6.6 The Convolution Integral

we know $\mathcal{L}\{f(t) \pm g(t)\} = \mathcal{L}\{f(t)\} \pm \mathcal{L}\{g(t)\}$

$$\mathcal{L}^{-1}\{F(s) \pm G(s)\} = \mathcal{L}^{-1}\{F(s)\} \pm \mathcal{L}^{-1}\{G(s)\}$$

BUT $\mathcal{L}\{f(t) \cdot g(t)\} \neq \mathcal{L}\{G(t)\}\,\mathcal{L}\{F(s)\}$

$$\mathcal{L}^{-1}\{F(s) \cdot G(s)\} \neq \mathcal{L}^{-1}\{F(s)\} \cdot \mathcal{L}^{-1}\{G(s)\}$$

e.g. $\mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} \neq \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} \mathcal{L}\left\{\frac{1}{s-1}\right\}$

$$= \mathcal{L}^{-1}\left\{-\frac{1}{s} + \frac{1}{s-1}\right\}$$

$$= -1 + e^{t}$$

so what <u>is</u> $\mathcal{L}^{-1}\{F(s) \cdot G(s)\}$?

$$\mathcal{L}^{-1}\{F(s)\cdot G(s)\} = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = \int_0^t f(\tau)\,g(t-\tau)\,d\tau = f(t) * g(t)$$

Convolution Integral

τ: dummy variable → integration variable

<u>example</u> $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)(s+2)}\right\}$

two ways: first with partial fraction expansion

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\} = e^{-t} - e^{-2t}$$

now with convolution integral

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s+1}}_{F(s)} \cdot \underbrace{\frac{1}{s+2}}_{G(s)}\right\}$$

$f(t) = e^{-t}$ $\quad$ $g(t) = e^{-2t}$

$$= \int_0^t e^{-(t-\tau)} e^{-2\tau} \, d\tau$$

or

τ is variable

t is "constant"

(for integration purposes)

$$= \int_0^t e^{-\tau} e^{-2(t-\tau)} \, d\tau$$

$$= \int_0^t e^{-t} e^{\tau} e^{-2\tau} \, d\tau$$

$$= \int_0^t \underbrace{e^{-t}}_{\text{"constant"}} e^{-\tau} \, d\tau = e^{-t}\left(-e^{-\tau}\right)\Big|_{\tau=0}^{\tau=t}$$

$$= e^{-t}\left(-e^{-t}+1\right) = e^{-t} - e^{-2t}$$

matches answer of partial fraction way.

what is convolution integral doing?

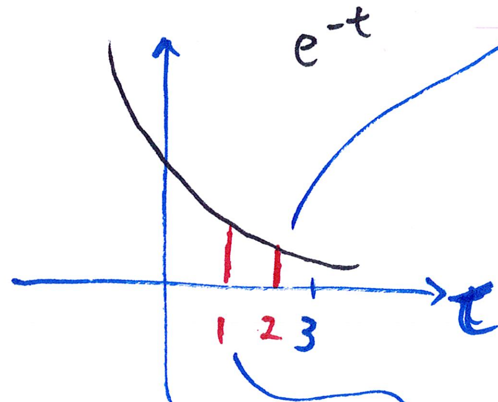
$$\int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

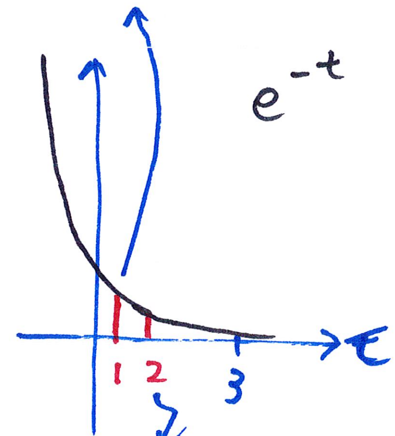
let $f(t) = e^{-t}$ $\quad g(t) = e^{-2t}$ $\quad$ and $t = 3$

$$\int_0^3 \underbrace{e^{-(3-\tau)}}_{\text{from end of first}} \underbrace{e^{-2\tau}}_{\text{from beginning of second}} d\tau$$

when $\tau = 1$

integrand is $e^{-2} \cdot e^{-2}$

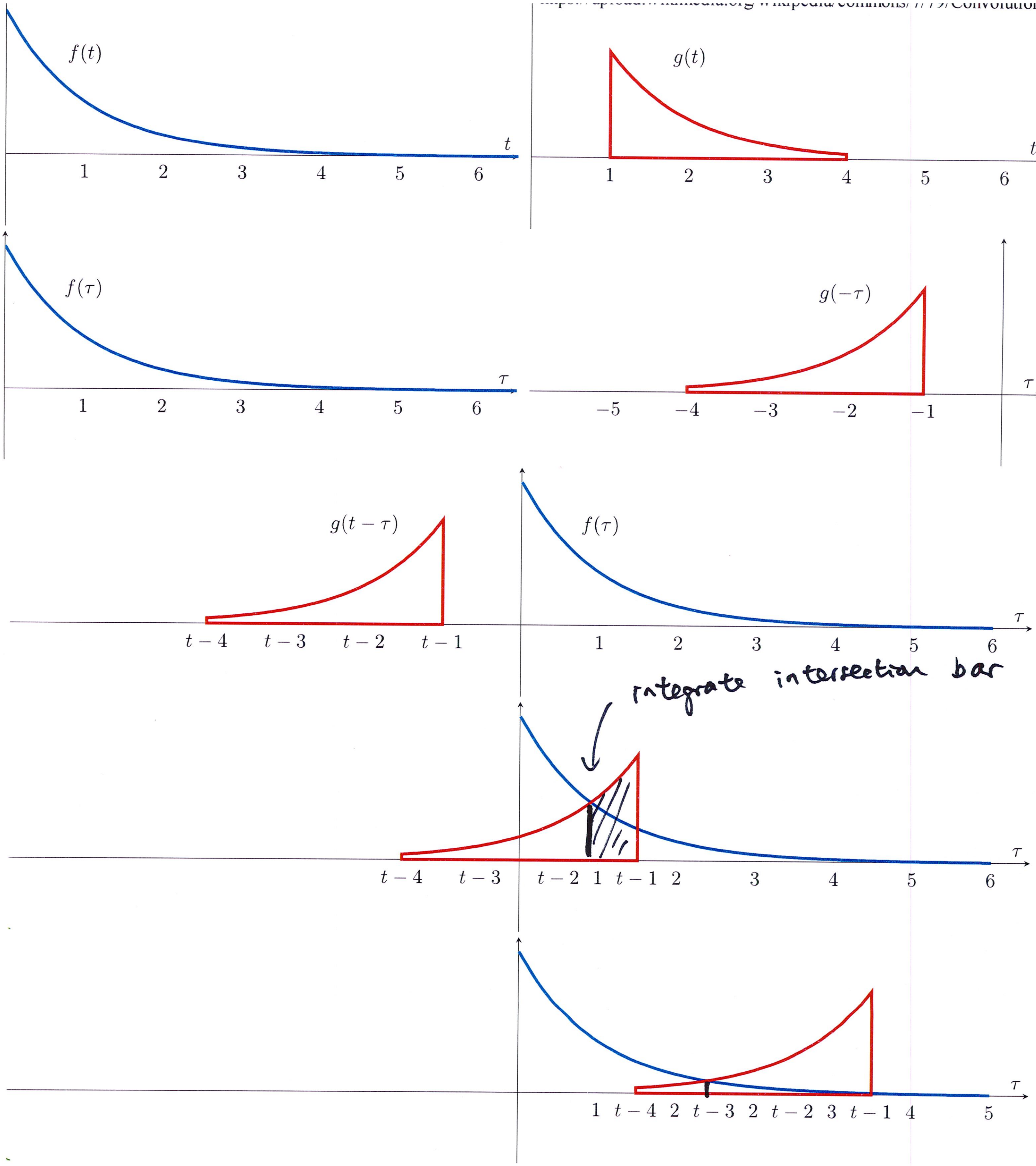




$\tau = 2$ $\quad e^{-(3-2)} \cdot e^{-2(2)}$

integration steps through functions in <u>opposite</u> directions

Integrate the product.



$f(t)$
$g(t)$
$f(\tau)$
$g(-\tau)$
$g(t-\tau)$
$t-4$ $t-3$ $t-2$ $t-1$
integrate intersection bar

<u>example</u> $y'' - 2y' + y = 4t^2 \qquad y(0) = y'(0) = 0$

$$s^2 Y - 2sY + Y = 4 \cdot \frac{2}{s^3} = \frac{8}{s^3}$$

$$Y(s^2 - 2s + 1) = \frac{8}{s^3}$$

$$Y = \frac{8}{s^3(s^2-2s+1)} = \frac{8}{s^3(s-1)^2}$$

$$y = \mathcal{L}^{-1}\left\{ \underbrace{\frac{8}{s^3}}_{4t^2} \cdot \underbrace{\frac{1}{(s-1)^2}}_{te^t} \right\}$$

$$y = \int_0^t 4(t-\tau)^2 \cdot \tau e^{\tau}\, d\tau = \int_0^t 4\tau^2 (t-\tau) e^{(t-\tau)}\, d\tau$$

the one w/ $t-\tau$
is the one we
integrate backwards
(end first)