5. $y'' - 2y' - 3y = -3te^{-t}$

homogeneous: $y_1 = e^{-t}$ $y_2 = e^{3t}$

$y = c_1 e^{-t} + c_2 e^{3t} + Y$

RHS: te^{-t} exponential

first deg

$Y = t(At+B)e^{-t} = Ate^{-t} + \underbrace{Be^{-t}}_{\text{duplicating } y_1}$

$= At^2e^{-t} + Bte^{-t}$

21. $y'' + 3y' = 2t^4 + t^2 e^{-3t} + \sin 3t$

homogeneous: $y_1 = 1 \qquad y_2 = e^{-3t}$

$$y = C_1 + C_2 e^{-3t} + Y$$

$$Y = t\left(At^4 + Bt^3 + Ct^2 + Dt + E\right) + t\left(Ft^2 + Gt + H\right)e^{-3t}$$

$$+ I \sin 3t + J \cos 3t$$

---

repeated roots: $\cancel{y'' - 4y = 0 \qquad r^2 = 4 \qquad r = \pm 2}$

$y'' + 2y' + y = 0 \qquad r^2 + 2r + 1 = 0$

$r = -1, -1$

$y_1 = e^{-t}$

$y_2 = v e^{-t} \qquad V = At + B$

$= Ate^{-t} + Be^{-t}$

gen. soln:

$y = C_1 e^{-t} + C_2 \left(Ate^{-t} + Be^{-t}\right)$

$= C_1 e^{-t}$

3.6 Variation of Parameters

another way to solve nonhomogeneous eqs.

$$y'' - 5y' + 6y = 2e^{t}$$

first, find homogeneous solutions: $y_1 = e^{2t}$ $y_2 = e^{3t}$

normally, $y = \underline{\underline{c_1}}\, y_1 + \underline{\underline{c_2}}\, y_2 + Y$

parameters

but now we look at it as: $y = \underline{\underline{u_1}}(t)\, y_1 + \underline{\underline{u_2}}(t)\, y_2$

find u_1, u_2 by subbing into DE

$$y = u_1 e^{2t} + u_2 e^{3t}$$

$$y' = 2u_1 e^{2t} + u_1' e^{2t} + 3u_2 e^{3t} + u_2' e^{3t}$$

looking ahead: Two unknowns (u_1, u_2) ONE eq. (DE)

$\rightarrow$ impose extra condition: $\boxed{u_1' e^{2t} + u_2' e^{3t} = 0}$

$$y' = 2u_1 e^{2t} + 3u_2 e^{3t}$$

$$y'' = 4u_1 e^{2t} + 2u_1' e^{2t} + 9u_2 e^{3t} + 3u_2' e^{3t}$$

DE: $y'' - 5y' + 6y = 2e^t$

$$\cancel{4u_1 e^{2t}} + 2u_1' e^{2t} + \cancel{9u_2 e^{3t}} + 3u_2' e^{3t}$$

$$\cancel{-10u_1 e^{2t}} \cancel{-15u_2 e^{3t}}$$

$$\cancel{+6u_1 e^{2t}} \cancel{+6u_2 e^{3t}} = 2e^t$$

condition imposed earlier: $u_1' e^{2t} + u_2' e^{3t} = 0$ — ①

DE: $2u_1' e^{2t} + 3u_2' e^{3t} = 2e^t$ — ②

mult. ① by 2

$$2u_1' e^{2t} + 2u_2' e^{3t} = 0 \quad \text{— ③}$$

② - ③

$$u_2' e^{3t} = 2e^t$$

$$u_2' = 2e^{-2t}$$

$$u_2 = -e^{-2t} + c_2$$

from 1

$$u_1' e^{2t} = -u_2' e^{3t}$$

$$= -2e^{-2t} e^{3t} = -2e^{t}$$

$$u_1 = -2e^{t} + c_1$$

solution : $y = u_1 y_1 + u_2 y_2$

$$= (-2e^{t} + c_1)\, e^{2t} + (-e^{-2t} + c_2)\, e^{3t}$$

$$= c_1 e^{2t} + c_2 e^{3t} - 2e^{3t} - e^{t}$$

$$y = \underbrace{c_1 e^{2t} + c_2 e^{3t}}_{\text{homogeneous}} - \underbrace{e^{t}}_{\text{particular}}$$

for ANY $y'' + p(t) y' + q(t) y = g(t)$ we <u>always</u>

ends up solving

$$\boxed{\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t) \end{aligned}}$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

unique solution if

$$\underbrace{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}_{\text{Wronskian}} \neq 0$$

<u>example</u> $y'' - 2y' + y = e^t$

homogeneous: $y_1 = e^t$ $y_2 = te^t$

if using undetermined coeff., Y has duplications

using variation of parameters, u_1, u_2 take care of duplications

$$u_1' e^t + u_2' t e^t = 0 \qquad - (1)$$

$$u_1' e^t + u_2'(te^t + e^t) = e^t \qquad - (2)$$

(2) - (1)

$$u_2' e^t = e^t$$

$$u_2' = 1 \qquad u_2 = t + c_2$$

from (1)

$$u_1' = -u_2' t$$

$$= -t \qquad u_1 = -\tfrac{1}{2}t^2 + c_1$$

Solution:

$$y = u_1 y_1 + u_2 y_2$$

$$= \left(-\tfrac{1}{2}t^2 + c_1\right)e^t + (t + c_2)te^t$$

$$= c_1 e^t + c_2 te^t - \tfrac{1}{2}t^2 e^t + t^2 e^t$$

$$= \underbrace{c_1 e^t + c_2 te^t}_{\text{homogeneous}} + \underbrace{\tfrac{1}{2}t^2 e^t}_{\text{particular}}$$

example

$$(t^2-1)y'' - 2ty' + 2y = (t^2-1)^2$$

NOT $g(t)$

$y_1 = t$

$y_2 = t^2+1$

$$y'' - \frac{2t}{t^2-1}y' + \frac{2}{t^2-1} = \boxed{t^2-1}$$

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = g(t)$$

$=$ RHS in DE in standard form

y'' has 1 as coeff.