10.

$$x_1 = -7e^{-t} + 6e^{-2t}$$

$$x_2 = 7e^{-t} + 9e^{-2t}$$

graph x_1 vs x_2

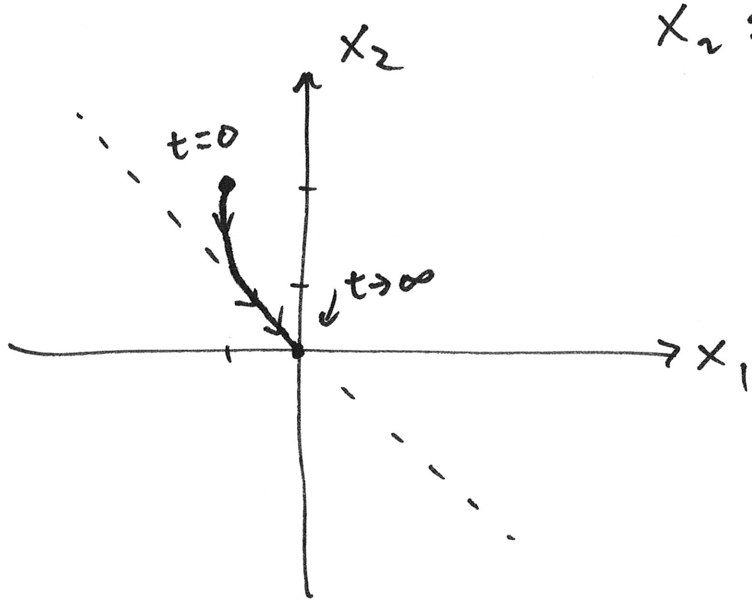
when t is large, $e^{-2t} \ll e^{-t}$

$$x_1 \approx -7e^{-t}$$

$$x_2 \approx 7e^{-t}$$

$$\frac{x_2}{x_1} \approx -1$$

$$x_2 \approx -x_1$$



23. $\vec{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$

verify $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^t$ is a solution.

$\vec{x}' = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} (te^t + e^t)}$

M. Tank 1: 50 gal, 10 oz salt

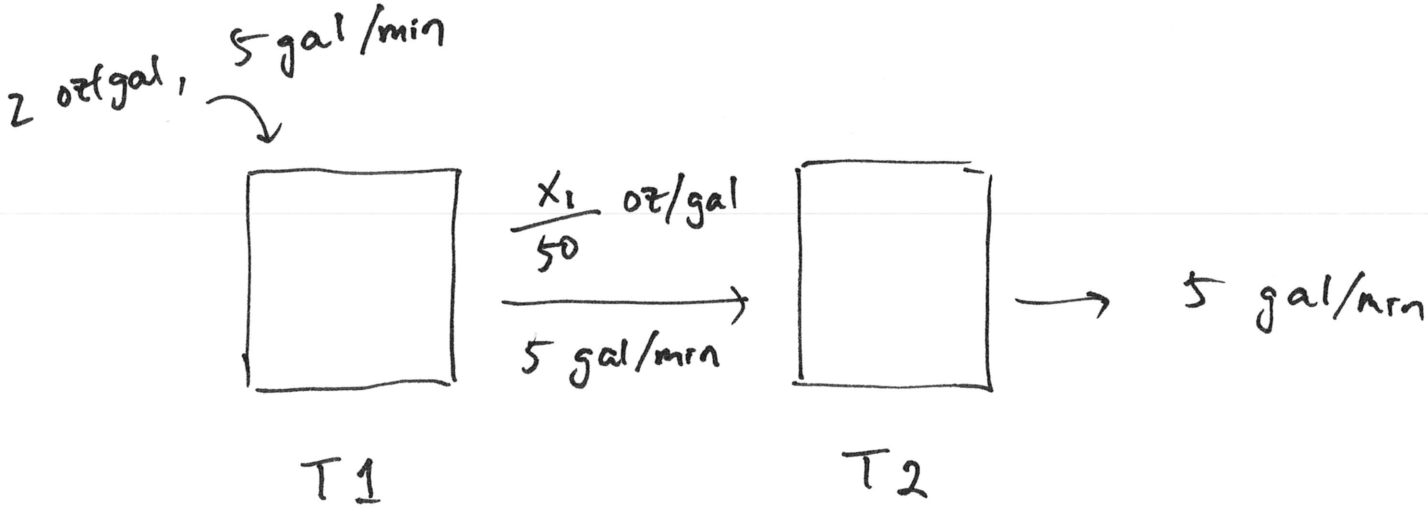
Tank 2: 20 gal, 15 oz salt

2 oz/gal, 5 gal/min into T1, stir, then let out into T2 at 5 gal/min

Solution out T2 at 5 gal/min.

$x_1(t)$: salt in T1

$x_2(t)$: " " T2



$$\frac{dx_1}{dt} = (2)(5) - \left(\frac{x_1}{50}\right)(5) = 10 - 0.1x_1$$

$$\frac{dx_2}{dt} = \left(\frac{x_1}{50}\right)(5) - \left(\frac{x_2}{20}\right)(5) = 0.1x_1 - 0.25x_2$$

7.3 Linear independence, Eigenvalues, Eigenvectors

vectors $\vec{X}^{(1)}, \vec{X}^{(2)}, \vec{X}^{(3)}, \ldots, \vec{X}^{(n)}$ are <u>linearly independent</u>

if $c_1 \vec{X}^{(1)} + c_2 \vec{X}^{(2)} + \cdots + c_n \vec{X}^{(n)} = \vec{0}$ means

$c_1 = c_2 = \cdots = c_n = 0$

→ the only way to add up to zero vector is if all vectors are multiplied by zero

e.g. $\vec{X}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{X}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

true only if $c_1 = c_2 = 0$

→ can't produce any of these $\vec{X}^{(n)}$ by linear combinations of the others

<u>example</u> are $\vec{x}^{(1)} = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$ and $\vec{x}^{(2)} = \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}$

linearly independent?

yes, if $c_1 \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

means $c_1 = c_2 = 0$

find c_1, c_2

$$\begin{bmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Gaussian elimination

$$\left[\begin{array}{cc|c} e^{-t} & e^{2t} & 0 \\ -e^{-t} & 2e^{2t} & 0 \end{array}\right]$$

reduce to row-echelon form

get left sub matrix close to identity

$$\left[\begin{array}{cc|c} e^{-t} & e^{2t} & 0 \\ 0 & 3e^{2t} & 0 \end{array}\right]$$ add R1 to R2

$$\left[\begin{array}{cc|c} e^{-t} & e^{2t} & 0 \\ 0 & 1 & 0 \end{array}\right]$$ multiply R2 by $\frac{1}{3e^{2t}}$

last row means $1 \cdot C_2 = 0 \rightarrow C_2 = 0$

row 1 means $e^{-t} c_1 + e^{2t} C_2 = 0$

$$e^{-t} C_1 = 0 \rightarrow C_1 = 0$$

so the vectors <u>are</u> linearly ~~indede~~ independent.

<u>example</u> are $\begin{bmatrix} e^{-t} \\ 2e^{-t} \end{bmatrix}$, $\begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$, $\begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$

linearly independent?

solve $c_1 \begin{bmatrix} e^{-t} \\ 2e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$

if there is a solution for c_1, c_2, then they are NOT linearly independent

$$\left[\begin{array}{cc|c} e^{-t} & e^{-t} & 0 \\ 2e^{-t} & e^{-t} & e^{-t} \end{array}\right]$$

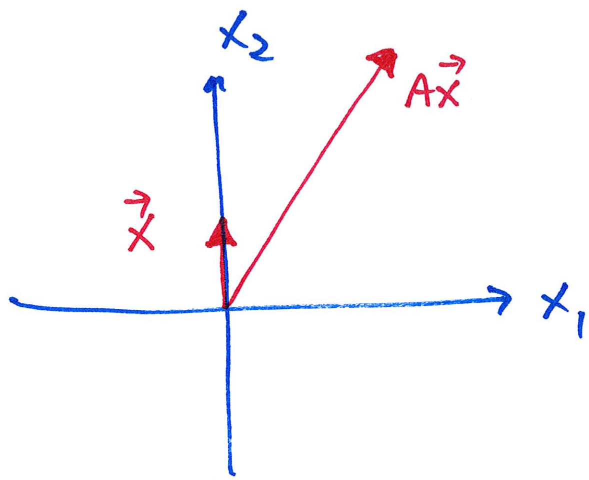
$$\left[\begin{array}{cc|c} e^{-t} & e^{-t} & 0 \\ 0 & -e^{-t} & e^{-t} \end{array}\right] \quad -2R1 + R2$$

$$\left.\begin{array}{l} -e^{-t} c_2 = e^{-t} \rightarrow c_2 = -1 \\ e^{-t} c_1 + e^{-t} c_2 = 0 \rightarrow c_1 = 1 \end{array}\right\} \text{solution exists, so NOT independent}$$

Eigenvalues and Eigenvectors

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$ is linear transformation of the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$



note the direction changed.

some vectors preserve their directions → eigenvectors

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad A\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(can only change magnitude OR turn 180° around) the change in magnitude is eigenvalue

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

eigenvalue corresponding to eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

find them: solve $A\vec{x} = \lambda \vec{x}$

$$\boxed{\begin{array}{l} (A - \lambda I)\vec{x} = \vec{0} \\ \text{and} \quad \det(A - \lambda I) = 0 \end{array}}$$

finds e-vectors

finds e-values (do this first)

<u>example</u>

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

e-values: $\begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = 0$

$(1-\lambda)(3-\lambda) - (0)(2) = 0$

$(1-\lambda)(3-\lambda) = 0$ so $\boxed{\lambda = 1, \quad \lambda = 3}$

each one has an e-vector

solve: $(A - \lambda I)\vec{x} = \vec{0}$

<u>$\lambda = 1$</u>

$$\left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow$$ ~~$x_2 = r$ (free variable)~~

~~first row~~:

first row: $2x_2 = 0 \rightarrow x_2 = 0$

bottom row: $x_1 = r$ (free variable)

so the e-vector corresponding to e-value $\boxed{\lambda = 1}$

is $\vec{x} = \begin{bmatrix} r \\ 0 \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

for simplicity, choose $r = 1$, so $\boxed{\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

Similarly, for $\lambda = 3$, $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

7.4 Basic Theory of Systems of Eqs

if $\vec{x}' = A\vec{x}$ and A is $n \times n$, then there are

n linearly independent solutions $\vec{x}^{(1)}, \vec{x}^{(2)}, \dots, \vec{x}^{(n)}$

(just like nth-order eq has n solutions)

then general solution is just a linearly combination

of them: $\vec{x} = c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} + c_3 \vec{x}^{(3)} + \cdots + c_n \vec{x}^{(n)}$

the Wronskian of these solutions is <u>non-zero</u>

because they are linearly independent.

$$W = \begin{vmatrix} \vec{X}^{(1)} & \vec{X}^{(2)} & \vec{X}^{(3)} & \cdots & \vec{X}^{(n)} \end{vmatrix}$$

solutions as columns.

Computer Project 3. Predator-Prey Equations

Goal: Investigate the qualitative behavior of a *nonlinear* system of differential equations.

Tools needed: `pplane8`

Description: A farmer has ladybugs and aphids in her fields. The helpful ladybugs (predator) eat the destructive aphids (prey) who devour her crops.

$$\text{Let } \begin{cases} x(t) = \text{aphid population (in millions) at time } t, \\ y(t) = \text{ladybug population (in millions) at time } t. \end{cases}$$

The farmer knows that the growth rates of the aphid and ladybug populations are given respectively by

$$\begin{cases} \dfrac{dx}{dt} = x(1-y), \\ \dfrac{dy}{dt} = y(x-1). \end{cases}$$

Questions: Assume there are initially 800,000 aphids and 400,000 ladybugs in all that follows below.

1. Use `pplane8` to plot the trajectory through $(0.8, 0.4)$. As t increases, describe what happens to each population. Is the aphid population ever smaller than 300,000? Are the aphids ever eradicated? Does the ladybug population ever exceed 2 million?

2. A fellow farmer suggests that she use pesticide to kill the aphids. She is reluctant because it also kills the helpful ladybugs and she prefers to have some ladybugs remaining to eat other destructive insects. If she were to use a pesticide, the growth rates would then become

$$\begin{cases} \dfrac{dx}{dt} = x(1-y) - s\,x, \\ \dfrac{dy}{dt} = y(x-1) - s\,y. \end{cases} \tag{$*$}$$

where $s \geq 0$ is a measure of the "strength" of the pesticide – the larger the s, the stronger the pesticide. Currently there are only two commercially available strengths: $s = 0.5$ and $s = 0.75$. Plot the trajectories for the new system of equations $(*)$ with these values of s. Will the aphids ever be totally eliminated?

3. If she knows her crops will survive if the aphid population never exceeds 2.6 million, which strength (if any) would you recommend she use: $s = 0.0$ (no pesticide), $s = 0.5$, $s = 0.75$?

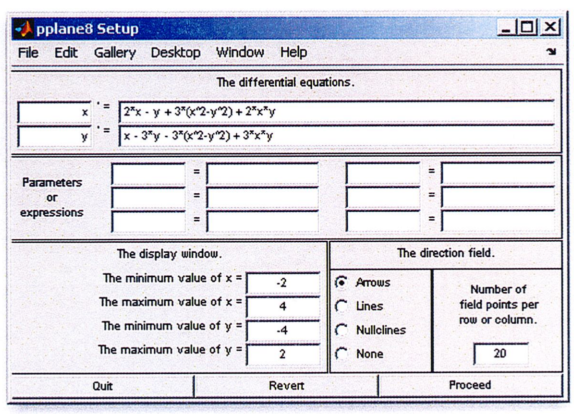
4. By special permission, she could get a pesticide with the maximum strength of $s = 1.5$. Plot this trajectory. What happens to the ladybugs and aphids if she uses this pesticide?

pplane8 Phase Portraits

- The routine pplane8 is already loaded on all ITaP machines as standard software. (If you are using your own copy of MATLAB you may need to download pplane8 from http://math.rice.edu/~dfield)
- You may also access MATLAB through the *Software Remote*:

  http://goremote.ics.purdue.edu

- To access pplane8, at a MATLAB prompt type: pplane8
- A popup window will appear similar to that for dfield8 (see below).
- Correctly enter your system of differential equations and the range of values of the independent and dependent variables. Hit **Proceed** and a graphics window will appear with the direction field of your system of differential equations. Click the mouse at any point and the corresponding trajectory will be plotted.
- There are several options available in the graphics display window: printing, keyboard input of initial conditions, inserting text, erasing solutions, zoom, etc.
- You may have up to six (6) parameters to quickly vary your system of differential equations.
- pplane8 is very similar to dfield8 in its syntax and use.



Quiz 10 MA 266, JULY 22

SECTION: NAME:

Problem 1.(10 points.)
Express the solution of the given initial value problem in terms of a convolution integral.

$$y^{(4)} - y = g(t), y(0) = y'(0) = y''(0) = y'''(0) = 0$$

Problem 2.(10 points.)
Tank 1 initially contains 20 gals water with 15 oz of salt in it, while Tank 2 initially contains 50 gals of water with 10 oz of salt in it. Water containing 2 oz/gal of salt flows into Tank 1 at a rate of 5 gal/min and the well-stirred mixture flows from Tank 1 into Tank 2 at the same rate of 5 gal/min. The solution in Tank 2 flows out to the ground at a rate of 3 gal/min. If $x_1(t)$ and $x_2(t)$ represent the number of ounces of salt in Tank 1 and Tank 2, respectively, **set up but do not solve** an initial value problem describing this system.