Supp. C

$$y^2 \frac{dy}{dx} + \frac{y^3}{x} = \frac{2}{x^2} \qquad (x>0)$$

$$u = y^3$$

$$\frac{du}{dx} = 3y^2 \frac{dy}{dx} \quad \rightarrow \quad \frac{dy}{dx} = \frac{1}{3y^2}\frac{du}{dx}$$

$$\cancel{y^2}\,\frac{1}{3\cancel{y^2}}\frac{du}{dx} + \frac{u}{x} = \frac{2}{x^2}$$

$$\frac{1}{3}u' + \frac{1}{x}u = \frac{2}{x^2} \qquad \text{linear}$$

$$u' + \frac{3}{x}u = \frac{6}{x^2}$$

$$\mu = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

$$x^3u' + 3x^2u = 6x$$

$$\frac{d}{dx}\left(x^3 u\right) = 6x$$

$$x^3 u = 3x^2 + C$$

$$u = \frac{3}{x} + \frac{C}{x^3}$$

$$\boxed{y^3 = \frac{3}{x} + \frac{C}{x^3}}$$

32. $$\frac{dy}{dx} = \frac{x^2+3y^2}{2xy} \qquad y = xv \rightarrow v = \frac{y}{x}$$

$$= \frac{\frac{x^2+3y^2}{x^2}}{\frac{2xy}{x^2}} = \frac{1+3v^2}{2v}$$

$$\frac{dy}{dx} = \frac{d}{dx}(xv) = x\frac{dv}{dx} + v$$

new DE: $x\frac{dv}{dx} + v = \frac{1+3v^2}{2v}$

$$x\frac{dv}{dx} = \frac{1+3v^2}{2v} - \frac{2v^2}{2v} = \frac{1+v^2}{2v}$$

$$\frac{2v}{1+v^2}\,dv = \frac{1}{x}\,dx$$

$$\ln|1+v^2| = \ln|x| + K$$

$$1+v^2 = e^{\ln|x|+K} = e^{\ln x}\cdot e^{K}$$

$$= Cx$$

$$v^2 = Cx - 1$$

$$\frac{y^2}{x^2} = Cx - 1 = y^2 = Cx^3 - x^2$$

22. $y' = \dfrac{3x^2}{3y^2 - 4}$ $\quad y(1) = 0$

$\vdots$

$y^3 - 4y = x^3 - 1$ $\quad$ interval this is valid?

y' undefined (vert. tangents)

$y^2 = \frac{4}{3}$ $\quad y = \pm 1.1547 \rightarrow x = ?$

$y = -1.1547,$ $\quad x^3 = 1 + (y^3) - 4(y)$

$x \approx -1.28$

2.3 Modeling with First Order Equations

Example 1. A 200-liter water tank initially contains 100 L of pure water. A mixture containing a concentration of 2 g/L of salt enters the tank at a rate of 3 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression for the amount of salt in the tank at any time.

let $y(t)$ be amount of salt (in grams)

need a DE

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= \underbrace{(2\ g/L)(3\ L/min)}_{\text{in}} - \underbrace{\left(\frac{y\ g}{100\ L}\right)(3\ L/min)}_{\text{out}}$$

$$\frac{dy}{dt} = 6 - 0.03y$$

linear and separable

$$\frac{1}{6-0.03y}\,dy = dt$$

$$\vdots$$

$$y = 200 + Ce^{-0.03t}$$

IC: $y(0)=0$ "pure water"

$$\boxed{y = 200 - 200e^{-0.03t}}$$

$$\lim_{t \to \infty} y = 200$$

tank has same concentration as flow in as $t \to \infty$

Example 2. Same set-up as in example 1. But what if the well-stirred mixture is let out of the tank at a rate of 2 L/min instead of 3 L/min? What is the amount of salt in the tank right before the moment the tank overflows?

$y(t)$: salt in tank (in grams)

$$\frac{dy}{dt} = \underbrace{(2\ g/L)(3\ L/min)}_{\text{in}} - \left(\frac{y}{100+t}\right)(2)$$

volume of water

net gain of 1 L/min

5 min → 105 L

15 min → 115 L

t min → (100+t) L

$$y' = 6 - \frac{2}{100+t}y$$

first order linear

$$y' + \frac{2}{100+t}y = 6$$

$$\mu = e^{\int \frac{2}{100+t}dt} = e^{2\ln(100+t)} = (100+t)^2$$

IC: $y(0) = 0$

⋮

$$\boxed{y(t) = 2(100+t) - 200(100)^2(100+t)^{-2}}$$

overflows $t > 100$

$y(100) = 350\ g$

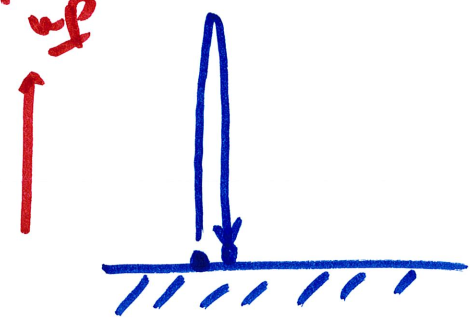
$$\text{concentration} = \frac{350\ g}{200\ L} = 1.75\ g/L$$

Example 3. An object of mass 1 kg is tossed straight up from ground level with an initial velocity of 5 m/s. Force due to air resistance can be modeled as , in the direction opposite to the velocity. Find expressions for the velocity and height as a functions of time.

$\frac{|v|}{20}$

y positive up

define positive direction

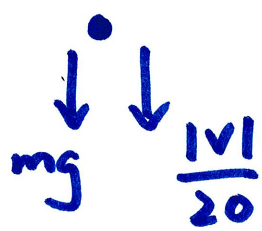


drag changes direction!

ground

$y = 0$

upward part:



downward part

up part: $F = ma$

$$m\frac{dv}{dt} = -mg - \frac{|v|}{20}$$

$v > 0$ going up so $|v| = v$

$$\frac{dv}{dt} = -g - \frac{v}{20m}$$

down part: $F = ma$

$$m\frac{dv}{dt} = -mg + \frac{|v|}{20}$$

$v < 0$ going down

$|v| = -v$

e.g. $v = -2$

$|-2| = -(-2)$

$= 2$

$$\frac{dv}{dt} = -g - \frac{v}{20m}$$

so one DE for both parts

$$\frac{dv}{dt} = -g - \frac{v}{20m}$$

$m = 1$

$g = 9.8$

$v(0) = 5$

linear and separable

$\vdots$

$$v(t) = -20mg + (5 + 20mg)e^{-\frac{1}{20m}t}$$

$$\boxed{v(t) = -196 + 201e^{-0.05t}}$$ velocity

$y(t) = \int v(t)\,dt$ $\quad y(0) = 0$ (ground)

$$\boxed{y(t) = -196t - 4020e^{-0.005t} + 4020}$$

when does max height occur? $v = 0$

solve for t

$t \approx 0.504$

max height: $y(t)$

when does it hit ground?

solve $y = 0 \rightarrow$ 2 values of t

$\hookrightarrow$ 2nd t