

4.1+4.2 Higher-Order Linear Equations

recall $y' + p(t)y = g(t)$, $y(t_0) = y_0$ has
unique solution on interval where $p(t), g(t)$
are continuous and containing t_0 .

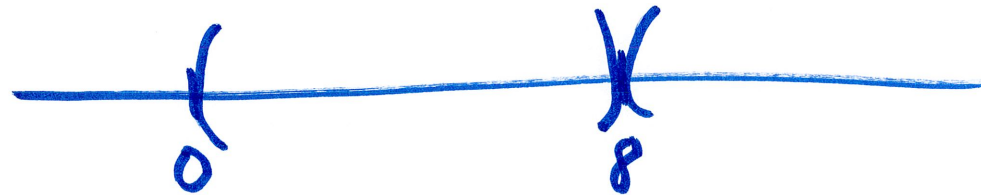
$y'' + p(t)y' + q(t)y = g(t)$, $y(t_0) = y_0$, $y'(t_0) = y_0'$
has unique solution on interval where
 $p(t), q(t), g(t)$ are continuous and containing
 $t = t_0$.

for n^{th} -order linear eq.

$y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + \dots + p_n(t)y = g(t)$
 $y(t_0) = y_0$, $y'(t_0) = y_0'$, $y''(t_0) = y_0''$, ..., $y^{(n-1)}(t_0) = y_0^{(n-1)}$
has unique solution on interval where
 p_1, p_2, \dots, p_n and $g(t)$ are continuous,
containing $t = t_0$.

example $y^{(5)} + \frac{t^2}{t-8} y''' + \frac{\sin t}{t-8} y'' + \frac{\ln t}{t-8} y = \frac{\sqrt{t}}{t-8}$

$$t \neq 8, t > 0$$



intervals where there are unique solutions:

$$(0, 8) \text{ or } (8, \infty)$$

need to pick.

Solution of constant-coeff eg.

example $2y''' - 4y'' - 2y' + 4y = 0$

characteristic eg: $2r^3 - 4r^2 - 2r + 4 = 0$

$$r^3 - 2r^2 - r + 2 = 0$$

$$r^2(r-2) - (r-2) = 0$$

$$(r-2)(r^2-1) = 0 \quad r = -1, 1, 2$$

$$y = c_1 e^{-t} + c_2 e^t + c_3 e^{2t}$$

example

$$y^{(4)} + 4y''' + 4y'' = 0$$

$$r^4 + 4r^3 + 4r^2 = 0$$

$$r^2(r^2 + 4r + 4) = 0$$

$$r^2(r+2)^2 = 0 \quad r = 0, 0, -2, -2$$

$$y = C_1 + C_2 t + C_3 e^{-2t} + C_4 t e^{-2t}$$

example

$$y''' - 5y'' + 3y' + y = 0$$

$$r^3 - 5r^2 + 3r + 1 = 0$$

by inspection, $r=1$ is a root

$$(r-1)(ar^2 + br + c) = 0$$

factor by grouping
is harder

$$ar^3 + br^2 + cr - ar^2 - br - c =$$

$$r^3 - 5r^2 + 3r + 1$$

$$a=1, \quad b-a=-5, \quad c-b=3, \quad -c=1$$

$$b=-4 \qquad c=-1$$

$$(r-1)(r^2 - 4r - 1) = 0$$

$$\downarrow$$

$$r=1$$

$$r = \frac{4 \pm \sqrt{16 + 4}}{2} = \frac{4 \pm 2\sqrt{5}}{2}$$

$$= 2 \pm \sqrt{5}$$

$$y = c_1 e^t + c_2 e^{(2+\sqrt{5})t} + c_3 e^{(2-\sqrt{5})t}$$

example An 8th-order DE has the following roots to the characteristic eq.

$$r = 2, 3, 3, 3, 2+3i, 2-3i, 2+3i, 2-3i$$

What is the general solution?

$$\begin{aligned} y = & c_1 e^{2t} + c_2 e^{3t} + c_3 t e^{3t} + c_4 t^2 e^{3t} \\ & + c_5 e^{2t} \cos 3t + c_6 e^{2t} \sin 3t \\ & + c_7 t e^{2t} \cos 3t + c_8 t e^{2t} \sin 3t \end{aligned}$$

example

$$y''' + y'' = 0 \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 2$$

$$r^3 + r = 0$$

$$r(r^2 + 1) = 0$$

$$r = 0, i, -i$$

$$y = C_1 + C_2 \cos t + C_3 \sin t$$

$$0 = C_1 + C_2$$

$$y' = -C_2 \sin t + C_3 \cos t$$

$$1 = C_3$$

$$y'' = -C_2 \cos t - C_3 \sin t$$

$$2 = -C_2 \quad C_1 = 2$$

$$\hookrightarrow y = 2 - 2 \cos t + \sin t$$

in general, solve system of n equations
for n unknowns for an n^{th} order eq.