25. $t^2y'' + 3ty' + y = 0 \qquad t > 0 \qquad y_1 = t^{-1}$

$$y_2 = vy_1 = vt^{-1}$$

$$y_2' = -vt^{-2} + v't^{-1}$$

$$y_2'' = 2vt^{-3} - v't^{-2} - v't^{-2} + v''t^{-1}$$

$$= 2vt^{-3} - 2v't^{-2} + v''t^{-1}$$

$$\cancel{2vt^{-1}} - 2v' + v''t - \cancel{3vt^{-1}} + 3v' + \cancel{vt^{-1}} = 0$$

$$v''t + v' = 0 \qquad v'' = \frac{d(v')}{dt}$$

$$t\,\frac{d(v')}{dt} = -(v')$$

$$\frac{1}{(v')}\,d(v') = -\frac{1}{t}\,dt$$

$$\ln(v') = -\ln t + C \rightarrow v' = Ct^{-1}$$

$$v = C_1 \ln t + C_2$$

23. $t^2y'' - 4ty' + 6y = 0 \qquad t > 0 \qquad y_1 = t^2$

$$y_2 = vt^2$$

$$y_2' = 2vt + v't^2$$

$$y_2'' = v''t^2 + 4v't + 2v$$

$$v''t^4 + \cancel{4v't^3} + \cancel{2vt^2} - \cancel{8vt^2} - \cancel{4v't^3} + \cancel{6vt^2} = 0$$

$v''t^4 = 0 \rightarrow v'' = 0$ since $t \neq 0$ in general

$$v' = C_1$$

$$v = C_1 t + C_2$$

15. c).

$y = e^{-3/2 t} - \frac{5}{2} t e^{-3/2 t}$ from part a)

find (t_0, y_0) of minimum point

$y' = 0$

$y' = \frac{1}{4} e^{-3t/2} (15t - 16)$ $y' = 0 \rightarrow$ $t = 16/15$

$y = -\frac{5}{3} e^{-8/5}$

3.5 Nonhomogeneous Eqs: Method of Undetermined Coefficients

linear, constant-coefficient, but not homogeneous

$$y'' + ay' + by = \underbrace{g(t)}$$

Undetermined coeff can handle: polynomial, sin or/and cos, exponentials

example $y'' - y' - 2y = e^{3t}$

Solution: $y = y_h + Y$

particular solution due to $g(t)$

homogeneous part

here, $y_h = c_1 e^{-t} + c_2 e^{2t}$

$y = c_1 e^{-t} + c_2 e^{2t} + Y$

particular solution Y will resemble $g(t)$

here, $g(t) = e^{3t}$, assume $Y = \underbrace{A}_{\text{undetermined coeff.}} \underbrace{e^{3t}}_{\text{form from } g(t)}$

Y is a solution, so satisfies DE

$Y = Ae^{3t}$ $\quad Y' = 3Ae^{3t}$ $\quad Y'' = 9Ae^{3t}$

DE $\quad y'' - y' - 2y = e^{3t}$

$9Ae^{3t} - 3Ae^{3t} - 2Ae^{3t} = e^{3t}$ $\quad \to A = 1/4$

$$\boxed{y = c_1 e^{-t} + c_2 e^{2t} + \frac{1}{4} e^{3t}}$$

example $y'' - y' - 2y = e^{2t}$ same LHS as prev. example

$$y = c_1 e^{-t} + c_2 e^{2t} + Y$$

following last example, assume $Y = Ae^{2t}$

but note this form duplicates part of homogeneous solution (similar to repeated roots)

modify Y: $\boxed{Y = Ate^{2t}}$

$$Y' = 2Ate^{2t} + Ae^{2t}$$

$$Y'' = 4Ate^{2t} + 4Ae^{2t}$$

sub into DE

$$\cancel{4Ate^{2t}} + 4Ae^{2t} - \cancel{2Ate^{2t}} - Ae^{2t} - \cancel{2Ate^{2t}} = e^{2t}$$

$$3Ae^{2t} = e^{2t} \qquad A = 1/3$$

$$\boxed{y = c_1 e^{-t} + c_2 e^{2t} + \frac{1}{3} te^{2t}}$$

example

$$\cancel{y'' - 4y' = e^{2t}}$$

$$y'' - 4y = e^{2t}$$

$$y = c_1 e^{2t} + c_2 t e^{2t} + Y$$

$$Y = A e^{2t} t^2$$

keep multiplying by t until not duplicating ANY part of homo. solution example $\cancel{y''+k}$ $y''-y'-2y = \cos t$

same LHS as example 1

$y = c_1 e^{-t} + c_2 e^{2t} + Y$

if $g(t)$ is $\cancel{\sin}$, $\cancel{\cos}$ or $\cancel{\cos}$, assume
($\sin(\alpha t)$ written over the first struck term, $\cos(\alpha t)$ over the last)

$Y = A\cos(\alpha t) + B\sin(\alpha t)$

sin, cos always show up together in Y, even if $g(t)$ only has one of them

here, $\left.\begin{aligned} Y &= A\cos t + B\sin t \\ Y' &= -A\sin t + B\cos t \\ Y'' &= -A\cos t - B\sin t \end{aligned}\right\}$ plug into DE

$-A\cos t - B\sin t + A\sin t - B\cos t - 2A\cos t - 2B\sin t = \cos t$

$(-3A-B)\cos t + (A-3B)\sin t = 1\cos t + 0\sin t$

$$\left.\begin{aligned} -3A - B &= 1 \\ A - 3B &= 0 \end{aligned}\right\} \quad A = -3/10, \quad B = -1/10$$

$$\boxed{y = C_1 e^{-t} + C_2 e^{2t} - \frac{3}{10}\cos t - \frac{1}{10}\sin t}$$

<u>example</u> $y'' + y = \sin t \qquad r^2 + 1 = 0 \quad r = \pm i$

$$y = C_1 \cos t + C_2 \sin t + Y$$

$$Y = (A\cos t + B\sin t)\,t$$

to avoid duplicating homo. solution

<u>example</u>

$$y'' - y' - 2y = 1$$

LHS same as first example
RHS polynomial

$$y = c_1 e^{-t} + c_2 e^{2t} + Y$$

Y has form of polynomial of the same degree as $g(t)$

$$Y = A \qquad (0^{th} \text{ degree})$$

<u>example</u>

$$y'' - y' - 2y = t^3$$

$$y = c_1 e^{-t} + c_2 e^{2t} + Y$$

$$Y = At^3 + Bt^2 + Ct + D \qquad (3^{rd} \text{ deg.})$$

example $y'' - y' = t^2$

$$y = C_1 + C_2 e^t + Y$$

$$Y = \underbrace{(At^2 + Bt + C)}_{\text{2nd deg.}}\, t$$

because C duplicates C_1 in homo. solution

example $y'' + y = t \sin t$

$$y = C_1 \cos t + C_2 \sin t + Y$$

$$Y: \quad t(\sin t)$$

first deg. polynomial

sin or cos t

$$Y = t(At + B)\sin t + t(Ct + D)\cos t$$

to eliminate duplication