4.3 The Method of Undetermined Coefficients

$$y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + \cdots + p_n(t)y = g(t)$$

usually used if $p_1, p_2, p_3, \ldots$ are <u>constants</u>

AND $g(t)$ is of the right form

→ polynomial, sin or cos, exponential

+ products of these

<u>example</u> $y^{(4)} + y''' = \sin 2t$

$$r^4 + r^3 = 0 \qquad r = 0, 0, 0, -1$$

$$y = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t} + Y$$

particular solution

$$Y = A \sin 2t + B \cos 2t$$

$$Y' = 2A\cos 2t - 2B\sin 2t$$

$$Y'' = -4A\sin 2t - 4B\cos 2t$$

$$Y''' = -8A\cos 2t + 8B\sin 2t$$

$$Y^{(4)} = 16A\sin 2t + 16B\cos 2t$$

$$16A\sin 2t + 16B\cos 2t - 8A\cos 2t + 8B\sin 2t = \sin 2t$$

$$16A + 8B = 1$$

$$16B - 8A = 0$$

$$32B - 16A = 0$$

$$40B = 1 \qquad B = \frac{1}{40} \qquad A = \frac{1}{20}$$

$$y = C_1 + C_2 t + C_3 t^2 + C_4 e^{-t} + \frac{1}{20}\sin 2t + \frac{1}{40}\cos 2t$$

example $y''' + 2y'' - y' - 2y = e^t + t^2$

$$r^3 + 2r^2 - r - 2 = 0$$

$$r^2(r+2) - (r+2) = 0$$

$$(r+2)(r^2-1) = 0 \qquad r = -2, -1, 1$$

$$y = c_1 e^t + c_2 e^{-t} + c_3 e^{-2t} + Y$$

$$Y = \underbrace{At^2 + Bt + C}_{\text{for } t^2 \text{ in } g(t)} + \underbrace{Dte^t}_{\text{for } e^t\text{, extra } t \text{ because of } e^t \text{ in homogeneous solution}}$$

$$Y' = 2At + B + Dte^t + De^t$$

$$Y'' = 2A + Dte^t + 2De^t$$

$$Y''' = Dte^t + 3De^t$$

DE: $y''' + 2y'' - y' - 2y = e^t + t^2$

$$\cancel{Dte^t} + 3De^t + 4A + \cancel{2Dte^t} + 4De^t$$
$$- 2At - B - \cancel{Dte^t} - De^t$$
$$- 2At^2 - 2Bt - 2C - \cancel{2Dte^t} = e^t + t^2$$

$$6De^t - 2At^2 - 2At - 2Bt + 4A - B - 2C = e^t + t^2$$

$6D = 1 \qquad D = 1/6$

$-2A = 1 \qquad A = -1/2 \qquad\qquad Y = At^2 + Bt + C + Dte^t$

$-2A - 2B = 0 \qquad B = 1/2$

$4A - B - 2C = 0 \qquad C = -\frac{5}{4}$

$$\boxed{y = c_1 e^t + c_2 e^{-t} + c_3 e^{-2t} - \frac{1}{2}t^2 + \frac{1}{2}t - \frac{5}{4} + \frac{1}{6}te^t}$$

IC's: $y(0)=0$, $y'(0)=1$, $y''(0)=0$

$y(0)=0$

$$0 = c_1 + c_2 + c_3 - \frac{5}{4} \rightarrow c_1 + c_2 + c_3 = \frac{5}{4}$$

$$y' = c_1 e^t - c_2 e^{-t} - 2c_3 e^{-2t} - t + \frac{1}{2} + \frac{1}{6}te^t + \frac{1}{6}e^t$$

$$1 = c_1 - c_2 - 2c_3 + \frac{1}{2} + \frac{1}{6} \rightarrow c_1 - c_2 - 2c_3 = \frac{1}{3}$$

$$y'' = c_1 e^t + c_2 e^{-t} + 4c_3 e^{-2t} - 1 + \frac{1}{6}te^t + \frac{1}{3}e^t$$

$$0 = c_1 + c_2 + 4c_3 - 1 + \frac{1}{3} \rightarrow c_1 + c_2 + 4c_3 = \frac{2}{3}$$

$$\begin{cases} c_1 + c_2 + c_3 = \frac{5}{4} \\ c_1 - c_2 - 2c_3 = \frac{1}{3} \\ c_1 + c_2 + 4c_3 = \frac{2}{3} \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5/4 \\ 1 & -1 & -2 & 1/3 \\ 1 & 1 & 4 & 2/3 \end{array}\right]$$

from Matlab, $c_1 = 25/36$, $c_2 = 3/4$, $c_3 = -7/36$

$$y = \frac{25}{36}e^{t} + \frac{3}{4}e^{-t} - \frac{7}{36}e^{-2t} - \frac{1}{2}t^2 + \frac{1}{2}t - \frac{5}{4} + \frac{1}{6}te^{t}$$

<u>example</u> $\quad y''' - 3y'' + 2y' = \underline{\underline{t + e^{t}}}$ $\qquad$ form of Y only

$$r^3 - 3r^2 + 2r = 0$$

$$r(r^2 - 3r + 2) = 0$$

$$r(r-2)(r-1) = 0 \qquad r = 0, 1, 2$$

$$y = c_1 + c_2 e^{t} + c_3 e^{2t} + Y$$

$$Y = (At + B)t + Ce^{t} \cdot t + \cancel{Dte^{2t}}$$

example roots of characteristic eq. are

$$r = 0, 0, 0, \pm i, \pm i, -1$$

$$g(t) = \underbrace{t^2 + 4}_{\text{2nd deg polynomial}} + e^t + \sin t + \cos 2t$$

$$Y = ?$$

$$y = C_1 + C_2 t + C_3 t^2 + C_4 \cos t + C_5 \sin t$$

$$+ C_6 t \cos t + C_7 t \sin t + C_8 e^{-t} + Y$$

$$Y = (At^2 + Bt + C)t^3 + De^t + (E\cos t + F\sin t)t^2$$

$$+ G\cos 2t + H\sin 2t$$