1. $y'' + y = \begin{cases} 1 & 0 \le t < 3\pi \\ 0 & 3\pi \le t < \infty \end{cases}$

$R\cos(\omega_0 t - \delta)$

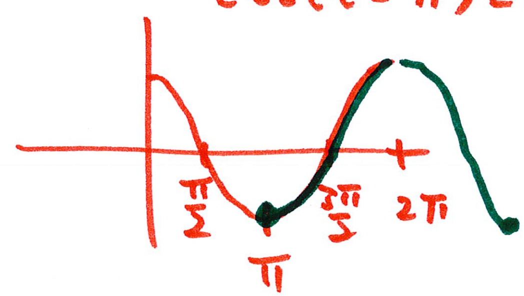
$\vdots$

$$Y = \frac{1}{s^2+1} + \frac{1}{s} - \frac{s}{s^2+1} - e^{-3\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

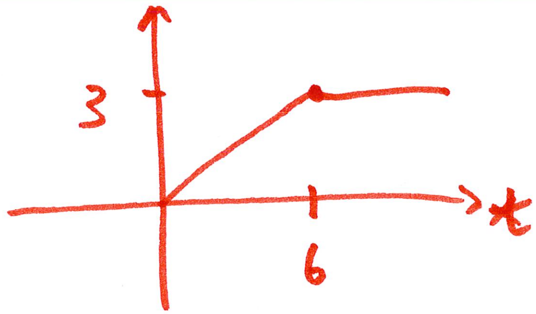
$1 \qquad \cos t$

$$y = \sin t + 1 - \cos t - u_{3\pi}(t)\left(1 - \cos(t - 3\pi)\right)$$

$\cos(t-\pi) = -\cos t$



9. $y''+y = \begin{cases} t/2 & 0 \le t < 6 \\ 3 & t \ge 6 \end{cases}$ $\qquad y(0)=0, \quad y'(0)=1$



$-\frac{1}{2}t+3$ $\quad$ L 6 units

$= -\frac{1}{2}(t+6)+3$

$= -\frac{1}{2}t$

$$= \frac{1}{2}t + u_6(t)\left(-\frac{1}{2}t+3\right)$$

$$s^2Y - \cancel{sy(0)} - y'(0) + Y = \frac{1}{2s^2} + e^{-6s}\left(-\frac{1}{2s^2}\right)$$

$$(s^2+1)Y = 1 + \frac{1}{2s^2} + e^{-6s}\left(-\frac{1}{2s^2}\right)$$

$$Y = \frac{1}{s^2+1} + \frac{1}{2s^2(s^2+1)} - e^{-6s}\frac{1}{2s^2(s^2+1)}$$

12. $y^{(4)} - y = u_1(t) - u_2(t)$ $\qquad y(0) = y'(0) = y''(0) = y'''(0) = 0$

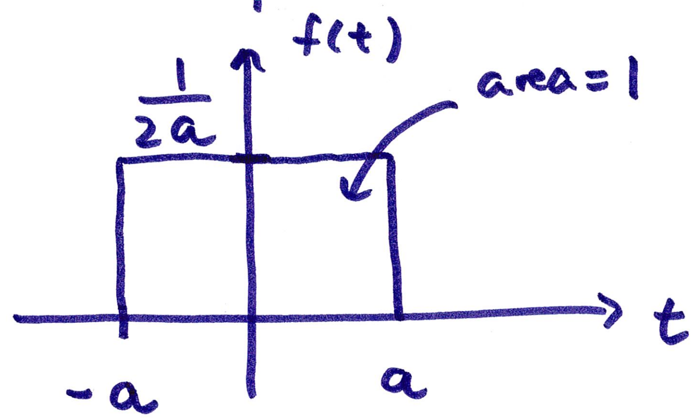
$$(s^4 - 1)Y = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \qquad \frac{1}{s(s^2+1)(s-1)(s+1)}$$

$$Y = \frac{e^{-s}}{s(s^4-1)} - \frac{e^{-2s}}{s(s^4-1)}$$

6.5 Impulse Function

impulse : short-acting force (e.g. bat hitting baseball)

model with step functions



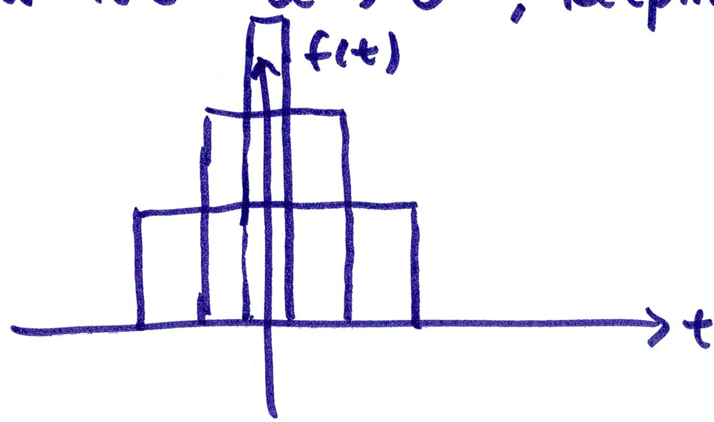
step up at $t=-a$

step down at $t=a$

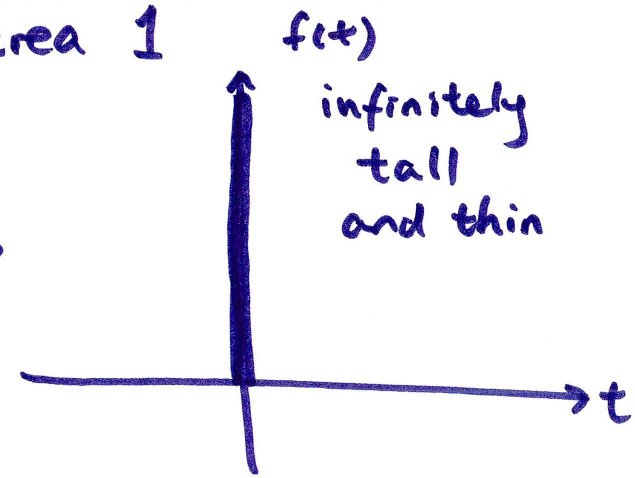
rectangle area = 1

$$f(t) = \frac{1}{2a} u_{-a}(t) - \frac{1}{2a} u_a(t)$$

now let $a \to 0$, keeping area 1



$\Rightarrow$



this is called the unit impulse function

or Dirac delta function

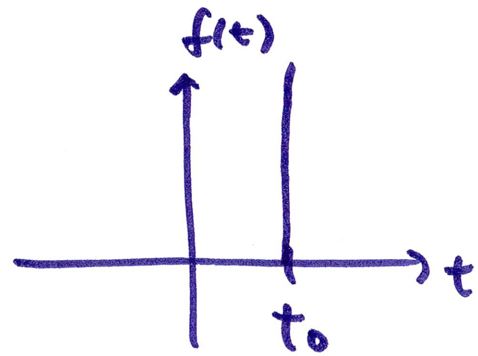
$$\delta(t) = 0 \qquad t \neq 0$$

but $$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$\delta(t-t_0)$ • puts the impulse at $t = t_0$

$$\delta(t-t_0) = 0 \qquad t \neq t_0$$

$$\int_{-\infty}^{\infty} \delta(t-t_0)\, dt = 1$$



$$\boxed{\mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s}}$$

note similarity and difference to $\mathcal{L}\{u_c(t)\} = \dfrac{e^{-cs}}{s}$

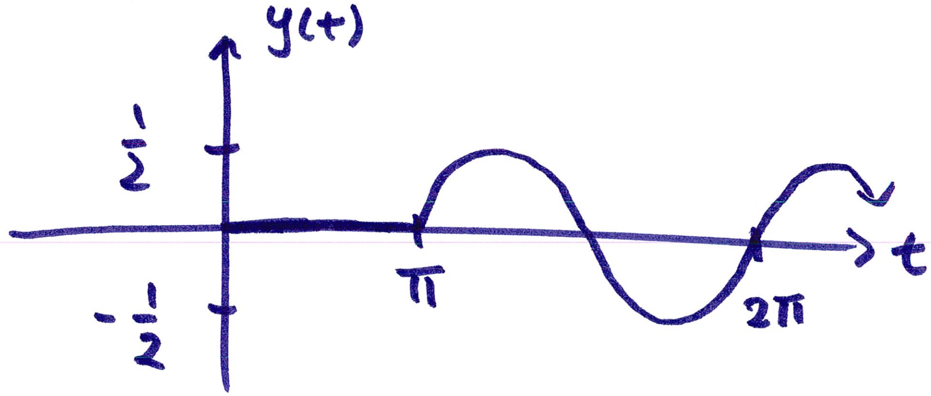
<u>example</u> $y'' + 4y = \delta(t-\pi)$ $\quad y(0) = y'(0) = 0$

$$s^2 Y + 4Y = e^{-\pi s}$$

$$Y = e^{-\pi s} \frac{1}{s^2+4} \quad \rightarrow \frac{1}{2}\sin 2t$$

$$y(t) = u_\pi(t)\left[\frac{1}{2}\sin 2(t-\pi)\right]$$

$$= \begin{cases} 0 & t < \pi \\ \frac{1}{2}\sin 2(t-\pi) = \frac{1}{2}\sin 2t & t \geq \pi \end{cases}$$



<u>example</u>

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi) \qquad y(0) = y'(0) = 0$$

$$(s^2+4)Y = e^{-\pi s} - e^{-2\pi s}$$

$$Y = e^{-\pi s}\frac{1}{s^2+4} - e^{-2\pi s}\frac{1}{s^2+4}$$

$$y = u_\pi(t)\cdot\frac{1}{2}\sin 2(t-\pi) - u_{2\pi}(t)\cdot\frac{1}{2}\sin 2(t-2\pi)$$

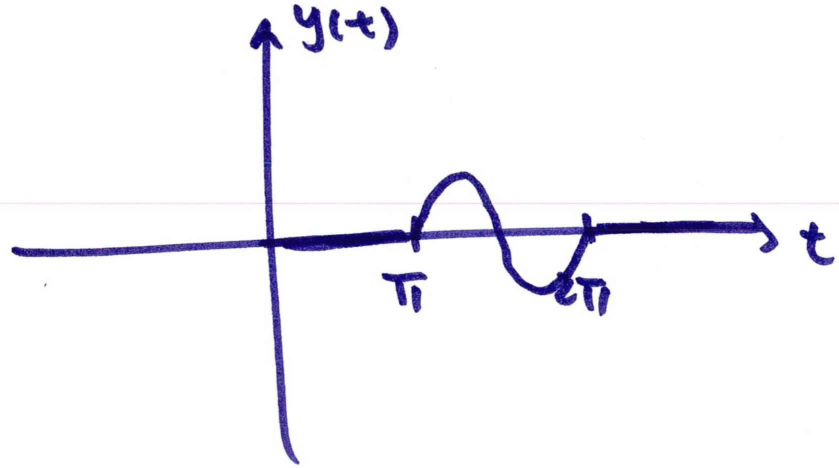
$$= u_\pi(t)\cdot\frac{1}{2}\sin 2t - u_{2\pi}(t)\cdot\frac{1}{2}\sin 2t$$

$$= \begin{cases} 0 & t < \pi \\ \frac{1}{2}\sin 2t & \pi \le t < 2\pi \\ 0 & t \ge 2\pi \end{cases}$$

period = π

$\sin 2(t-\pi)$
$= \sin(2t - 2\pi)$
$= \sin 2t$

$\sin 2(t-2\pi)$
$= \sin(2t - 4\pi)$
$= \sin 2t$



<u>example</u> $\quad y'' + 2y' + 3y = \sin t + \delta(t - 3\pi)$

$$y(0) = y'(0) = 0$$

$$(s^2 + 2s + 3)Y = \frac{1}{s^2+1} + e^{-3\pi s}$$

$$Y = \frac{1}{(s^2+1)(s^2+2s+3)} + e^{-3\pi s}\frac{1}{s^2+2s+3}$$

$$Y = \frac{1}{4}\frac{1}{s^2+1} - \frac{1}{4}\frac{s}{s^2+1} + \frac{1}{4}\frac{s+1}{s^2+2s+3} + e^{-3\pi s}\frac{1}{s^2+2s+3}$$

$$\frac{1}{s^2+1} \downarrow \sin t \qquad \frac{s}{s^2+1} \downarrow \cos t$$

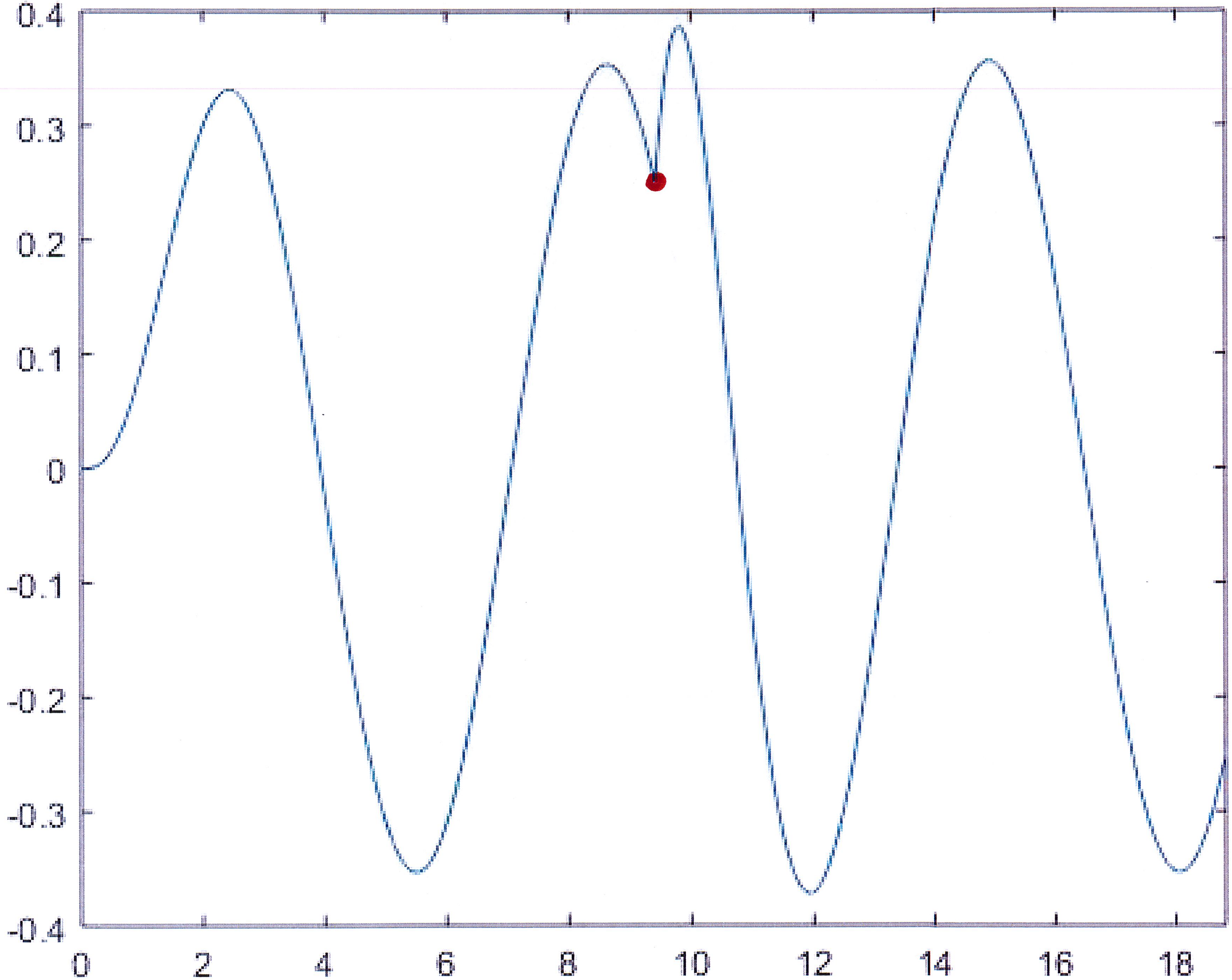
$$\frac{s+1}{s^2+2s+3} \downarrow \frac{s+1}{(s+1)^2+2} \qquad e^{-t}\cos\sqrt{2}t$$

$$\frac{1}{s^2+2s+3} = \frac{1}{(s+1)^2+2} \qquad \frac{1}{\sqrt{2}}e^{-t}\sin\sqrt{2}t$$

$$y = \underbrace{\frac{1}{4}\sin t - \frac{1}{4}\cos t}_{\text{particular due to } \sin t} + \underbrace{\frac{1}{4}e^{-t}\cos\sqrt{2}\,t}_{\text{homogeneous w/ IC's}}$$

$$+ \underbrace{u_{3\pi}(t)\,\frac{1}{\sqrt{2}}\,e^{-(t-3\pi)}\sin\sqrt{2}\,(t-3\pi)}_{\text{particular due to } \delta(t-3\pi)}$$

$$= \begin{cases} \frac{1}{4}\sin t - \frac{1}{4}\cos t + \frac{1}{4}e^{-t}\cos\sqrt{2}\,t & t < 3\pi \\[2em] \frac{1}{4}\sin t - \frac{1}{4}\cos t + \frac{1}{4}e^{-t}\cos\sqrt{2}\,t \\ \quad + \frac{1}{\sqrt{2}}e^{-(t-3\pi)}\sin\sqrt{2}(t-3\pi) & t \geq 3\pi \end{cases}$$


0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0
2
4
6
8
10
12
14
16
18