

$$y'' = r^2 e^{rt}$$

$$r^2 e^{rt} + 2re^{rt} - 3e^{rt} = 0$$

$$e^{rt}(r^2 + 2r - 3) = 0$$

since $e^{rt} \neq 0$, $r^2 + 2r - 3 = 0$

$$(r+3)(r-1) = 0$$

$$r = -3, \text{ or } r = 1$$

$y'' + 2y' - 3y = 0$ has e^{-3t} and e^{t} as solutions