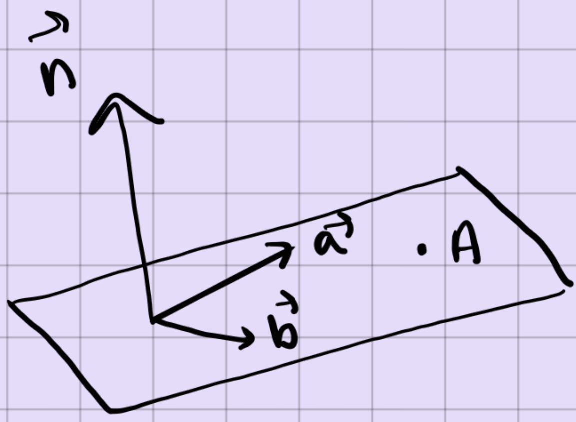
Planes



$$\vec{n} = \vec{b} \times \vec{a}$$

$$\vec{n} \cdot \vec{a} = 0 \quad \text{(Plane eqn)}$$

$n_1 (x-a) + n_2 (y-b), n_3 (z-c) = 0$, (x,y,z) (a,b,c) are points on the plane

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1) Plane perpendicular to $2x + y - 2z = 2$ and $x + 3z = 4$, passing through $(2,1,1)$.

$$\vec{n} = \langle 2, 1, -2 \rangle \times \langle 1, 0, 3 \rangle$$

$$= \langle 3, -2-6, -1 \rangle$$

$$= \langle 3, -8, -1 \rangle$$

a) $3(x-2) - 8(y-1) - 1(z-1) = 0$

$$3x - 6 - 8y + 8 - z + 1 = 0$$

$$3x - 8y - z = -3$$ ✓

b) $3x - 8y - z = \langle 3, -8, -1 \rangle \cdot \langle 2, 1, 1 \rangle$

$$3x - 8y - z = 6 - 8 - 1$$

$$3x - 8y - z = -3$$ ✓

2) $\langle a, b, c \rangle$ is a <u>unit vector</u>. Given a normal vector $\langle a, b, c \rangle$ and a point (a, b, c), find the plane containing both.

$$a(x-a) + b(y-b) + c(z-c) = 0$$

$$ax - a^2 + by - b^2 + cz - c^2 = 0$$

$$ax + by + cz = a^2 + b^2 + c^2$$

$\sqrt{a^2+b^2+c^2} = 1$

$\hookrightarrow a^2+b^2+c^2 = 1$

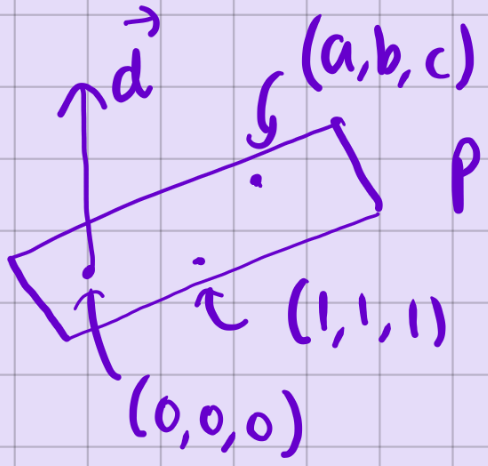
$\therefore ax + by + cz = 1$

3) 3 points on plane P :

$(0,0,0)$, $(1,1,1)$, (a,b,c)

The line passing through $(0,0,0)$ and $(1,2,a)$ is perpendicular to P.

$\vec{d} = \langle 1, 2, a \rangle$



$\langle 1, 2, a \rangle \cdot \langle 1, 1, 1 \rangle = 0$

$1 + 2 + a = 0 \Rightarrow \boxed{a = -3}$

4) Blue plane : $x + 3y - 2z = 6$

Yellow plane: $2x + y + z = 3$

Line of intersection (green) :

$$\begin{aligned} x + 3y - 2z &= 6 \\ 2x + y - z &= 3 \end{aligned}$$

Say $z = 0$:

$2(x + 3y = 6) \rightarrow 2x + 6y = 12$

$2(x + 3y = 6)$ $\quad\quad$ $\cancel{2x} + 6y = 12$

$2x + y = 3$ $\quad\quad$ $-\cancel{2x} - y = -3$

$5y = 9$

$x = \frac{3}{5}$ $\quad\quad$ $y = \frac{9}{5}$

Point of intersection : $(3/5, 9/5, 0)$

$\langle 1, 3, -2 \rangle \times \langle 2, 1, -1 \rangle$

$= \langle -3 + 2, -4 + 1, 1 - 6 \rangle$

$= \langle -1, -3, -5 \rangle$

Blue plane : $x + 3y - 2z = 6$

Yellow plane : $2x + y + z = 3$

Line : $\langle -1, -3, -5 \rangle \lambda + \langle 3/5, 9/5, 0 \rangle$

Projectile : $\langle 1, 2, 1 \rangle t + \langle 1, -2, -1 \rangle = r(t)$

In this system, what is the order of intersection for the projectile ?

Projectile : $x = t+1$
$y = 2t-2$
$z = t-1$

Blue plane :

$$t+1 + 6t-6 - 2t+2 = 6$$

$$5t = 9 \Rightarrow \boxed{t = 9/5}$$

Yellow plane :

$$2t+\cancel{2} + 2t-\cancel{2} + t-1 = 3$$

$$5t = 4 \Rightarrow \boxed{t = 4/5}$$

$t_{yellow} < t_{Blue} \Rightarrow$ Passes through Π_{yellow} first.

↳ Passes through both planes as t has an unrestricted domain.

5) ℓ_1 passes through $(-1,1,2)$, and is perpendicular to $x-2y+2z=8$. When does it

intersect the yz-plane ?

$\ell_1 : \langle 1,-2,2\rangle \lambda + \langle -1,1,2\rangle$

$\Rightarrow x = \lambda - 1 = 0 \Rightarrow \boxed{\lambda = 1}$

$\Rightarrow y = -2\lambda + 1 = -1$

$\Rightarrow z = 2\lambda + 2 = 4$

$\langle 0,-1,4\rangle$

Quadric surfaces :

1) $ax^2 + by^2 + cz^2 = L$

↳ Divide by L to get $\frac{ax^2}{L} + \frac{by^2}{L} + \frac{cz^2}{L} = 1$

↳ if one coefficient is negative, the surface is a hyperboloid of one sheet.

↳ if either two coefficients are negative, the surface is a hyperboloid of two sheets.

2) $x^2 + y^2 - z^2$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

↳ Cone :

↳ xy : circle / ellipse

↳ yz : hyperbola

↳ xz : hyperbola

3) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

↳ elliptic paraboloid

↳ xy : circle / ellipse

↳ yz : parabola

↳ xz : parabola

4) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

↳ Hyperbolic paraboloid

↳ xy : hyperbola

↳ yz : parabola

↳ xz : parabola

Vector valued functions, and motion :

$r(t) = \langle t^3, t^4, t^5 \rangle$ displacement

$v(t) = \langle 3t^2, 4t^3, 5t^4 \rangle$ velocity

$a(t) = \langle 6t, 12t^2, 20t^3 \rangle$ acceleration

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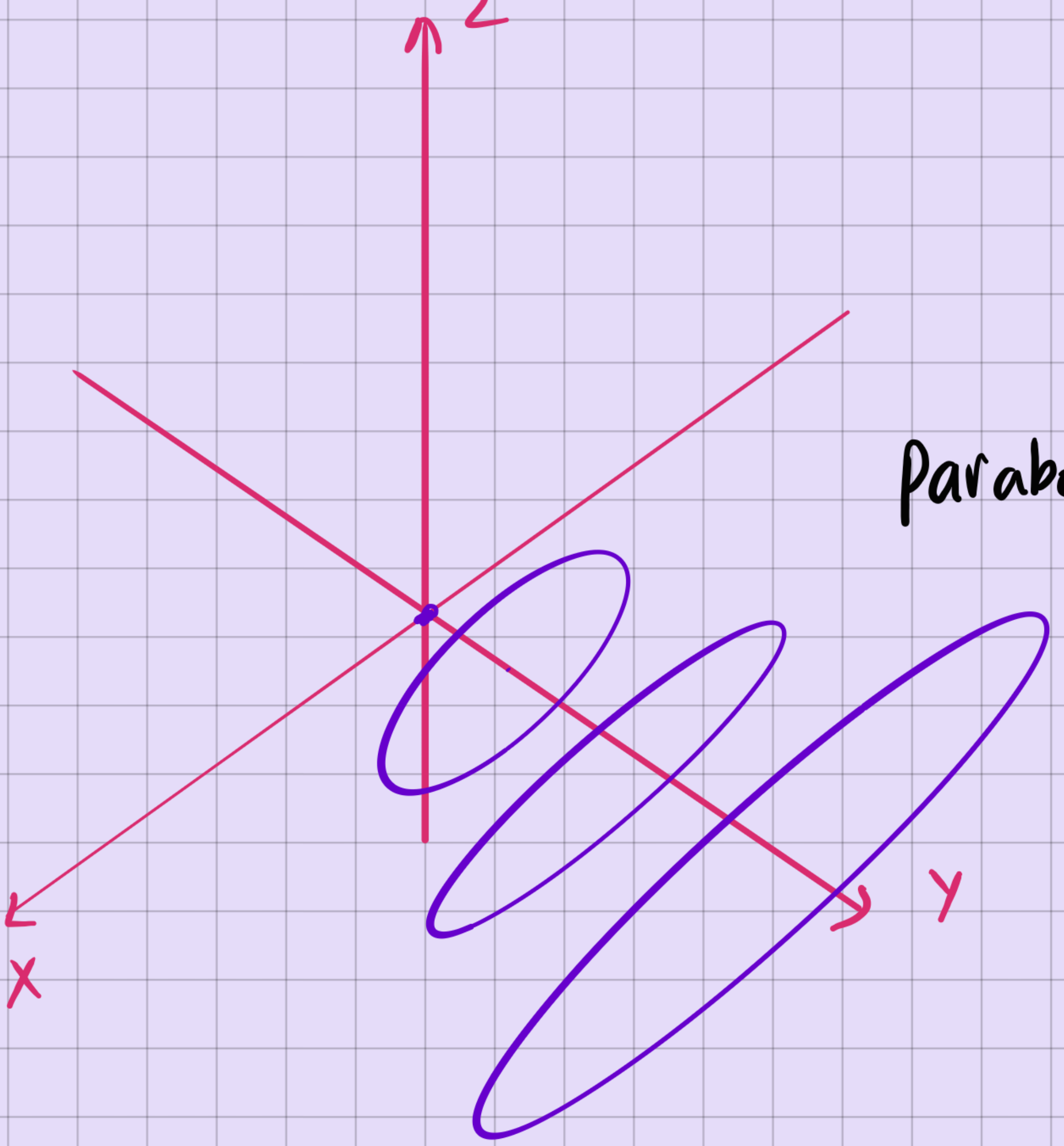
1) $r(t) = \langle t\sin t, 3t^2, -t\cos t \rangle$

↳ $x = t\sin t$
$y = 3t^2$
$z = -t\cos t$

$xy : \langle t\sin t, 3t^2 \rangle$

$xz : \langle t\sin t, -t\cos t \rangle$ ← circular

$yz : \langle 3t^2, -t\cos t \rangle$



Paraboloid

2) $x + y + z = 1$
$x - 2y + 2z = 4$, say $\underline{z = t}$

$\cancel{x} + y = 1 - t$
$- \cancel{x} \underset{+}{-} 2y = \underset{-}{4} \underset{+}{-} 2t$

$3y = -3 + t$

$y = \underline{t - 3}$

$x + \underline{t} - 1 = 1 - t$

3 $\quad\quad$ 3

$$\boxed{x = 2 - \frac{4t}{3}}$$

$$z = -3t \;,\; y = -t-1 \;,\; x = 2+4t$$

3) $y^2 + z^2 = 1 \quad , \quad x + y + 2z = 3$

Intersection : $\quad z = \sin(t)$

$y^2 = 1 - \sin^2(t) = \cos^2(t)$

$y = \cos(t)$

$x = 3 - 2\sin(t) - \cos(t)$

$\langle 3 - 2\sin(t) - \cos(t), \cos(t), \sin(t) \rangle$

4) $a(t) = \langle 6, 6t, 0 \rangle$

$r(1) = \langle 2, 1, 2 \rangle \quad , \quad v(0) = \langle 0, 0, 1 \rangle ,$

find $|r(2)|$.

$$v(t) = \langle 6t, 3t^2, 0\rangle + \langle 0, 0, 1\rangle$$

$$= \langle 6t, 3t^2, 1\rangle$$

$$r(t) = \langle 3t^2, t^3, t\rangle + C$$

$$r(1) = \langle 3, 1, 1\rangle + C = \langle 2, 1, 2\rangle$$

$$C = \langle -1, 0, 1\rangle$$

$$r(t) = \langle 3t^2 - 1, t^3, t + 1\rangle$$

$$r(2) = \langle 11, 8, 3\rangle \Rightarrow |r(2)| = \sqrt{121 + 64 + 9}$$

$$= \underline{\sqrt{194}}$$

5) $r(t) = \langle t\cos t, t\sin t, 9 - t^2\rangle$, $\underline{(s(1))}$

$$v(t) = \langle \cos t - t\sin t, \sin t + t\cos t, -2t\rangle$$

$$s(t) = |v(t)|$$

$$= \sqrt{\cos^2 t - \cancel{2t\cos t\sin t} + t^2\sin^2 t + \sin^2 t + \cancel{2t\cos t\sin t} + t^2\cos^2 t}$$

$$\sqrt{\quad + 4t^2}$$

$$= \sqrt{1 + 1 + 4t^2}$$

$$s(1) = \sqrt{6}$$

6) $a(t) = \langle 0, 2t, \sqrt{2} \rangle$, $v(0) = \langle 1, 0, 0 \rangle$

$$v(t) = \langle 0, t^2, \sqrt{2}t \rangle + \langle 1, 0, 0 \rangle$$

$$= \langle 1, t^2, \sqrt{2}t \rangle$$

distance travelled from $t=0$ to $t=3$:

$$\int_0^3 |v(t)| \, dt$$

$$= \int_0^3 \sqrt{1 + t^4 + 2t^2} \, dt$$

$$= \int^3 \left(\cancel{\sqrt{t^2+1}}\right)^{\cancel{2}} dt$$

0

$$= \frac{t^3}{3} + t \Big|_0^3 = 9 + 3 = \underline{\underline{12}}$$

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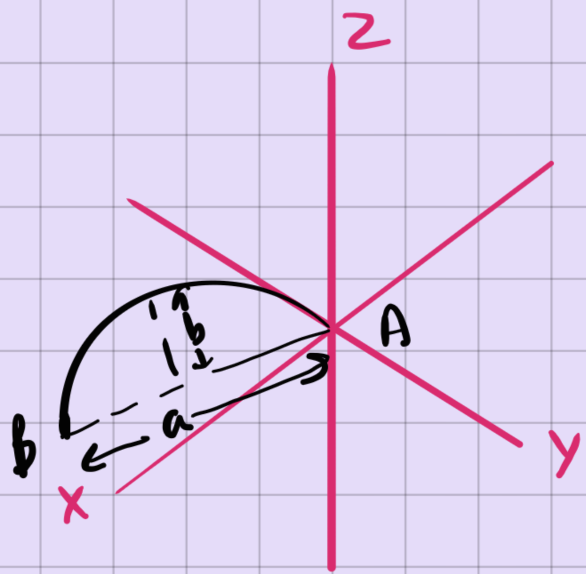
speed, $s(t) = |\vec{v}(t)|$

distance travelled from $t = t_1$ to $t = t_2$:

$$\int_{t_1}^{t_2} s(t)\, dt$$

Projectile motion :



a) Flight duration :

$r(t) = \langle x, y, z \rangle$, $\boxed{z = 0}$

time at which $z = 0$

time at which $z = 0$.

b) Range, a :

Given t_f at which $z = 0$,

$$|\vec{r}(t_f)| = a$$

c) Max height, b :

$$\vec{v}(t) = \vec{r}\,'(t)$$

↳ z component $= 0$, $t = t_b$

$r(t_b)$ is highest position, with the z component = altitude peak.

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7) $\vec{r}\,''(t) = \langle 12t, 12t^2, 1\rangle$, $r(0) = \langle 0,1,0\rangle$, $\vec{r}\,'(0) = \langle 0,0,-1\rangle$. Find $\vec{r}(1)$.

$$\vec{r}\,'(t) = \langle 6t^2, 4t^3, t\rangle + \overset{\langle 0,0,-1\rangle}{C_1}$$

$$\vec{r}\,'(t) = \langle 6t^2, 4t^3, t-1\rangle$$

$$\vec{r}(t) = \langle 2t^3, t^4, \frac{t^2}{2} - t \rangle + C_2^{\langle 0,1,0\rangle}$$

$$\vec{r}(t) = \langle 2t^3, t^4+1, \frac{t^2}{2} - t \rangle$$

$$\vec{r}(1) = \langle 2, 2, \frac{-1}{2} \rangle$$

8) $\vec{r}(t) = \langle 2t, 1-2t, 5+t \rangle$

$$\vec{v}(t) = \langle 2, -2, 1 \rangle$$

$$|\vec{v}(t)| = \sqrt{9} = 3$$

$$\int_0^a 3 \, dt = 3$$

$$3a = 3 \Rightarrow \boxed{a = t = 1}$$

9) $a(t) = \langle 0, 0, e^t \rangle$, $v(0) = \langle 0, 0, 0 \rangle$,

$r(0) = \langle 0, 1, 1 \rangle$

$v(t) = \langle 0, 0, e^t - 1 \rangle$

$v(t) = \langle 0, 0, e^t - 1 \rangle$

$r(t) = \langle 0, 1, e^t - t \rangle$

$r(2) = \langle 0, 1, e^2 - 2 \rangle$

10) $\vec{r}(t) = \langle 5t, 1-3t, 5+4t \rangle$

$\vec{v}(t) = \langle 5, -3, 4 \rangle \Rightarrow |\vec{v}(t)| = \sqrt{50}$

$$\int_0^t \sqrt{50}\, dt = 2$$

$$\sqrt{50}\, t = 2 \Rightarrow \boxed{t = \frac{2}{\sqrt{50}}}$$

$$r\left(t = \frac{2}{5\sqrt{2}}\right)_x = \boxed{\sqrt{2}}$$

11) $r_1(t) = \langle t+1, 2\sqrt{t}, \sqrt{2}\, t \rangle$

$r_2(t) = \langle 2t, t^2+1, t^2 - 2t + \sqrt{2} + 1 \rangle$

Find the angle b/w the tangent

vectors at point of intersection.

$$\rightarrow r_1'(t) = \langle 1, t^{-1/2}, \sqrt{2} \rangle$$

$$\rightarrow r_2'(t) = \langle 2, 2t, 2t-2 \rangle$$

$$r_1(t) = r_2(t)$$

$$\Rightarrow t+1 = 2t \rightarrow \underline{\underline{t=1}}$$

$$\Rightarrow 2\sqrt{t} = t^2+1 \rightarrow \underline{\underline{t=1}}$$

$$\Rightarrow \sqrt{2}\cancel{t} = t^2 - 2\cancel{t} + \sqrt{2} + \cancel{1} \rightarrow \underline{\underline{t=1}}$$

$$r_1'(1) = \langle 1, 1, \sqrt{2} \rangle$$

$$r_2'(1) = \langle 2, 2, 0 \rangle$$

$$\cos\theta = \frac{\cancel{4}}{\cancel{2} \cdot \cancel{2}\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$

12) $\vec{a}(t) = \langle 0, 0, -9.8 \rangle$

$\vec{v}(t) = \langle 0, 0, -9.8t \rangle + C_1$ $\quad v(0) = \langle 0, 0, 19.6 \rangle$

$= \langle 0, 0, 19.6 - 9.8t \rangle$

$19.6 = 9.8t \Rightarrow \boxed{t = 2}$

$|\vec{v}(t)| = 19.6 - 9.8t$ ← velocity z-compon $= 0$

$$d = \int_0^2 19.6 - 9.8t \quad dt$$

max height

$= 19.6(2) - 9.8(2) = 19.6m$

Arc length

Given $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$

$|\vec{r}\,'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$

$$\boxed{s(t) = \int_a^t |\vec{r}\,'(u)| \; du}$$

↳ after evaluating $s(t)$ in terms of t.

↳ find t in terms of s

↳ substitute in $\vec{r}(t)$ to get $\vec{r}(s)$

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1) $\vec{r}(t) = \langle 2t, t^2, \ln(t) \rangle$

$\vec{r}\,'(t) = \langle 2, 2t, 1/t \rangle$

$|\vec{r}\,'(t)| = \sqrt{4 + 4t^2 + 1/t^2}$ $\qquad \dfrac{4t^2 + 4t^4 + 1}{t^2}$

$$= \sqrt{\frac{(2t^2+1)^2}{t^2}} = \frac{2t^2+1}{t}$$

$$= 2t + t^{-1}$$

$$s(t) = \int_1^t 2u + u^{-1} \; du$$

$$= u^2 + \ln(u) \Big|_1^t = t^2 + \ln(t) - 1 - \cancel{\ln(1)}^{\to 0}$$

$$s(t) = t^2 + \ln(t) - 1$$

Arclength from $1 \leq t \leq 2$:

$\hookrightarrow$ $s(2) = 4 - 1 + \ln(2)$

$$= \underline{3 + \ln(2)}$$

2) $r(t) = \langle 3\sin(2t), 4, 3\cos(2t) \rangle$ $\quad 0 \leq t \leq \pi/3$

$$r'(t) = \langle 3\cos(2t)\cdot 2, 0, -3\sin(2t)\cdot 2 \rangle$$

$$r'(t) = \langle 6\cos(2t), 0, -6\sin(2t) \rangle$$

$$|\vec{r}\,'(t)| = 6$$

$$\int_0^{\pi/3} 6 \, dt = 6t \Big|_0^{\pi/3} = \underline{\underline{2\pi}}$$

3) $a(t) = \langle 0, 2t, \sqrt{2} \rangle$, $v(0) = \langle 1, 0, 0 \rangle$.

Find distance travelled for $0 \le t \le 3$.

$$v(t) = \langle 0, t^2, \sqrt{2}t \rangle + C, \quad C = \langle 1,0,0 \rangle$$

$$v(t) = \langle 1, t^2, \sqrt{2}t \rangle$$

$$|\vec{v}(t)| = \sqrt{1 + t^4 + 2t^2} = \sqrt{(t^2+1)^2}$$

$$\int_0^3 t^2+1 \; dt = \frac{t^3}{3} + t \Big|_0^3$$

$$= 9 + 3 = \underline{\underline{12}}$$

4) $r(t) = \langle 3t, 4\sin t, 4\cos t \rangle$, distance $= 20$.

$$v(t) = \langle 3, 4\cos t, -4\sin t \rangle$$

$$|v(t)| = \sqrt{25} = 5$$

$$\int_0^\alpha 5 \; dt = 20$$

$$5\alpha = 20 \Rightarrow \boxed{\alpha = 4}$$

5) $r(t) = \langle 5t,\ 1-3t,\ 5+4t \rangle$

$r'(t) = \langle 5,\ -3,\ 4 \rangle \Rightarrow |r'(t)| = \sqrt{50}$

$$s(t) = \int_0^t \sqrt{50}\, du = \sqrt{50}\, t = s$$

$$t = \frac{s}{\sqrt{50}}$$

$$r(s) = \left\langle \frac{\cancel{5}s}{8\sqrt{2}},\ 1 - \frac{3s}{5\sqrt{2}},\ 5 + \frac{4s}{5\sqrt{2}} \right\rangle$$

$$r(s=2) = \left\langle \sqrt{2},\ 1 - \frac{3\sqrt{2}}{5},\ 5 + \frac{2\sqrt{2}}{5} \right\rangle$$

Level curves :

↳ Given a surface, set the z-value to be a constant and plot the xy-cartesian plane's shape.

$$f(x,y) = \sqrt{x^2+4y^2+4} - x = k$$

$$\hookrightarrow \cancel{x^2} + 4y^2 + 4 = (k+x)^2 = k^2 + 2kx + \cancel{x^2}$$

$$\Rightarrow 4y^2 + 4 = k(k+2x)$$

$$\Rightarrow y^2 = \frac{k(k+2x)}{4} - 1$$

For $k=0$, y DNE.

For $k \neq 0$, y^2 is in terms of the x.

$\hookrightarrow$ parabolas

Thus, the level curves are parabolas.

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1) $f(x,y) = \sqrt{x^2+y^2+1} + x = k$

$$x^2 + y^2 + 1 = (k-x)^2$$

$$\cancel{x^2} + y^2 + 1 = k^2 - 2x + \cancel{x^2}$$

$$y^2 = k^2 - 2x - 1$$

↳ $k = 0 \Rightarrow y$ DNE

↳ $k \neq 0 \Rightarrow k > \sqrt{2x+1}$

↳ parabola

2) $f(x,y) = \cos(x) = k$

↳ $x = \cos^{-1}(k)$

↳ vertical lines

<u>Curvature</u>:

Given $r(t) = \langle x(t), y(t), z(t) \rangle$

↳ $T(t) = \dfrac{r'(t)}{|r'(t)|} = \dfrac{\langle x'(t), y'(t), z'(t) \rangle}{\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}}$

Unit tangent vector

$$K = \frac{|T'(t)|}{|r'(t)|} = \frac{|\vec{r}''(t) \times \vec{r}'(t)|}{|\vec{r}'|^3}$$

1) $r(t) = \langle 2\sin t, 1, 2\cos t \rangle$, Find $K(\pi/4)$

$r'(t) = \langle 2\cos t, 0, -2\sin t \rangle$

$|r'(t)| = \sqrt{4} = 2$

$T(t) = \langle \cos t, 0, -\sin(t) \rangle$

$T'(t) = \langle -\sin t, 0, -\cos t \rangle$

$$K(t) = \frac{|\langle -\sin t, 0, -\cos t \rangle|}{|\langle 2\cos t, 0, -2\sin t \rangle|}$$

$$= \frac{1}{2}$$

2) $r(t) = \langle 1, 5t^2, 4t \rangle$, Find $K(0)$

$r'(t) = \langle 0, 10t, 4 \rangle$, $r''(t) = \langle 0, 10, 0 \rangle$

$r''(t) \times r'(t) = \langle 0, 10, 0 \rangle \times \langle 0, 10t, 4 \rangle$

$= \langle 40, 0, 0 \rangle$

$|r''(t) \times r'(t)| = 40$

$$|r'(t)| = \sqrt{100t^2 + 16}$$

$$|r'(t)|^3 = (100t^2 + 16)^{3/2}$$

$$K(t) = \frac{40}{(100t^2+16)^{3/2}}, \quad K(0) = \frac{\cancel{40}\ 10}{4 \times 4 \times \cancel{4}}$$

$$= \frac{10}{16} = \frac{5}{8}$$

3) $r(t) = \langle 5\sin t,\ 3\cos t,\ 4\cos t \rangle$

$$r'(t) = \langle 5\cos t,\ -3\sin t,\ -4\sin t \rangle$$

$$|r'(t)| = 5$$

$$T(t) = \langle \cos t,\ -\tfrac{3}{5}\sin t,\ -\tfrac{4}{5}\sin t \rangle$$

$$T'(t) = \langle -\sin t,\ -\tfrac{3}{5}\cos t,\ -\tfrac{4}{5}\cos t \rangle$$

$$K(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{1}{5}$$

4) $r(t) = \langle t,\ t^2/2,\ t^3/3 \rangle$ $\qquad K(1)$

$r'(t) = \langle 1, t, t^2 \rangle$

$r''(t) = \langle 0, 1, 2t \rangle$

$$r''(t) \times r'(t) = \langle 1, t, t^2 \rangle \times \langle 0, 1, 2t \rangle$$
$$= \langle 2t^2 - t^2, -2t, 1 \rangle$$
$$= \langle t^2, -2t, 1 \rangle$$

$|r'(t)| = \sqrt{1 + t^2 + t^4}$

$$K(t) = \frac{\sqrt{t^4 + 4t^2 + 1}}{\left(\sqrt{1 + t^2 + t^4}\right)^{3/2}}, \quad K(1) = \frac{\sqrt{6}}{3\sqrt{3}}$$
$$= \frac{\sqrt{2}}{3}$$

Partial derivatives :

Say $f(x,y) = \ln(x^2 + y^2 + 2)$

treat 'y' as constant

$$f_x = \frac{\partial f}{\partial x} = \frac{1}{x^2 + y^2 + 2} \cdot \frac{d}{dx}(x^2 + y^2 + 2)$$

$$= \frac{2x}{x^2+y^2+2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{0 - 2y(2x)}{(x^2+y^2+2)^2}$$

$$f_{xy} = \frac{-4xy}{(x^2+y^2+2)^2}$$

<u>Notation:</u>

$$f_x = \frac{\partial f}{\partial x}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial f}{\partial y \, \partial x}$$

So, differentiating with respect to x, then y, is f_{xy} or $\frac{\partial f}{\partial y \, \partial x}$.

↳ The last differentiation is the leftmost.

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1) $f(x,y) = e^{x+3y-3} \sin(\pi xy)$

$f_x = e^{x+3y-3} \sin(\pi xy) + \pi y \cos(\pi xy)\, e^{x+3y-3}$

$f_x(1,1) = \cancel{e \sin(\pi)}^{\to 0} + \pi \cos(\pi)\, e$

$= -e\pi$

2) $f(x,y) = x\sin(xy^2)$, $f_{xy}(\pi,1) = ?$

$f_x = \sin(xy^2) + (xy^2)\cos(xy^2)$

$f_{xy} = \cos(xy^2)\cdot(2xy) + (2xy)\cos(xy^2) - (xy^2)\sin(xy^2)\cdot(2xy)$

$f_{xy}(\pi,1) = \cos(\pi)\cdot 2\pi + 2\pi\cdot\cos(\pi) - \cancel{(2\pi)(0)(2\pi)}^{\to 0}$

$= -4\pi$

3) $f(x,y) = \ln(xy^2 + x)$

$f_x = \dfrac{1}{xy^2 + x} \cdot (y^2 + 1) = \dfrac{\cancel{y^2+1}}{x\cancel{(y^2+1)}}$

$= \dfrac{1}{x}$

$f_{xy} = \dfrac{\partial}{\partial y}\left(\dfrac{1}{x}\right) = 0$

4) Suppose the graph of $z = g(x,y)$ intersects the plane $x=0$ along the curve $z = y^3 + 2y^2 + 1$. Find

$g_y\ (0,2).$

At $(0,2)$, $g(x,y) = y^3+2y^2+1$

$\hookrightarrow g_y = 3y^2+4y \Big|_{y=2} = \underline{\underline{20}}$

5) $f(x,y,z) = e^{xy^2} + \sin(x^3+y^3) + \ln(y^7) + \tan(9x^3).$

No z in the RHS.

Find $f_{xyz}(0,3,7)$.

$f_x = y^2e^{xy^2} + 3x^2\cos(x^3+y^3) + 27x^2\sec^2(9x^3)$

$f_{xy} = 2ye^{xy^2} + 2xy^3e^{xy^2} - 3x^2\sin(x^3+y^3)\cdot 3y^2$

$f_{xyz} = 0$

6) $f(x,y,z) = \dfrac{xz}{\sqrt{y^2-z}}$

$f_x = \dfrac{z\sqrt{y^2-z}}{y^2-z} = \dfrac{z}{\sqrt{y^2-z}}$

$f_{xy} = \dfrac{-z}{\cancel{2}\sqrt{y^2-\ldots}}\ \cancel{2}y = -yz$

$$\frac{\sqrt{y^2-z}}{y^2-z} \qquad \frac{\quad}{(y^2-z)^{3/2}}$$

$$f_{xyz} = \frac{-y\,(y^2-z)^{3/2} - yz\left(\frac{3}{2}\right)\sqrt{y^2-z}}{(y^2-z)^{9/4}}$$

$$f_{xyz}(1,2,3) = -2 - 9 = \underline{\underline{-11}}$$

Chain rule

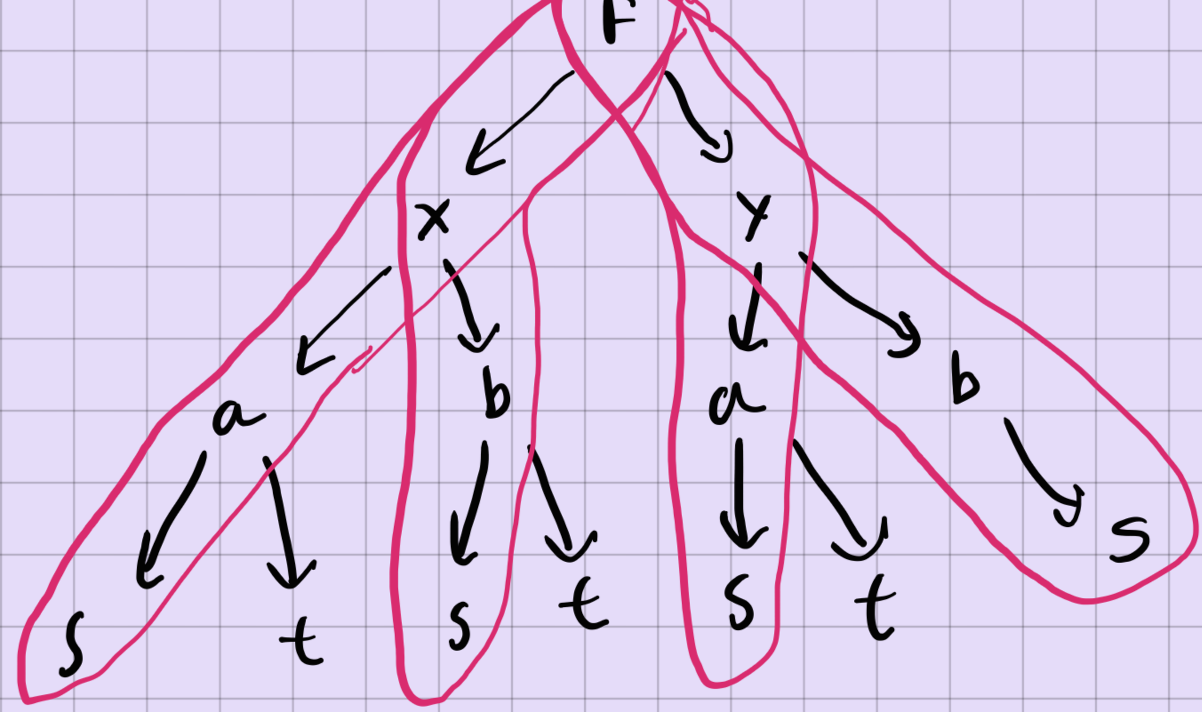
$$\frac{\partial f}{\partial s} = \frac{d\partial}{\partial x} \cdot \frac{\partial x}{\partial s}$$

↳ Say $f(x,y) = x^2 - y$, $x = s^2$, $y = 2t$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s}$$

$$= 2x \cdot 2s = 2s^2 \cdot 2s = \underline{\underline{4s^3}}$$

A good practice is drawing a tree diagram :



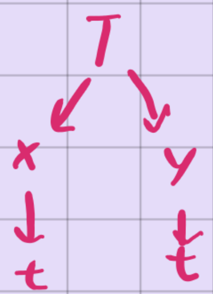
$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial a}\frac{\partial a}{\partial s} + \frac{\partial F}{\partial x}\frac{\partial x}{\partial b}\frac{\partial b}{\partial s}$$

$$+ \frac{\partial F}{\partial y}\frac{\partial y}{\partial a}\frac{\partial a}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial b}\frac{\partial b}{\partial s}$$

---

1) Temperature at a point (x,y) is $T(x,y) = x^3 y$. Find the rate of change of temperature wrt. time at $t=2$ given $r(t) = \langle t, t^2 \rangle$.

$x = t, \; y = t^2$



$$\Rightarrow \frac{\partial T}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial t}$$

$$\Rightarrow 3x^2 y \quad + x^3 (2t)$$

$\Rightarrow 3t^4 + 2t^4 = 5t^4$. At $t=2$, 5×16
$= 80$

2) Let $f(x,y,z)$ be a function which is differentiable at $(1,1,1)$, and $f_x(1,1,1) = -6$, $f_y(1,1,1) = 2$, $f_z(1,1,1) = -1$.

Let $\gamma(t) = \langle x(t), y(t), z(t) \rangle$ be the equation of a differentiable curve, and $\gamma(0) = \langle 1,1,1 \rangle$ and $\frac{d\gamma}{dt}(0) = \langle 3,-3,1 \rangle$.

Find $\frac{d}{dt} f(\gamma(t))$ at $t=0$.

$\hookrightarrow F = f(\gamma(t)) \rightarrow f(\langle 1,1,1 \rangle)$

$\hookrightarrow \frac{dF}{dt} \cdot \frac{d\gamma(t)}{dt}$ at $t=0$

$= \frac{dF}{dt} \cdot \langle 3,-3,1 \rangle$

$\langle f_x, f_y, f_z \rangle$

$= \langle -6, 2, -1 \rangle \cdot \langle 3, -3, 1 \rangle = -18 - 6 - 1$

$= -25$

3) $g(u,v) = f\left((u+2v)^3 + 1,\ e^{uv} - 1\right)$

	f	g	f_x	f_y
$(-1,0)$	8	1	4	2
$(0,0)$	1	3	5	7

Find $g_v(-1,0)$

$$f\left((u+2v)^3+1,\ e^{uv}-1\right) = f(x,y)$$

$\hookrightarrow$ $x = (u+2v)^3+1$, $y = e^{uv}-1$

$x = 0$, $y = 0$

$$\frac{\partial g}{\partial v} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dv}$$

$$= 5\cdot 6(u+2v)^2 + 7\cdot ue^{uv}$$

$$= 30 + 7(-1)$$

$$= \underline{\underline{23}}$$

4) $f(u,v) = e^v \sin(u)$, $u = s-t$, $v = t^2$

f

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial t}$$

$$\begin{array}{ccc} u & & v \\ \swarrow \ \downarrow & & \downarrow \\ s \quad t & & t \end{array}$$

$$= (\cos(u) e^{v})(-1) + (e^{v} \sin(u))(2t)$$

$$(s,t) = (1,1)$$

$$\frac{\partial f}{\partial t} = \cancel{\cos(0)}^{1} e^{1} (-1) + \cancel{e^{1} \sin(0)(2)} \to 0$$

$$= \boxed{-e}$$

5) $z = f(x,y)$, $f_x = xy^2$, $f_y = x^2 y + y^2$.

$x = s + 2t$, $y = t^2$. Find $z_t(\underset{s}{2}, \underset{t}{1})$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$= xy^2 \cdot (2) + (x^2 y + y^2)(2t)$$

$$f_t(2,1) = 4 \cdot 1 \cdot 2 + (16 \cdot 1 + 1)(2)$$

$$= 8 + 34 = \underline{42}$$

6) $z = e^{r} \cos\theta$, $r = 12st$, $\theta = \sqrt{s^2 + t^2}$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{dr}{ds} + \frac{\partial z}{\partial \theta}\cdot\frac{d\theta}{ds}$$

$$= e^r \cos\theta\,(12t) - e^r \sin\theta \left(\frac{\cancel{2}s}{\cancel{2}\sqrt{s^2+t^2}}\right)$$

$$= e^{12st}\cos\left(\sqrt{s^2+t^2}\right)(12t) - \frac{e^{12st}\sin\left(\sqrt{s^2+t^2}\right)s}{\sqrt{s^2+t^2}}$$

$$= e^{12st}\left(12t\cos\left(\sqrt{s^2+t^2}\right) - \frac{s\sin\left(\sqrt{s^2+t^2}\right)}{\sqrt{s^2+t^2}}\right)$$

Implicit differentiation :

Say $\cos(xyz) = x + 3y + 2z$. Find $\frac{\partial z}{\partial y}$ if z is a function of x and y.

$$\Rightarrow f(x,y,z) = x + 3y + 2z - \cos(xyz)$$

$$\to \partial F + \partial F \cdot \partial z = 0$$

$$\partial y \quad \partial z \quad \partial y$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

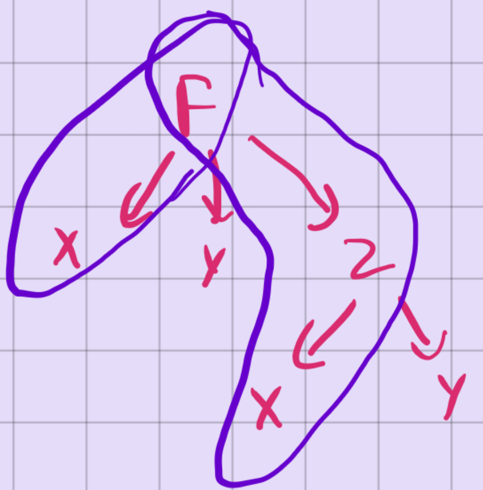
$$= \frac{-(3 + xz\sin(xyz))}{2 + xy\sin(xyz)}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(0,1)} = \frac{-3-0}{2+0} = -\frac{3}{2}$$

---

1) $\tan(xyz) = x^2 - z^2$

↳ $F(x,y,z) = x^2 - z^2 - \tan(xy\overset{\pi}{z})$



Note: $F(x,y,z) = z$ as well.

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$= \frac{-(2x - yz\sec^2(xy\overset{\pi}{z})\pi)}{-2z - \sec^2(xy\overset{\pi}{z})xy\pi}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,2,1)} = \frac{-2 + 2\cancel{\sec^2(2\pi)}^{1}}{-2 - 2\cancel{\sec^2(2\pi)}^{1}}$$

$$= \frac{-2+2\pi}{-2-2\pi} = \frac{1-\pi}{1+\pi}$$

2) $(x^2+y^2)^3 = 8x^2y^2$. Find $\left.\frac{dy}{dx}\right|_{(-1,1)}$

$\rightarrow F(x,y) = 8x^2y^2 - (x^2+y^2)^3 = 0$

$$8(2xy^2 + 2yy'x^2) - 3(x^2+y^2)^2(2x+2yy') = 0$$

At $(-1,1)$:

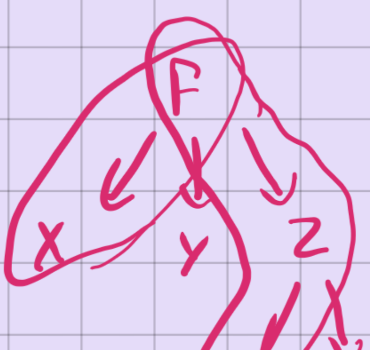
$$8(2(-1)(1) + 2(1)(1)y') - 3(1+1)^2(-2+2y') = 0$$

$\Rightarrow 8(-2+2y') - 12(-2+2y') = 0$

$\Rightarrow -4(-2+2y') = 0$

$\Rightarrow \boxed{y' = 1}$

3) $x + y + z + \sin(xyz) = 0$



$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

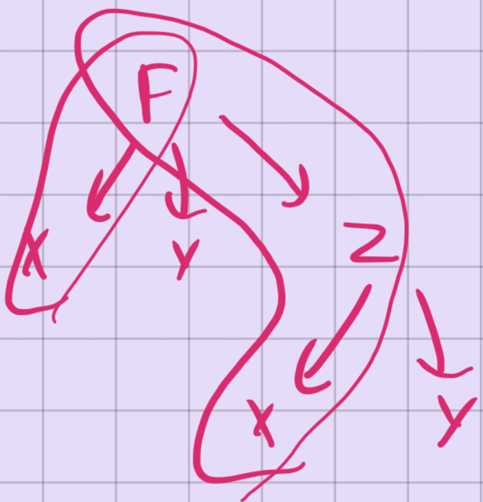
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$= \frac{-(1 + yz\cos(xyz))}{1 + xy\cos(xyz)}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,1,\pi/2)} = \frac{-(1 + \frac{\pi}{2}(0))}{1 + 0}$$

$$= \underline{\underline{-1}}$$

4) $xyz - y\sin(xy) - z^2 = 0$. Find $\left.\frac{\partial z}{\partial x}\right|_{(1,\pi,\pi)}$



$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$$

$$= \frac{-(yz - y^2\cos(xy))}{xy - 2z}$$

At $(1,\pi,\pi)$:

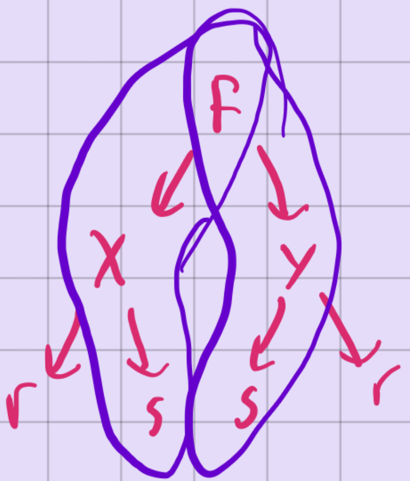
$$\hookrightarrow \frac{-(\pi^2 + \pi^2 \overset{-1}{\cancel{\cos(\pi)}})}{\pi - 2\pi} = \frac{\cancel{-}2\pi^{\cancel{2}}}{\cancel{-}\cancel{\pi}}$$

$= 2\pi$

5) $z = f(x,y)$, $x = s^2 - r$, $y = r^2 s$.

$f_x = z_x = e^x + y^4$, $f_y = z_y = 4xy^3$.

Find $\dfrac{\partial z}{\partial s}$ at $r=-1$, $s=1$.



$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= (e^x + y^4)(2s) + (4xy^3)(r^2)$$

$$= (e^2 + 1)(2) + (4(2)(1)^3)(1)$$

$$= 2e^2 + 2 + 8 = 2e^2 + 10$$

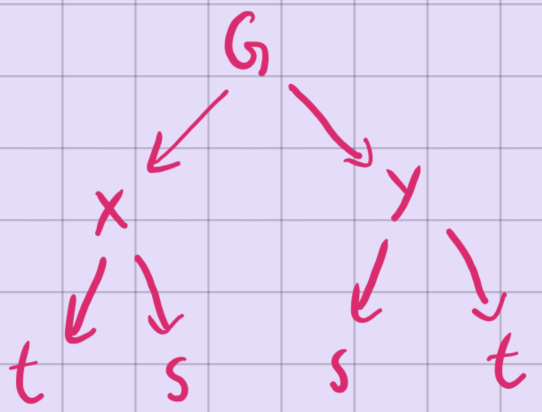
6) Let $g(x,y) = 7x^2y^3$, $x = \cos(t) + \sin(s)$,

$y = \sin(t) + \cos(s)$.

Find a point in space (s,t) such that the directional derivative of g in the direction of a general vector $\vec{v}$ is 0.

$$D_{\vec{v}}\, g(x,y) = \nabla g \cdot \hat{v} = 0$$

$\hookrightarrow \nabla g = 0$ or $\hat{v} = 0$ or $\cos\theta = 0$



$$\nabla g = \langle G_t , G_s \rangle$$

$$G_t = \frac{\partial G}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial G}{\partial y}\frac{\partial y}{\partial t}$$

$$= 14xy^3(-\sin(t))$$

$$+ 21y^2x^2(\cos(t))$$

$$G_s = \frac{\partial G}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial G}{\partial y}\frac{\partial y}{\partial s}$$

$$= 14xy^3(\cos(s)) + 21x^2y^2(-\sin(s))$$

$$\nabla g =$$

$$\langle 21x^2y^2\cos(t) - 14xy^3\sin(t),\ 14xy^3\cos(s) - 21x^2y^2\sin(s) \rangle$$

$\Rightarrow$ $x = \cos(t) + \sin(s)$

$y = \sin(t) + \cos(s)$

At $x = 0$ or $y = 0$, $\nabla g = \langle 0, 0 \rangle$

$y = 0 \Rightarrow \quad \sin(t) + \cos(s) = 0$

$t = 0, \quad s = \pi/2$

$\underline{(\pi/2, 0)}$

Limits :

↳ Solving limits has 3 possible methods :

1) Direct substitution.

2) Simplification, then substitution.

3) Trying different paths until two paths give different results, implying limit does not exist – DNE.

→ $$\lim_{(x,y)\to(0,0)} \frac{\sin(4x^2 + 8y^2)}{x^2 + 2y^2}$$

↳ $4x^2 + 8y^2 = t$, $x \to 0$ and $y \to 0$ – $t \to 0$

↳ $x^2 + 2y^2 = t/4$

$$\Rightarrow \lim_{t \to 0} \frac{\sin(t)}{t/4}$$

$$\Rightarrow \lim_{t \to 0} 4 \frac{\sin t}{t} = 4 \cdot 1 = 4$$

Note that for a function to be continuous, $f(a,b) = \lim_{x,y \to a,b} f(x,y)$. If the function is

continuous, it may or may not be differentiable, but if a function is differentiable, it is continuous.

$$f(x,y) = \frac{xy}{x^2+y^2} \quad (x,y) \neq (0,0)$$

and

$$0 \quad (x,y) = (0,0)$$

Is the function continuous at $(0,0)$?

$$\rightarrow f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y) = (0,0)$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

↳ Solving along different paths:

⇒ $y = x$

↳ $$\lim_{x\to 0} \frac{x(x)}{x^2+(x)^2} = \frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2}$$

⇒ $y = 0$

↳ $$\lim_{x\to 0} \frac{0}{x^2} = \frac{0}{0}$$

Thus limit DNE.

↳ function is not continuous at $(0,0)$.

---

1) $$f(x,y) = \frac{x^2-y^2}{x-y} = x+y \quad \text{if} \quad x \neq y$$

$$= cx \quad \text{if} \quad x = y$$

What value of c makes $f(x,y)$ continuous?

↳ Going from the left of $x=y$

Say $x = y = 3$

Going from the left of $x=y$ to the value $x=y$, what value for c gives the same values?

$$\lim_{(x,y)\to(3,3)} x+y = \underline{\underline{6}} \quad (= 2x)$$

$\hookrightarrow$ $f(x,y)$ when $x \neq y$ $=$ $f(x,y)$ when $x=y$.

$$\lim_{(x,y)\to(3,3)} cx = \underline{\underline{3c}}$$

$\hookrightarrow$ $3c = 6 \Rightarrow \underline{\underline{c=2}}$

2)

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 3a(x^2+y^2) - y^4}{(x^2+y^2)} = 12. \text{ Find } \underline{\underline{a}}$$

$$\hookrightarrow \frac{x^4}{x^2+y^2} - \frac{3a\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}} - \frac{y^4}{(x^2+y^2)}$$

$$\hookrightarrow \frac{(x^2-y^2)\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}} - 3a$$

$$\hookrightarrow \lim_{(x,y)\to(0,0)} \cancel{x^2} - \cancel{y^2} - 3a = 12$$

$$\underline{a = -4}$$

3)

$$\lim_{(x,y)\to(0,0)} \frac{3x-2y}{\sqrt{x^2+y^2}} \qquad (\mathrm{I})$$

a) $y = x$

$$\lim_{x\to 0} \frac{3x-2x}{\sqrt{x^2+x^2}} = \frac{\cancel{x}}{\sqrt{2}\,\cancel{x}} = \frac{\sqrt{2}}{2}$$

b) $y = 0$

$$\lim_{x\to 0} \frac{3\cancel{x}}{\sqrt{\cancel{x^2}}} = 3$$

Thus

(I) DNE.

$$\lim_{(x,y)\to(0,0)} \frac{e^{x+y}}{1+e^{x-y}} \qquad (\mathrm{II})$$

$$= \frac{e^0}{1+e^0} = \frac{1}{2}$$

4)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+xy+y^2}$$ (I)

a) $x=y$:

$$\lim_{y\to 0} \frac{y^2}{y^2+y^2+y^2} = \frac{\cancel{y^2}}{3\cancel{y^2}} = \boxed{\frac{1}{3}}$$

b) $x=0$:

$$\lim_{y\to 0} \frac{0}{0+0+y^2} = \boxed{\frac{0}{0}}$$

DNE

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{e^x+e^y} = \frac{0}{2} = \boxed{0}$$

5) $$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+3y^2)}{7x^2+21y^2}$$

$$u = x^2+3y^2, \quad 7u = 7x^2+21y^2$$

$$(x,y)\to(0,0) \Rightarrow u\to 0$$

$$\lim_{u\to 0} \frac{1-\cos(u)}{7u} = \left[\frac{0}{0}\right]$$ Use L'H

$$\lim_{u \to 0} \frac{\sin(u)}{7} = 0$$

Principal unit vector :

$$r(t) = \langle \cos t + \sin t,\ \cos t - \sin t,\ t \rangle$$

$$r'(t) = \langle -\sin t + \cos t,\ -\sin t - \cos t,\ 1 \rangle$$

$$|r'(t)|^2 = \cos^2 t - \cancel{2\cos t \sin t} + \sin^2 t + \sin^2 t + \cos^2 t + \cancel{2\cos t \sin t} + 1 = 3$$

$$|r'(t)| = \sqrt{3}$$

$$T(t) = \frac{1}{\sqrt{3}} \langle \cos(t) - \sin(t),\ -\sin(t) - \cos(t),\ 1 \rangle$$

$$T'(t) = \frac{1}{\sqrt{3}} \langle -\sin(t) - \cos(t),\ -\cos(t) + \sin(t),\ 0 \rangle$$

$$|T'(t)| = \sqrt{\frac{1}{3}\left(\sin^2(t) + \cancel{2\sin(t)\cos(t)} + \cos^2(t) + \sin^2 t + \cos^2 t - \cancel{2\sin t \cos t}\right)}$$

$$= \sqrt{\frac{2}{3}}$$

$$N(t) = \frac{1}{\cancel{\sqrt{3}}} \times \frac{\cancel{\sqrt{3}}}{\sqrt{2}} \left\langle -\sin(t) - \cos(t), -\cos(t) + \sin(t), 0 \right\rangle$$

$$= -\frac{1}{\sqrt{2}} (\sin(t) + \cos(t))\, \hat{i} + \frac{1}{\sqrt{2}} (\sin(t) - \cos(t))\, \hat{j}$$

$$\boxed{N(t) = \frac{T'(t)}{|T'(t)|}, \quad T(t) = \frac{r'(t)}{|r'(t)|}}$$

<u>Normal vector</u> to a <u>surface</u> :

$$r(a,b) = \langle x(a,b), y(a,b), z(a,b) \rangle$$

$$\hookrightarrow r_a = \frac{\partial}{\partial a}\left[r(a,b) \right]$$

$$r_b = \frac{\partial}{\partial b}\left[r(a,b) \right]$$

$$\boxed{\text{Normal vector} = r_a \times r_b}$$

---

1) $f(x,y) = (7 - x^2 - y^2)^{1/2}$. Find max rate of change at (-2, 1)

$$f_x = \frac{-\cancel{2}x}{\cancel{2}\sqrt{7-x^2-y^2}} = \frac{-x}{\sqrt{7-x^2-y^2}}$$

$$f_y = \frac{-y}{\sqrt{7-x^2-y^2}}$$

$$\nabla f(x,y) = \left\langle \frac{-x}{\sqrt{7-x^2-y^2}} , \frac{-y}{\sqrt{7-x^2-y^2}} \right\rangle$$

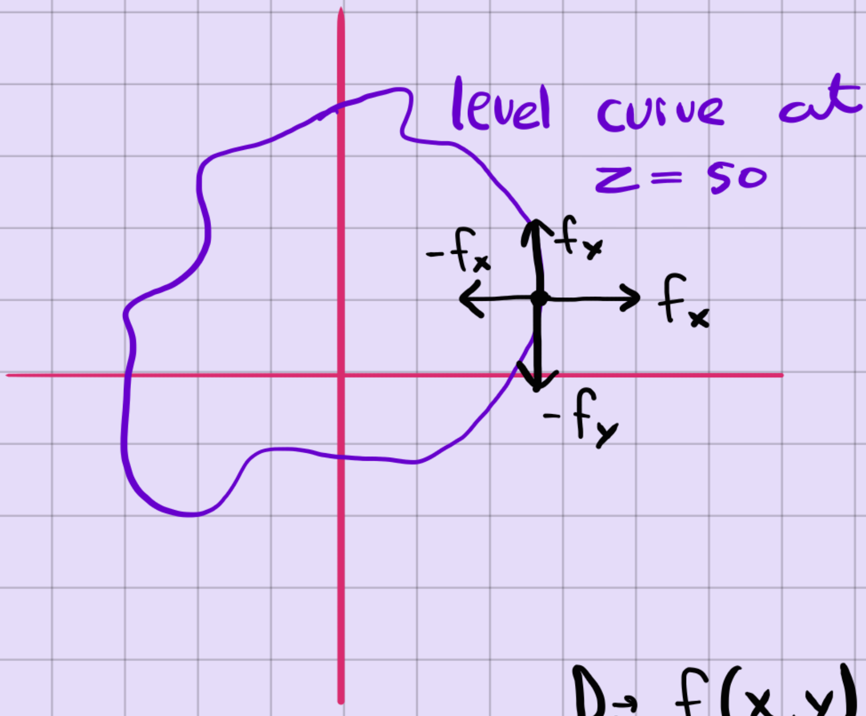
Max = magnitude $\nabla f(x,y)$

$$\nabla f(-2,1) = \left\langle \frac{2}{\sqrt{2}} , \frac{-1}{\sqrt{2}} \right\rangle$$

$$\sqrt{\frac{4}{2} + \frac{1}{2}} = \frac{\sqrt{5}}{\sqrt{2}}$$

<u>Directional</u> <u>derivatives</u> :

↳ How does the f value change in any direction ?



What about other directions?

↳ say $\vec{u} = \langle a, b \rangle$

$$D_{\vec{u}} f(x,y) = f_x a + f_y b$$

"gradient" $= \nabla f(x,y) \cdot \langle a, b \rangle$

---

1) $f(x,y) = xe^{y^2} + e^{x+y}$, at $(0,0)$ in direction of $\vec{u} = \langle 3, -4 \rangle$

$$\nabla f(x,y) = \langle e^{y^2} + e^{x+y}, 2xye^{y^2} + e^{x+y} \rangle$$

$$\nabla f(0,0) = \langle 1 + 1, 2(0) + 1 \rangle = \langle 2, 1 \rangle$$

$$\hat{u} = \langle 3/5, -4/5 \rangle$$

$$D_{\hat{u}} f(0,0) = \frac{6}{5} - \frac{4}{5} = \frac{2}{5}$$

2) $f(x,y,z) = x^2y + y^2z$ at $(1,2,3)$ towards

$\langle 3,1,5 \rangle$.

$$\nabla f(x,y,z) = \langle 2xy,\ x^2+2yz,\ y^2 \rangle$$

$$\nabla f(1,2,3) = \langle 4,\ 1+12,\ 4 \rangle$$

$$= \langle 4, 13, 4 \rangle$$

$$\vec{u} = \langle 2,-1,2 \rangle \rightarrow \hat{u} = \left\langle \frac{2}{3}, \frac{-1}{3}, \frac{2}{3} \right\rangle$$

$$D_{\hat{u}} f(3,1,5) = \frac{8}{3} - \frac{13}{3} + \frac{8}{3} = \frac{3}{3} = \underline{\underline{1}}$$

3) $D(x,y) = 3x^2y + 5y^2$

$$D_x = 6y \quad , \quad D_y = 3x^2 + 10y$$

$$\nabla D(x,y) = \langle 6yx,\ 3x^2 + 10y \rangle$$

$2 \qquad 1$

$$= \langle 12,\ 12+10 \rangle = \langle 12, 22 \rangle$$

$$\vec{u} = \langle 3, 1 \rangle \Rightarrow \hat{u} = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle$$

$$D_{\hat{u}} D(x,y) = \frac{36}{\sqrt{10}} + \frac{22}{\sqrt{10}} = \frac{58}{\sqrt{10}}$$

4) $f(x,y) = xy + x^3$

$$\nabla f(x,y) = \langle y+3x^2, x \rangle \Big|_{(1,2)}$$

$$= \langle 5, 1 \rangle$$

$$D_{\hat{u}} f(x,y) = \langle 5, 1 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$$

$$= \frac{5}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 2\sqrt{2}$$

5) $f(x,y) = \cos(2x+3y)$. Find the rate of change of $f(x,y)$ at $(2,1)$ towards the origin.

$$\nabla f(x,y) = \langle -2\sin(2x+3y), -3\sin(2x+3y) \rangle$$

$$\vec{u} = \langle -2, -1 \rangle \Rightarrow \hat{u} = \left\langle -\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle$$

$$\nabla f(2,1) = \langle -2\sin(7), -3\sin(7) \rangle$$

$$D_{\hat{u}} f(2,1) = \frac{4}{\sqrt{5}} \sin(7) + \frac{3\sin(7)}{\sqrt{5}}$$

$$= \frac{7\sin(7)}{\sqrt{5}}$$

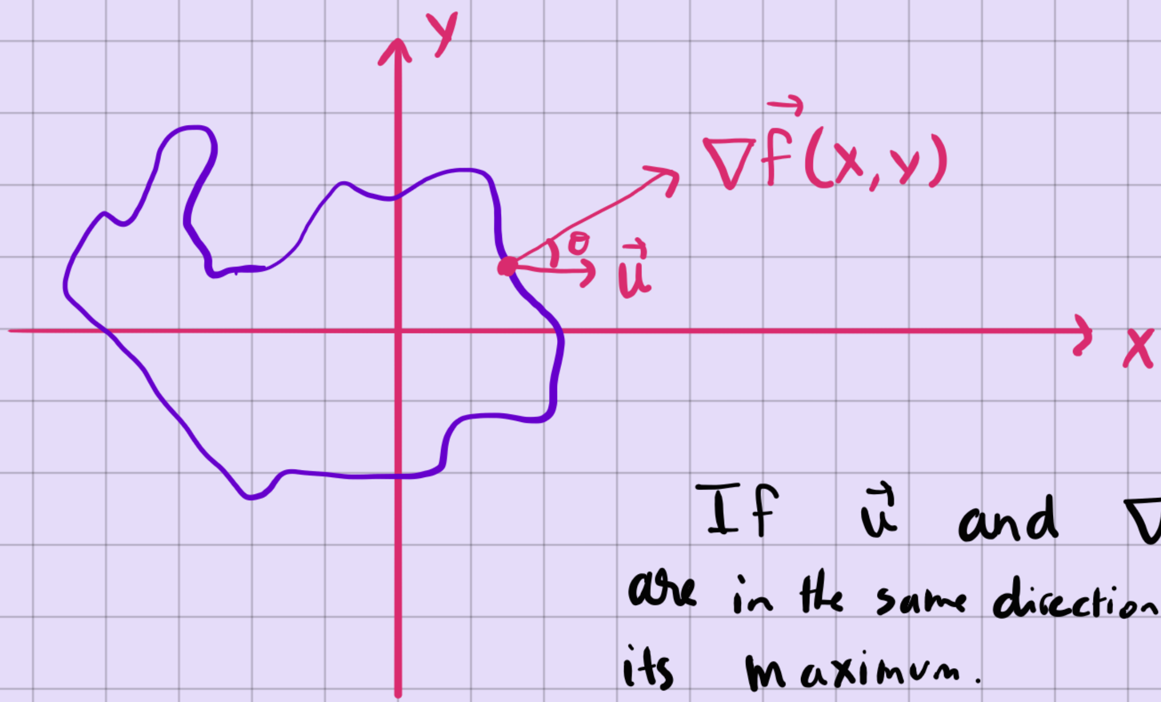
The gradient function :

$$\nabla f(x,y) = \langle f_x(x,y),\ f_y(x,y) \rangle$$

$$D_{\hat{u}} f = \vec{u} \cdot \nabla f$$

$$\hookrightarrow D_{\hat{u}} f = \underbrace{|u|}_{\text{unit vector}} |\nabla f| \cos\theta \qquad \hookrightarrow -1 \leq \cos\theta \leq 1$$

$$\Rightarrow -|\nabla f| \leq D_{\hat{u}} f \leq |\nabla f|$$



If $\vec{u}$ and $\nabla \vec{f}(x,y)$ are in the same direction, $D_{\hat{u}} f$ is at its maximum.

$\hookrightarrow \cos(0^\circ) = 1$

If $\vec{u}$ and $\nabla \vec{F}(x,y)$ are in opposite direction

$\theta = 180° \Rightarrow D_{\hat{u}} f(x, y)$ is its lowest.

↳ $\cos(180°) = -1$

If $\theta = 0°$, $\vec{u}$ is in the direction of greatest ascent. This means you are going uphill at the greatest rate.

If $\theta = 180°$, $\vec{u}$ is in the direction of greatest descent. This means you are going downhill at the greatest rate.

If $\theta = 90°$, $\vec{u}$ is in the direction such that the motion is along the same level curves, moving nor uphill nor downhill.

↳ $\vec{u} \parallel$ tangent of level curve

↓

The slope of a level curve at (x, y) = $\dfrac{-f_x}{f_y}$

---

1) $f(x,y) = x^2 y + e^{xy}\sin(y) + 15$. Find the direction of steepest increase at $(1, 0)$.

$$\nabla f(x,y) = \langle 2xy + y\sin(y)e^{xy},\ x^2 + \cos(y)e^{xy} + xe^{xy}\sin(y) \rangle$$

$\nabla f(1,0) = \langle 0+0,\ 1+1(1)+0 \rangle$

$= \langle 0, 2 \rangle$

Greatest ascent implies $\hat{u}$ is in the same direction as $\nabla f(1,0) = \langle 0,2 \rangle$.

$\vec{u} = \langle 0,2 \rangle \Rightarrow \hat{u} = \langle 0,1 \rangle$

2) $f(x,y) = x^2 e^{x+y}$. Find vector in the direction of most rapid descent at $(1,1)$.

$\nabla f(x,y) = \langle 2x e^{x+y} + x^2 e^{x+y},\ e^{x+y} \rangle$

$\nabla f(1,1) = \langle 2e^2 + e^2,\ e^2 \rangle$

$= \langle 3e^2, e^2 \rangle$

Most rapid descent means $\vec{u} = -\nabla \vec{f}(1,1)$

$\vec{u} = \langle -3e^2, -e^2 \rangle$, $|\vec{u}| = \sqrt{10}\, e^2$

$\hat{u} = \left\langle \frac{-3}{\sqrt{10}}, \frac{-1}{\sqrt{10}} \right\rangle$

3) Let $z(x,y) = x^2 + 2y^2$. $\vec{u}$ is the unit vector in the direction of greatest descent at $(1,2)$. Find $D_{\vec{u}} f(2,1)$ for $f(x,y) = 3xy^2$.

$$\nabla z(x,y) = \langle 2x, 4y \rangle \Big|_{(1,2)} = \langle 2, 8 \rangle$$

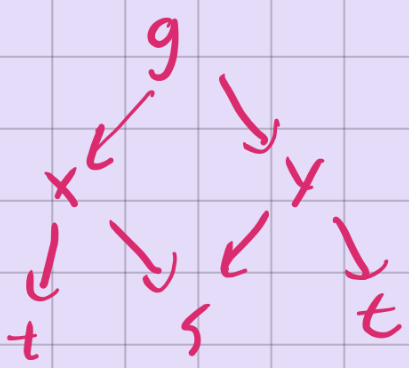
$$\vec{u} = -\langle 2, 8 \rangle \Rightarrow |\vec{u}| = \sqrt{68} \Rightarrow \hat{u} = \left\langle \frac{-2}{\sqrt{68}}, \frac{-8}{\sqrt{68}} \right\rangle$$

$$\nabla f(x,y) = \langle 3y^2, 6xy \rangle \Big|_{(2,1)} = \langle 3, 12 \rangle$$

$$D_{\hat{u}} f(2,1) = \langle 3, 12 \rangle \cdot \left\langle \frac{-2}{\sqrt{68}}, \frac{-8}{\sqrt{68}} \right\rangle$$

$$= \frac{-6}{\sqrt{68}} - \frac{96}{\sqrt{68}} = \frac{-102}{\sqrt{68}}$$

4) $g(x,y) = 7x^2y^3$, $x = \cos(t) + \sin(s)$,
$y = \sin(t) + \cos(s)$. Find
$\nabla g(s,t)$.



$$\frac{dg}{dt} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= 14xy^3(-\sin(t)) + 21x^2y^2\cos(t)$$

$$\frac{dg}{ds} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial g}{\partial y} \cdot \frac{\partial x}{\partial s}$$

Substitute x, y to g_t and g_s.

$$= 14xy^3(\cos(s)) + 21x^2y^2(-\sin(s))$$

$\hookrightarrow \nabla g(s,t) = \langle g_s, g_t \rangle$

5) $f(x,y) = x^2 e^{x+y}$. Find $\hat{u}$ for most rapid descent at $(1,1)$.

$$\nabla f(x,y) = \langle 2x e^{x+y} + e^{x+y} x^2 ,\ x^2 e^{x+y} \rangle \Big|_{(1,1)}$$

$$= \langle 2e^2 + e^2 ,\ e^2 \rangle$$

$$= \langle 3e^2 ,\ e^2 \rangle$$

$\vec{u} = \langle -3e^2, -e^2 \rangle$, $|\vec{u}| = \sqrt{10}\, e^2$

$$\hat{u} = \left\langle \frac{-3}{\sqrt{10}}, \frac{-1}{\sqrt{10}} \right\rangle$$

6) $r(u,v) = \langle \sin(u), \cos(u)\sin(v), \sin(v) \rangle$. Find a normal vector at $(\pi/3, \pi/3)$.

$r_u = \langle \cos u ,\ -\sin u \sin v ,\ 0 \rangle$

$r_v = \langle 0 ,\ \cos u \cos v ,\ \cos v \rangle$

$$r_u \Big|_{(\frac{\pi}{3}, \frac{\pi}{3})} = \left\langle \frac{1}{2}, -\frac{3}{4}, 0 \right\rangle \ - \ \vec{r}_1$$

$$r_v \Big|_{(\frac{\pi}{3}, \frac{\pi}{3})} = \langle 0, \tfrac{1}{4}, \tfrac{1}{2} \rangle \qquad - \vec{r_2}$$

$$\vec{r_1} \times \vec{r_2} = \langle -\tfrac{3}{8}, -\tfrac{1}{4}, \tfrac{1}{8} \rangle (8)$$

$$= \langle -3, -2, 1 \rangle$$

7) $$\frac{x+7}{4} = \frac{y+2}{b} = \frac{z-1}{2} = \lambda.$$

Find b such that the line is parallel to the plane $3x - 2y + 2z = 14$.

$\vec{n} = \langle 3, -2, 2 \rangle \qquad \vec{d} = \langle 4, b, 2 \rangle$

$$\vec{n} \cdot \vec{d} = 0 \Rightarrow 12 - 2b + 4 = 0$$
$$\Rightarrow b = 8$$

Tangent plane

Given $z = f(x, y)$, we can use the tangent plane to approximate the function value.

↳ A tangent plane contains all possible tangent lines to the surface at the given point.

↳ The gradient function is orthogonal to all the tangent vectors.

↳ $\nabla f(x,y,z) \cdot \langle x_t, y_t, z_t \rangle = 0$

$F_x(x-x_0) + F_y(y-y_0) + F_z(z-z_0) = 0$

↳ $\langle F_x, F_y, F_z \rangle = \nabla F(x,y,z)$ is the normal vector.

---

1) $z = \ln(x-4y)$. Consider the tangent plane at $(9,2,0)$. What is the value of λ if $(2,1,\lambda)$ exists on the same tangent plane?

$F(x,y,z) = \ln(x-4y) - z$

$\nabla F(x,y,z) = \left\langle \frac{1}{x-4y}, \frac{-4}{x-4y}, -1 \right\rangle$

$\nabla F(9,2,0) = \left\langle \frac{1}{1}, \frac{-4}{1}, -1 \right\rangle$

$= \langle 1, -4, -1 \rangle$

$\Rightarrow 1(x-9) + (-4)(y-2) + (-1)(z-0) = 0$

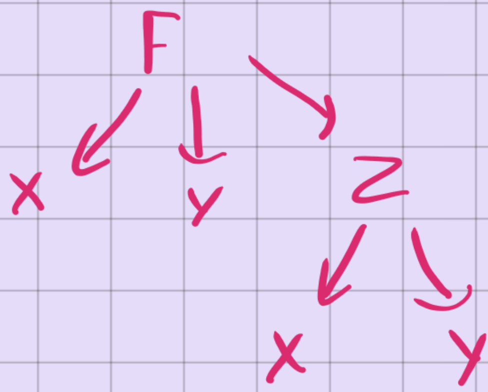
$\Rightarrow \quad x - 9 - 4y + 8 - z = 0$

$\Rightarrow \quad x - 4y - z = 1$

$(2, 1, \lambda) \Rightarrow 2 - 4 - \lambda = 1$

$\lambda = -3$

2) $F(x,y,z) = xyz - 2x\ln y - z^2 - 2 = 0$. z is implicitly defined such that $z = z(x,y)$ at $(3,1,1)$. Find tangent plane at $(3,1)$.



$$\frac{\partial F}{\partial x} = yz - 2\ln y$$

$$\frac{\partial F}{\partial y} = xz - \frac{2x}{y}$$

$$\frac{\partial F}{\partial z} = xy - 2z$$

$$\nabla F(3,1,1) = \langle 1 - 2(0),\ 3 - 6,\ 3 - 2 \rangle$$

$$= \langle 1, -3, -1 \rangle$$

$$(x-3) + (-3)(y-1) + (-1)(z-1) = 0$$

$$x - \cancel{3} - 3y + \cancel{3} - z + 1 = 0$$

$x - 3y - z = -1$

3) $x^2 - y^2 + z^2 + 2 = 0$. Find tangent plane at $(1,2,1)$

$\nabla F(x,y,z) = \langle 2x, -2y, 2z \rangle$

$\nabla f(1,2,1) = \langle 2, -4, 2 \rangle$

$2(x-1) - 4(y-2) + 2(z-1) = 0$

$2x - 4y + 2z = -4$

$x - 2y + z = -2$

4) $r(u,v) = \langle u, v, uv^2 \rangle$. Find the tangent plane at $(1,2,4)$.

$(u, v, r(u,v))$

$\hookrightarrow u = 1, \; v = 2$

$r_u = \langle 1, 0, v^2 \rangle$

$r_v = \langle 0, 1, 2uv \rangle$

$\vec{n} = r_u \times r_v$

$= \langle 1, 0, v^2 \rangle \times \langle 0, 1, 2uv \rangle$

$= \langle 1, 0, 4 \rangle \times \langle 0, 1, 4 \rangle$

$= \langle -4, -4, 1 \rangle$

$-4(x-1) - 4(y-2) + z - 4 = 0$

$-4x + \cancel{4} - 4y + 8 + z - \cancel{4} = 0$

$-4x - 4y + z = -8$

$\hookrightarrow \; 4x + 4y - z = 8$

5) $x^2 + y^2 - 4y + z - 3 = 0$

$F_x = 2x, \quad F_y = 2y - 4, \quad F_z = 1$

$\nabla F(x,y,z) = \langle 2x, 2y-4, 1 \rangle$

For the tangent plane to be horizontal, the $\vec{n}$ must be vertical.

$\hookrightarrow \; \hat{n} = \langle 0, 0, 1 \rangle$

$\underline{\underline{x = 0, \; y = 2}}$

6) $r(t) = \langle 2t-1, t^2, t^2-2 \rangle$. L is the tangent line to $r(t)$ at $(3,4,2)$. Find when the line L intersects the xy-plane.

$$r(t) = \langle 2t-1, t^2, t^2-2 \rangle$$

at $(3,4,2) \Rightarrow 2t-1 = 3 \Rightarrow \underline{\underline{t=2}}$

$$r'(t) = \langle 2, 2t, 2t \rangle$$

$$r'(2) = \langle 2, 4, 4 \rangle$$

$$L(t) = \langle 2, 4, 4 \rangle t + \langle 3, 4, 2 \rangle, \quad z = 0$$

$$\Rightarrow 4t + 2 = 0 \Rightarrow \underline{\underline{t = -\tfrac{1}{2}}}$$

$$L(-\tfrac{1}{2}) = \underline{\underline{\langle 2, 2, 0 \rangle}}$$

7) $z = \dfrac{9}{u+v^2}$. Find tangent plane at $u=2$, $v=1$.

$$\frac{9}{u+v^2} - z = 0 = F(u, v, z)$$

$$F_u = \frac{-9(1)}{(u+v^2)^2}, \quad F_v = \frac{-9(2v)}{(u+v^2)^2}, \quad F_z = -1$$

$$\nabla F(2,1,3) = \left\langle \frac{-9}{9}, \frac{-18}{9}, -1 \right\rangle$$

$$-(u-2) - 2(v-1) - (z-3) = 0$$

$$-u + 2 - 2v + 2 - z + 3 = 0$$

$$u + 2v + z = -7$$

Using the tangent plane for linear approximations.

↳ Express the tangent plane in form of

$$z = f_x(x - x_0) + f_y(y - y_0) + z_0$$

Say :

$$z = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4).$$

At $x = 3.01$, $y = 3.99$.

$$z = 5 + \frac{3}{5}(3.01 - 3) + \frac{4}{5}(3.99 - 4)$$

$$= 5 + \frac{3}{5}(0.01) - \frac{4}{5}(0.01)$$

1) $f(x,y) = g(x)\,h(y)$

$$\left.\begin{aligned} g_\Delta(x) &= g(x) + g'(x)\,\Delta g \\ h_\Delta(y) &= h(y) + h'(y)\,\Delta h \end{aligned}\right] \text{linear approx.}$$

$g_\Delta(x) = 2 + 3(0.1)$, $h_\Delta(y) = 5 + (-1)(0.2)$

$= 2.3$ $\qquad$ $= 4.8$

$f(x,y) = 2.3 \times 4.8 = \underline{\underline{11.1}}$

2) $\dfrac{\Delta x}{10} + \dfrac{\Delta y}{4} = 0.01 + 0.025$

$= 0.035$

$\dfrac{\Delta A}{40} = 0.035 \quad \Rightarrow \Delta A = \dfrac{\cancel{40} \times \cancel{35}^{\,7}}{\cancel{1000}_{\ \cancel{25}\ 5}}$

$\dfrac{7}{5} = 1.4$

3) $z = f(x,y) = x^2 y$

$x : 1 \to 1.01 \Rightarrow \Delta x = 0.01$

$y : 3 \to 2.91 \Rightarrow \Delta y = -0.09$

Find approx. change in z.

$x_\Delta = 2xy \cdot \Delta x \Rightarrow 2(1)(3) \times 0.01$

$y_\Delta = x^2 \cdot \Delta y \Rightarrow (1)^2 \times (-0.09)$

$\Delta z = x_\Delta + y_\Delta = -0.03$

For such differentials,

$$dz = f_x\, dx + f_y\, dy$$

at starting point

Min and Max :

1) Find the gradient function, $\nabla f(x,y)$.

2) Find critical points

3) Test.

↳ Boundary conditions

↳ 2nd derivative test

Discriminant

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

↳ $D > 0$ and $f_{xx} > 0$

↳ <u>relative</u> minimum

↳ $D > 0$ and $f_{xx} < 0$

↳ <u>relative</u> maximum

↳ $D < 0$

↳ saddle point

↳ not a min, nor a max

↳ $D = 0$

↳ inconclusive

---

1) $f(x,y) = x^2 + y^2$

$f_x = 2x$, $f_y = 2y$ ⇒ critical point $= (0,0)$

$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 0$

$D = 4, \quad f_{xx} = 2$

↳ relative min

2) $f(x,y) = x^3 - 48xy + 64y^3$

$f_x = 3x^2 - 48y = 0$

$f_y = -48x + 192y^2 = 0$

$\Rightarrow 3x^2 - 48y = 0 \rightarrow y = 3x^2/48$

$\Rightarrow 192y^2 - 48x = 0$

↳ $192\left(\dfrac{3x^2}{48}\right)^2 - 48x = 0$

$\cancel{4}\ \cancel{96} \qquad 3$

$\Rightarrow \dfrac{\cancel{192}\,(\cancel{9}x^4)}{\cancel{48}\cdot\cancel{48}} - 48x = 0$

$\cancel{12}\ 4$

$\cancel{24}$

$\Rightarrow \dfrac{3}{4}x^4 - 48x = 0$

$\Rightarrow x\left(\dfrac{3}{4}x^3 - 48\right) = 0$

16

$\underline{x=0}$ $\quad$ $x^3 = \dfrac{4\cancel{8} \cdot 4}{\cancel{3}}$

$\underline{y=0}$ $\quad$ $\underline{x = 4}$

$\underline{y = 1}$

$$f_{xx} = 6x \quad , \quad f_{yy} = 384y \, , \quad f_{xy} = -48$$

$$D = (6x)(384y) - (48)^2$$

$(x=0, y=0) \Rightarrow D<0 \Rightarrow$ saddle point

$(x=4, y=1) \Rightarrow D>0, \ f_{xx}>0 \Rightarrow$ relative min.

3) $f(x,y) = xy\,e^{-x^2-y^2}$

$$f_x = ye^{-x^2-y^2} + \left(-2x^2y\,e^{-x^2-y^2}\right) = 0$$

$$= e^{-x^2-y^2}(y)(1-2x^2) = 0$$

$$f_y = xe^{-x^2-y^2} + (xy)(-2y)\,e^{-x^2-y^2} = 0$$

$$= e^{-x^2-y^2}(x)(1-2y^2) = 0$$

$x = 0,\ x = \pm\frac{1}{\sqrt{2}},\quad y = 0,\ y = \pm\frac{1}{\sqrt{2}}$

(Pair them such that $f_y = f_x = 0$)

$\left(x = 0,\ y = 0\right),\quad \left(x = \pm\frac{1}{\sqrt{2}},\ y = \pm\frac{1}{\sqrt{2}}\right)$

↳ 4 pairs

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$$f_{xx} = 2xy\left(2x^2-3\right)e^{-x^2-y^2}$$

$$f_{yy} = 2xy\left(2y^2-3\right)e^{-x^2-y^2}$$

$$f_{xy} = \left(2x^2-1\right)\left(2y^2-1\right)e^{-x^2-y^2}$$

$(0,0)$: $f_{xx} = 0,\ f_{yy} = 0,\ f_{xy} > 0$

↳ $D = (0)(0) - f_{xy} \rightarrow D < 0$

↳ Saddle point

$\left(\pm\frac{1}{\sqrt{2}},\ \pm\frac{1}{\sqrt{2}}\right)$: $\boxed{f_{xy} = 0}$

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$: $f_{xx} < 0$, $f_{yy} < 0$

$\left(-\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ → ↳ $D > 0$, $f_{xx} < 0$

↳ relative max

For $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $f_{xx} > 0$, $f_{yy} > 0$

$\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ ↳ $D > 0$, $f_{xx} > 0$

↳ relative min

4) $f(x,y) = xy^4 - x - \frac{x^2}{2}$

$f_x = y^4 - 1 - x \quad = 0$

$f_y = 4xy^3 \quad = 0$

$x = 0$ | $y = 0$
$y = 1$ | $x = -1$
$y = -1$

$(0,1)$ $(0,-1)$ $(-1,0)$

		$(0,1)$	$(0,-1)$	$(-1,0)$
f_{xx}	$= -1$	-1	-1	-1
f_{yy}	$= 12xy^2$	0	0	0
f_{xy}	$= 4y^3$	4	-4	0
		$D<0$	$D<0$	$D=0$
		↳ Saddle	↳ Saddle	

5) $f(x,y) = \dfrac{y^3}{3} + x^2 + 4xy - 2x - 13y + 7$

$f_x = 2x + 4y - 2 = 0$

$f_y = y^2 + 4x - 13 = 0$

$x = \dfrac{2-4y}{2} = 1-2y \Rightarrow x = -17, \ 3$

$y^2 + 4(1-2y) - 13 = 0$
$y^2 + 4 - 8y - 13 = 0$
$y^2 - 8y - 9 = 0$

$y = \dfrac{8 \pm \sqrt{64+36}}{2} = \dfrac{8 \pm 10}{2} = 9, -1$

$P_c = (-17, 9), \ (3, -1)$

$f_{xx} = 2$ $\qquad$ $D = 4y - 16$

$f_{yy} = 2y$ $\qquad$ ↳ $(3,-1)$ is $D<0$

$f_{xy} = 4$ $\qquad$ ↳ saddle

↳ $(-17, 9)$ is $D>0$

↳ local min.

6) $f(x,y) = y^3 - 3y + 3x^2 y$

$f_x = 6xy \qquad = 0$

$f_y = 3y^2 - 3 + 3x^2 \qquad = 0$

(1) $6xy = 0$

↳ $x=0, \; y=0$

(2) $y^2 + x^2 = 1$

↳ $y=0, x=1$ or $x=0, y=1$

$P_c = (1,0)$ and $(0,1)$, $(0,-1)$ and $(-1,0)$.

$f_{xx} = 6y$

$D = 36y^2 - 36x^2$

$f_{yy} = 6y$

$f_{xy} = 6x$

$(1,0) \to D < 0$

↳ saddle

$(0,1) \to D > 0, \ f_{xx} > 0$

↳ local min.

$(-1,0) \to D < 0$

↳ saddle

$(0,-1) \to D > 0, \ f_{xx} < 0$

↳ local max.

Find max/min in a specific interval.

↳ Check critical points, and the boundaries.

$f(x,y) = y^3 - 3y + 3x^2 y$

↳ $f_x = 6xy \qquad = 0$

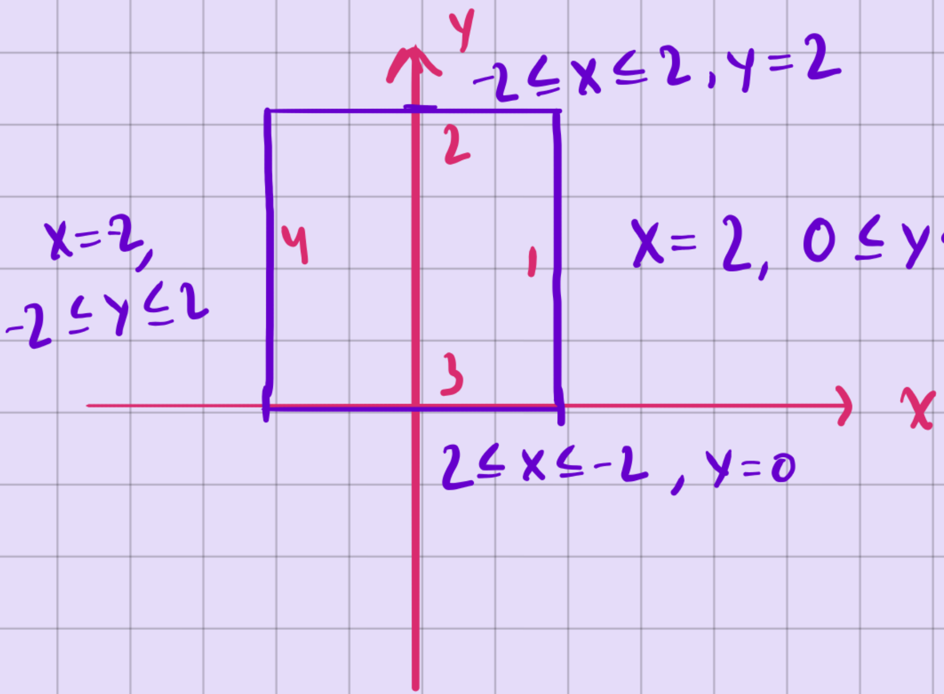
↳ $f_y = 3y^2 - 3 + 3x^2 \qquad = 0$

↳ $C_p = (0,1), (0,-1), (1,0), (-1,0)$

The boundary is given as:

The boundary is given as :

$-2 \le x \le 2$, and $0 \le y \le 2$



1) $f(2,y) = y^3 - 3y + 12y = y^3 + 9y$

$f'(2,y) = 3y^2 + 9 = 0$

$y^2 = -3$ ✗

2) $f(x,2) = 8 - 6 + 6x^2 = 6x^2 + 2$

$f'(x,2) = 12x = 0$

↳ $\boxed{x=0}$

3) $f(x,0) = 0$ ✗

4) $f(-2,y) = y^3 - 3y + 12y = y^3 + 9y$

x

Points of interest :

$(x=0\ ,\ y=2)$ (on the boundary)

critical points $(0,1),\ (0,-1),\ (-1,0)\ ,\ (1,0)$ (inside)

$(2,0),\ (-2,0),\ (2,2),\ (-2,-2)$ (ends)

$$f(x,y) = y^3 - 3y + 3x^2y$$

$\hookrightarrow$ $f(0,2) = 2$

$f(0,1) = \boxed{-2}$

$f(0,-1) = 2$

$f(-1,0) = 0$

$f(1,0) = 0$

$f(2,0) = 0$

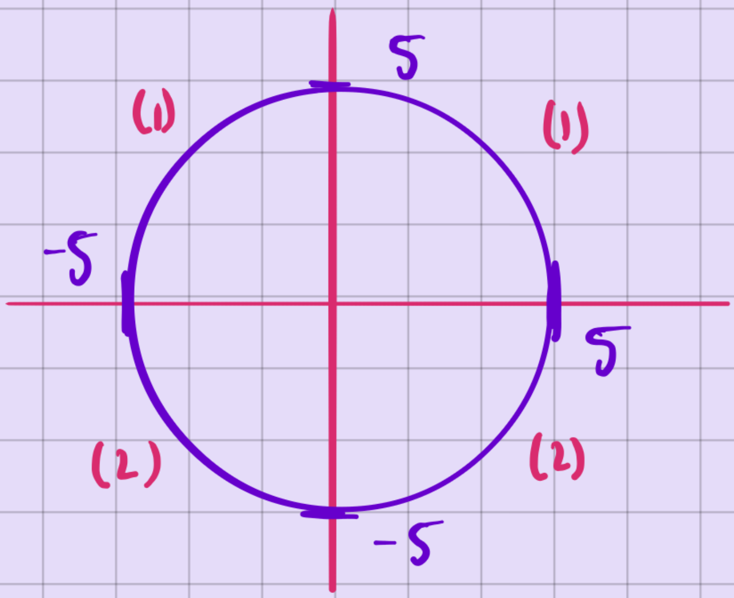
$f(-2,0) = 0$

$f(2,2) = \boxed{26}$

$f(-2,-2) = 22$

Max $= 26$, $m = -2$

1) $f(x,y) = x^2 + y^2 - 6x + 9$, above the region $x^2 + y^2 \leq 25$.



(1) $y = \sqrt{25 - x^2}$

(2) $y = -\sqrt{25 - x^2}$

$f(x,y) = \cancel{x^2} + 25 - \cancel{x^2} - 6x + 9$ $\quad 0 \leq y \leq 5,$
$-5 \leq x \leq 5$

$= 34 - 6x$

$f'(x,y) = -6 = 0$ ⊗

$f(x,y) = \cancel{x^2} + 25 \cancel{- x^2} - 6x + 9$ $\quad -5 \leq y \leq 0$
$-5 \leq x \leq 5$

$= 34 - 6x$

$f'(x,y) = -6 = 0$ ⊗

$f_x = 2x - 6$, $f_y = 2y$

$(x = 3,\ y = 0)$ $P_c = (3, 0)$

$P_b = (5, 0),\ (-5, 0)$

no $(0, \pm 5)$

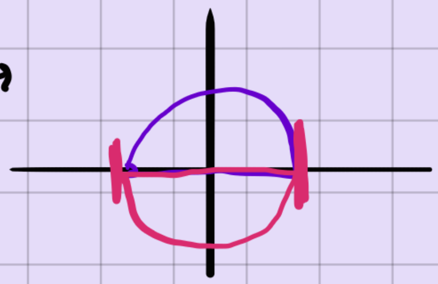
$f(3,0) = 0$

$f(5,0) = 4$

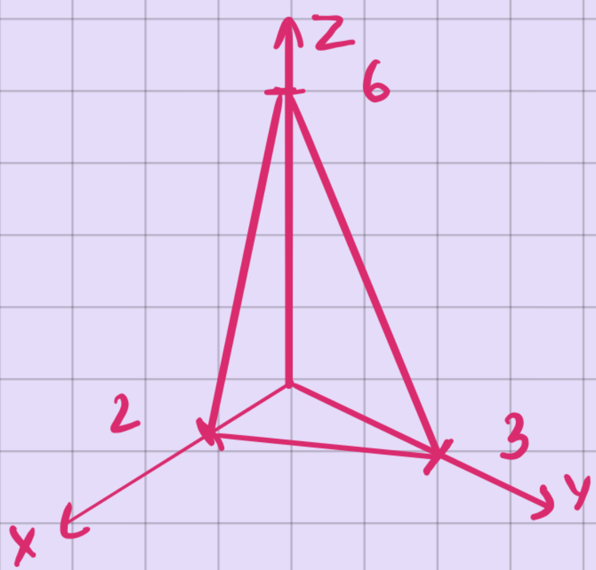
$f(-5,0) = 64$

since we divided the $x^2 + y^2 \leq 25$ into half circles.

↳



2) What point on $3x + 2y + z = 6$ is closest to the origin?



distance from any point on the plane to origin:

$$d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$

$$d = \sqrt{x^2 + y^2 + z^2}$$

since $3x+2y+z=6$

$\hookrightarrow z = 6-3x-2y$

$\Rightarrow d^2 = x^2+y^2+(6-3x-2y)^2 = f(x,y)$

$f_x = 2x + 2(6-3x-2y)(-3) = 0$

$f_y = 2y + 2(6-3x-2y)(-2) = 0$

$2x - 6(6-3x-2y)$

$\Rightarrow 2x-36+18x+12y = 0$

$\Rightarrow 20x+12y = 36 \quad (1)$

$5x+3y=9$

$2y - 4(6-3x-2y)$

$\Rightarrow 2y-24+12x+8y = 0$

$\Rightarrow 12x+10y = 24 \quad (2)$

$6x+5y=12$

Solve (1) and (2) to find P_c.

$6(5x+3y=9)$

$-5(6x+5y=12)$

$\Rightarrow$

$\cancel{30x}+18y=54$

$\cancel{-30x}-25y=-60$

$-7y = -6$

$\boxed{y = 6/7}$

$x = \left(\frac{84-30}{7}\right)\left(\frac{1}{6}\right)$

$\boxed{x = \frac{9}{7}}$

$f_{xx} = 2 + 18 = 20$

$f_{yy} = 2 + 8 = 10$

$f_{xy} = 12$

$D > 0,\ f_{xx} > 0$

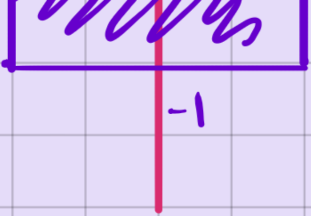
↳ local min.

The point closest to the origin is

$(9/7, 6/7, 3/7)$

3) $d(x,y) = \sqrt{(x-2)^2 + (y-2)^2 + 4}$ on the domain $-2 \le x \le 2$, $-1 \le y \le 1$.

$d_x = \dfrac{2(x-2)}{\sqrt{(x-2)^2 + (y-2)^2 + 4}} = 0$



$$d_y = \frac{2(y-2)}{\sqrt{(x-2)^2+(y-2)+4}} = 0$$

$P_c = (2,2)$ ✗

$d(2,y) = \sqrt{(y-2)^2+4}$ $\qquad -1 \le y \le 1$

$d(y)$

$d'(y) = 2(y-2) = 0$

$= (y=2)$ ✗

$d(-2,y) = \sqrt{18+(y-2)^2}$ $\qquad -1 \le y \le 1$

$d(y)$

$d'(y) = 2(y-2) = 0$

$y = 2$ ✗

$d(x,1) = \sqrt{(x-2)^2+5}$ $\qquad -2 \le x \le 2$

$d(x)$

$d'(x) = 2(x-2) = 0 \Rightarrow x = 2$ ✓

$\hookrightarrow \quad (2,1)$

$d(x,-1) = \sqrt{(x-2)^2 + 13} \qquad -2 \le x \le 2$

$d(x)$

$d'(x) = 2(x-2) = 0 \Rightarrow x = 2$ ✓

$\hookrightarrow \quad (2,-1)$

Boundary points :

$(2,1), (-2,-1), (2,-1), (-2,1)$

Points of interest :

$(2,1), (-2,1), (2,-1), (-2,-1)$

$f(2,1) = \sqrt{5}$ // Abs min

$f(-2,1) = \sqrt{21}$

$f(2,-1) = \sqrt{13}$

$f(-2,-1) = \sqrt{29}$ // Abs max

4) $f(x,y) = \sqrt{x^2+3y} = z$

$$\sqrt{x^2+3y} - z = 0 = F(x, y, z)$$

$$F_x = \frac{2x}{2\sqrt{x^2+3y}}, \quad F_y = \frac{3}{2\sqrt{x^2+3y}}, \quad F_z = -1$$

$$\nabla F(4,3,5) = \left\langle \frac{4}{5}, \frac{3}{10}, -1 \right\rangle$$

$$\frac{4}{5}(x-4) + \frac{3}{10}(y-3) - z + 5 = 0$$

Find $\sqrt{(4.02)^2 + 3(2.97)}$, $x = 4.02$, $y = (2.97)$

$$\frac{4}{5}(0.02) + \frac{3}{10}(-0.03) + 5 = z$$

$$z = \frac{4}{5} \times \frac{2}{100} - \frac{3}{10} \times \frac{3}{100} + \frac{500}{100}$$

$$= \frac{16}{1000} - \frac{9}{1000} + \frac{5000}{1000}$$

$$= \frac{5007}{1000} = \underline{\underline{5.007}}$$