I.

a) show that $y(t) = \int_0^t e^{-u^2}\,du$ is

Solution to $\frac{dy}{dt} = e^{-t^2}$ $\quad y(0)=0$

$$y'(t) = \frac{d}{dt}\int_0^t e^{-u^2}\,du = e^{-t^2}$$

IC: $y(0)=0$ $\quad y(0) = \int_0^0 e^{-u^2}\,du = 0$

b). $y(2) = \int_0^2 e^{-u^2}\,du$

solve $y' = e^{-t^2}$ using Euler w/ $h = 1/2$

start at $t=0$, then $t = 1/2$, $t=1$, $t=3/2$, $t=2$

20. $y' = 1 - t + y \qquad y(t_0) = y_0$

a) show that $y = (y_0 - t_0)e^{t-t_0} + t$

b). show that $y_K = (1+h)y_{K-1} + h - ht_{K-1}$

Euler update formula: $y_{n+1} = y_n + f(t_n, y_n)h$

let $n+1 = K$ $\qquad y_K = y_{K-1} + f(t_{K-1}, y_{K-1})h$

$$f(t,y) = 1 - t + y$$

$$f(t_{K-1}, y_{K-1}) = 1 - t_{K-1} + y_{K-1}$$

$$y_K = y_{K-1} + (1 - t_{K-1} + y_{K-1})h$$

$$= y_{K-1} + h - ht_{K-1} + hy_{K-1}$$

$$= (1+h)y_{K-1} + h - ht_{K-1}$$

d). $y_n = (1+h)^n (y_0 - t_0) + t_n$

let $h = \dfrac{t - t_0}{n}$

$$y_n = \left(1 + \frac{t-t_0}{n}\right)^n (y_0 - t_0) + t_n$$

$$\lim_{n\to\infty} y_n = \left[\lim_{n\to\infty} \left(1 + \frac{t-t_0}{n}\right)^n\right] (y_0 - t_0) + t_n$$

$$y(t) = e^{t-t_0} (y_0 - t_0) + t$$

3.1 2nd Order Homogeneous Eqs w/ Constant Coefficients

last lesson on exam 1.

2nd order linear : $y'' + p(t)\,y' + q(t)\,y = g(t)$

if $g(t) = 0$ for all t, then equation is homogeneous

does not mean function of $\frac{y}{x}$ as in first order

today, focus on $\left.\begin{array}{l} p(t) = a \\ q(t) = b \end{array}\right\}$ constants.

what kind of solutions?

simple one: $y'' - y = 0 \rightarrow y'' = y$

function equal to its own second derivative

$\underbrace{y = e^{t} \quad , \quad y = e^{-t}}$

or any constant-multiple of these

another one: $y'' - y' = 0 \rightarrow y'' = y'$

$y = e^{t} \quad , \quad y = 1 = e^{0t}$

note: solutions are in the form of $y = e^{rt}$

$r =$ constant

example $y'' + y' - 2y = 0$

assume solution $y = e^{rt}$ $y' = re^{rt}$ $y'' = r^2 e^{rt}$

plug into DE

$$r^2 e^{rt} + re^{rt} - 2e^{rt} = 0$$

$$e^{rt}(r^2 + r - 2) = 0$$

since $e^{rt} \neq 0$, $\underbrace{r^2 + r - 2}_{\text{characteristic polynomial}} = 0$

note similarity to DE.

$$(r+2)(r-1) = 0$$

so $r = -2$, $r = 1$

two solutions are: $y_1 = e^{-2t}$, $y_2 = e^t$

$$\boxed{\text{general solution: } y(t) = c_1 y_1 + c_2 y_2 = c_1 e^{-2t} + c_2 e^{t}}$$

c_1, c_2 constants

c_1, c_2 depend on two initial conditions

typically $y(t_0) = y_0$ and $y'(t_0) = y_0'$

example $6y'' - 5y' + y = 0 \qquad y(0) = 4, \quad y'(0) = 0$

$6r^2 - 5r + 1 = 0$ characteristic equation

$(2r - 1)(3r - 1) = 0$

$r = 1/2, \quad r = 1/3$

general solution: $y = c_1 e^{1/2\,t} + c_2 e^{1/3\,t}$

IC: $y(0) = 4 \qquad y' = \frac{1}{2} c_1 e^{1/2\,t} + \frac{1}{3} c_2 e^{1/3\,t}$

$4 = c_1 + c_2$ (1)

IC: $y'(0) = 0$

$0 = \frac{1}{2} c_1 + \frac{1}{3} c_2$ (2)

$0 = c_1 + \frac{2}{3} c_2$ (3)

① - ③ $\quad 4 = \frac{1}{3} c_2 \quad\quad c_2 = 12$

from ① $\quad c_1 = -8$

particular solution: $y = -8e^{\frac{1}{2}t} + 12e^{\frac{1}{3}t}$

$\lim_{t \to \infty} y =$ unbound (to $\pm\infty$)

if both roots of characteristic eqs are positive, the all nonzero solutions become unbounded as $t \to \infty$

if both negative, all nonzero solutions go to 0 as $t \to \infty$

If one positive, one negative, then the solution w/ pos. exponent becomes unbounded, the other goes to 0. gen. solution is unbounded, too.

example $y'' - y' - 2y = 0 \qquad y(0) = \alpha, \quad y'(0) = 2$

find α so $\lim_{t\to\infty} y = 0$ general solution

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0 \qquad r = -1, \; r = 2$

leads to unbounded solution

gen. solution

$y = c_1 e^{-t} + c_2 e^{2t} \qquad y' = -c_1 e^{-t} + 2c_2 e^{2t}$

IC: $y(0) = \alpha$

$\alpha = c_1 + c_2 \quad (1)$

IC: $y'(0) = 2$

$2 = -c_1 + 2c_2 \quad (2)$

① + ② $\quad \alpha + 2 = 3C_2$

$$C_2 = \frac{1}{3}\alpha + \frac{2}{3}$$

from ① $\quad C_1 = \alpha - C_2 = \frac{2}{3}\alpha - \frac{2}{3}$

solution: $y = \left(\frac{2}{3}\alpha - \frac{2}{3}\right)\underbrace{e^{-t}}_{\text{to } 0} + \left(\frac{1}{3}\alpha + \frac{2}{3}\right)\underbrace{e^{2t}}_{\text{unbounded}}$

$\alpha = -2$ will ensure $\lim_{t\to\infty} y = 0$

Exam 1: 11 questions

8 multiple-choice

BRNG 2280 during class time

on Tue. 6/28