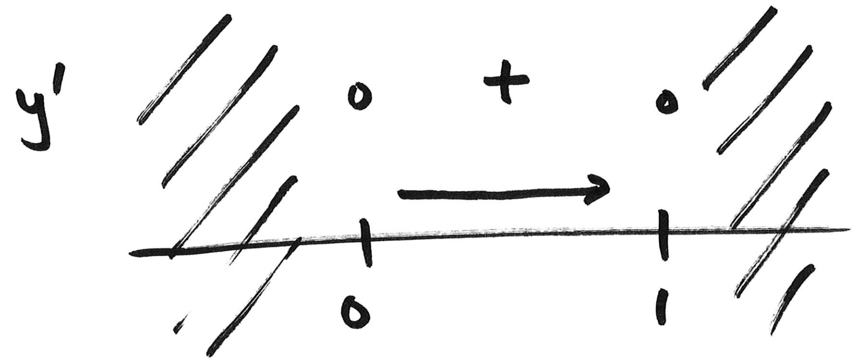
22. $\frac{dy}{dt} = \alpha y (1-y) \qquad y(0) = y_0$

y: proportion of infected

$x + y = 1$

x: susceptible

a) equilibrium: $y=0, \; y=1$



$y=0$ is unstable

$y=1$ is asymptotically stable

b). $\frac{1}{y(1-y)} dy = \alpha dt$

$\ln y - \ln(1-y) = \alpha t + K$

$\ln\left(\frac{y}{1-y}\right) = \alpha t + K$

$$\frac{y}{1-y} = e^{\alpha t + k} = Ce^{\alpha t} \qquad y(0) = y_0$$

$$\frac{y_0}{1-y_0} = C$$

$$\frac{y}{1-y} = \left(\frac{y_0}{1-y_0}\right)e^{\alpha t}$$

$$y = (1-y)\left(\frac{y_0}{1-y_0}\right)e^{\alpha t}$$

$$y = \left(\frac{y_0}{1-y_0}\right)e^{\alpha t} - \left(\frac{y_0}{1-y_0}e^{\alpha t}\right)y$$

$$y + \left(\frac{y_0}{1-y_0}\right)e^{\alpha t}y = \frac{y_0}{1-y_0}e^{\alpha t}$$

$$y\left[1 + \left(\frac{y_0}{1-y_0}\right)e^{\alpha t}\right] = \frac{y_0}{1-y_0}e^{\alpha t}$$

$$y = \frac{\frac{y_0}{1-y_0}e^{\alpha t}}{1 + \frac{y_0}{1-y_0}e^{\alpha t}} \qquad \lim_{t\to\infty} y = 1$$

17. a) $\frac{dy}{dt} = ry\ln\left(\frac{K}{y}\right) \qquad y(0) = y_0$

sub $u = \ln\left(\frac{y}{K}\right)$ eliminate $\frac{dy}{dt}$

$$\frac{du}{dt} = \frac{1}{\frac{y}{K}} \frac{y'}{K} = \frac{K}{y}\frac{y'}{K} = \frac{y'}{y}$$

$$y' = yu'$$

$$\ln\left(\frac{K}{y}\right) = \ln\left(\frac{y}{K}\right)^{-1}$$

$$= -\ln\left(\frac{y}{K}\right)$$

$$= -u$$

$$\cancel{y}u' = r\cancel{y}\ln\left(\frac{K}{y}\right)$$

$$u' = -ru$$

$$\vdots$$

$$u = Ce^{-rt}$$

$$\ln\left(\frac{y}{K}\right) = Ce^{-rt}$$

$$\ln\left(\frac{y_0}{K}\right) = C$$

$$\ln\left(\frac{y}{K}\right) = \ln\left(\frac{y_0}{K}\right)e^{-rt}$$

$$\frac{y}{K} = e^{\ln\left(\frac{y_0}{K}\right)e^{-rt}}$$

$$y = Ke^{\ln\left(\frac{y_0}{K}\right)e^{-rt}}$$

2.6 Exact Equations

an exact equation has the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

such that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ or $M_y = N_x$

has solution $\psi(x,y) = C$

where $\psi_x = M$ and $\psi_y = N$

---

it's called <u>exact</u> because the exact (or total) derivative of $\psi(x,y) = C$ is

$$\frac{\partial \psi}{\partial x}\cancelto{1}{\frac{dx}{dx}} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0$$

$$\psi_x + \psi_y\, y' = 0 \longrightarrow \psi_x\,dx + \psi_y\,dy = 0$$

from continuity of $\psi(x,y)$, we know

$$\underbrace{(\psi_x)}_{M}{}_y = \underbrace{(\psi_y)}_{N}{}_x$$

$$\Rightarrow \quad M_y = N_x$$

$\psi(x,y)$: potential function

<u>example</u> Solve $(2xy^2+2y) + (2x^2y+2x)y' = 0$

$$\underbrace{(2xy^2+2y)}_{M}dx + \underbrace{(2x^2y+2x)}_{N}dy = 0$$

exact? need $M_y = N_x$

$$M_y = 4xy + 2$$

$$N_x = 4xy + 2$$

so, yes, it is exact.

Solution is $\psi(x,y) = C$

where $\psi_x = M$, $\psi_y = N$

pick M to integrate with respect to x, treating y as constant

$$\psi(x,y) = \int M\,dx = \int (2xy^2 + 2y)\,dx$$

$$= x^2y^2 + 2xy + h(y)$$

function of y or constant (~~dispear~~ vanish when taken partial with respect to x)

$h(y) = ?$

note: $\psi_x = M$

$\psi_y = N$

$$\psi_y = 2x^2y + 2x + \frac{dh}{dy} = N$$

$$= 2x^2y + 2x$$

so $\frac{dh}{dy} = 0 \rightarrow h = K$ (constant)

$$\Psi(x, y) = x^2y^2 + 2xy + K$$

solution is $\Psi(x, y) = C$

$$x^2y^2 + 2xy + K = C$$

$$x^2y^2 + 2xy = C - K$$

$$\boxed{x^2y^2 + 2xy = C}$$

depends on IC

<u>example</u> $(e^x \sin y - 2y\sin x) + (e^x\cos y + 2\cos x + 3)y' = 0$

$M = e^x \sin y - 2y \sin x$

$N = e^x \cos y + 2\cos x + 3$

$M_y = N_x$?

$M_y = e^x \cos y - 2\sin x$

$N_x = e^x \cos y - 2 \sin x$

exact.

solution: $\psi = C$ $\quad \psi_x = M \quad \psi_y = N$

$\psi = \int M\,dx = \int (e^x \sin y - 2y \sin x)\,dx$

$= e^x \sin y + 2y\cos x + h(y)$

$\psi_y = N$

$e^x \cos y + 2\cos x + h'(y) = e^x\cos y + 2\cos x + 3$

$h(y) = 3y$

$$\psi(x,y) = e^x \sin y + 2y\cos x + 3y$$

solution: $\boxed{e^x \sin y + 2y\cos x + 3y = C}$

if eq is not exact, use other methods

(separable, linear, homogeneous, Bernoulli)

two