Lagrange Multipliers :

↳ A more powerful optimization technique

↳ find min/max within a boundary faster.

↳ "Constraint optimization"

Example :

↳ Find the max and min of $f(x,y) = x^2 + y^2$, given the constraint $xy = 1$.

Method 1 :

$$xy = 1 \Rightarrow x = \frac{1}{y}$$

$$f\left(\frac{1}{y}, y\right) = \left(\frac{1}{y}\right)^2 + y^2 = f(y)$$

$$f'(y) = 2y - \frac{2}{y^3} = 0$$

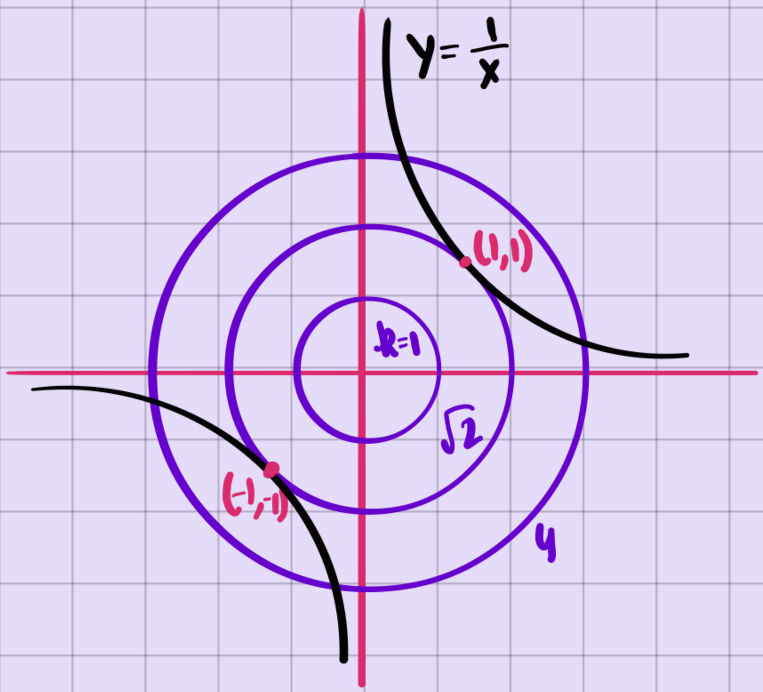
$$\Rightarrow 2y^4 - 2 = 0$$

$$\Rightarrow y^4 = 1 \Rightarrow \underline{y = \pm 1}, \ \underline{x = \pm 1}$$

Critical points : $(1,1)$ and $(-1,-1)$

↳ Use discriminant.

Method 2 :



$f(x,y) = x^2+y^2 = k$.

At critical points, the objective function and constraint function are tangent.

↳ Same gradient

$$\nabla f(x,y,z) = \underbrace{\lambda}_{\text{any constant}} \nabla g(x,y,z)$$

$f(x,y) = x^2+y^2$

$g(x,y) = xy-1$

$\nabla f(x,y) = \langle 2x, 2y \rangle$

$\nabla g(x,y) = \langle y, x \rangle$

↳ $\langle 2x, 2y \rangle = \lambda \langle y, x \rangle$

↳ $2x = \lambda y$ and $2y = \lambda x$

$$x = \frac{\lambda y}{2}$$

$$2\cancel{y} = \lambda\left(\frac{\lambda \cancel{y}}{2}\right)$$

$$4 = \lambda^2 \Rightarrow \boxed{\lambda = \pm 2}$$

$$x = \frac{2y}{2} \Rightarrow x = y \quad , \quad x = \frac{-2y}{2} \Rightarrow \boxed{x = -y}$$

↳ Substitute into $g(x,y) = 0$

↳ $xy = 1$

↳ $x = 1,\ y = 1$

↳ $x = -1,\ y = -1$

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1) Set $g(x,y) = 0$ (constraint function)

2) Find $\vec{\nabla} f = \lambda \vec{\nabla}_g$

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1) $f(x,y) = 4 - x^2 - y^2$, $g(x,y) = 4x^2 + y^2 - 4 = 0$

$\nabla f(x,y) = \langle -2x, -2y \rangle$, $\nabla g(x,y) = \langle 8x, 2y \rangle$

$$\langle -2x, -2y \rangle = \lambda \langle 8x, 2y \rangle$$

① $\lambda\, 8x = -2x \Rightarrow \lambda 8x + 2x = 0$

① $\lambda 8x = -2x \rightarrow \lambda 8x + 2x = 0$

$\hookrightarrow 2x(4\lambda + 1) = 0$

$\hookrightarrow x = 0$ or $\lambda = \frac{-1}{4}$

Use in $g(x,y)$

$\hookrightarrow g(0,y) = 0$

$\hookrightarrow \underline{\underline{y = \pm 2}}$

② $\lambda 2y = -2y$

$2y(\lambda + 1) = 0$

$\hookrightarrow y = 0$ or $\lambda = -1$

$\hookrightarrow g(x,0) = 0$

$\hookrightarrow \underline{\underline{x = \pm 1}}$

$(\pm 1, 0)$ and $(0, \pm 2)$

$\hookrightarrow$ Find $f(x,y)$ at these critical points.

2) $f(x,y,z) = xyz$ subject to $x^2 + y^2 + z^2 = 3$

$\hookrightarrow g(x,y,z) = x^2 + y^2 + z^2 - 3 = 0$

$\nabla f = \langle yz, xz, xy \rangle$, $\nabla g = \langle 2x, 2y, 2z \rangle$

$yz = 2\lambda x$ (1) $\lambda = \frac{yz}{2x}$

$xz = 2\lambda y$ (2) $\lambda = \frac{xz}{2y}$

$xy = 2\lambda z \quad (3) \quad \lambda = \frac{xy}{2z}$

$\Rightarrow \frac{y\cancel{z}}{2x} = \frac{x\cancel{z}}{2y} \quad , \quad \frac{\cancel{y}z}{2x} = \frac{x\cancel{y}}{2z} \quad , \quad \frac{\cancel{x}z}{2y} = \frac{\cancel{x}y}{2z}$

$\Rightarrow 2y^2 = 2x^2 \quad , \quad 2z^2 = 2x^2 \quad , \quad 2z^2 = 2y^2$

$\hookrightarrow \quad x = y = z \quad , \quad -x = -y = -z$

$g(x,y,z) = x^2 + y^2 + z^2 - 3 = 0$

$\hookrightarrow \quad x^2 + x^2 + x^2 - 3 = 0$

$\hookrightarrow \quad \cancel{3}x^2 = \cancel{3} \Rightarrow x^2 = 1 \Rightarrow \underline{\underline{x = \pm 1}}$

$x = y = z = \pm 1$

3) $f(x,y) = x + y$ on the curve $g(x,y) = x^2 + 2y^2 - 6 = 0$.
Find <u>max.</u>

$\nabla f = \langle 1, 1 \rangle \qquad \nabla g = \langle 2x, 4y \rangle$

$\Rightarrow 2x\lambda = 1 \rightarrow \lambda = \frac{1}{2x}$

$\Rightarrow 4y\lambda = 1 \rightarrow \lambda = \frac{1}{4y}$

$2x = 4y$

$\hookrightarrow \quad x = 2y$

$g(2y, y) = (2y)^2 + 2y^2 - 6 = 0$

$\hookrightarrow 4y^2 + 2y^2 = 6$

$\Rightarrow y = \pm 1 \Rightarrow x = \pm 2$

Critical points :

$(1,2), \ (1,-2), \ (-1,2), \ (-1,-2)$

$f(1,2) = \underline{\underline{3}}$ $\qquad f(-1,2) = 1$

$f(1,-2) = -1$ $\qquad f(-1,-2) = -3$

4) Find the x-component for the min and max for $f(x,y) = (x-2)^2 + (y-4)^2$ on the constraint $g(x,y) = x^2 + y^2 - 5 = 0$.

$\nabla f = \langle 2x-4, 2y-8 \rangle, \quad \nabla g = \langle 2x, 2y \rangle$

$\Rightarrow 2x - 4 = \lambda(2x), \quad 2y - 8 = \lambda(2y)$

$\Rightarrow \lambda = \dfrac{2x-4}{2x}, \quad \lambda = \dfrac{2y-8}{2y}$

$\Rightarrow (2x-4)(\cancel{2}y) = (2y-8)(\cancel{2}x)$

$\Rightarrow (x-2)(y) = (y-4)(x)$

$\Rightarrow \cancel{xy} - 2y = \cancel{xy} - 4x$

$\Rightarrow y = 2x$

$\hookrightarrow g(x, 2x) = x^2 + (2x)^2 = 5$

$\Rightarrow 5x^2 = 5$

$\Rightarrow \underline{\underline{x = \pm 1}}, \; y = \underline{\underline{\pm 2}}$

$f(x,y) = (x-2)^2 + (y-4)^2$

$\rightarrow (1,2) = 1 + 4 = 5$

$\rightarrow (1,-2) = 1 + 36 = 37$ max : $(-1,-2)$

$\rightarrow (-1,2) = 9 + 4 = 13$ min : $(1,2)$

$\rightarrow (-1,-2) = 9 + 36 = 45$

5) Find Max and min values of $f(x,y) = 2x + 6y$ on the constraint $g(x,y) = x^2 + y^2 - 10 = 0$.

$\nabla f = \langle 2, 6 \rangle$, $\nabla g = \langle 2x, 2y \rangle$

$\Rightarrow \lambda(2x) = 2$, $\lambda(2y) = 6$

$\Rightarrow \lambda = \frac{1}{x}$, $\lambda = \frac{3}{y}$

$\Rightarrow 3x = y$

$\hookrightarrow g(x, 3x) = 0$

$\Rightarrow x^2 + 9x^2 = 10$

$\Rightarrow \underline{x = \pm 1}, \underline{y = \pm 3}$

$(1,3) \rightarrow +20$

$(1,-3) \rightarrow -16$

$(-1,3) \rightarrow +16$

$(-1,-3) \rightarrow -20$

$\Rightarrow M = 20, \; m = -20$

6) Find shortest distance from $(1,1,0)$ to the plane $x+y+z=1$.

$$d = \sqrt{(x-1)^2 + (y-1)^2 + z^2}$$

$$\Rightarrow d^2 = f(x,y,z) = (x-1)^2 + (y-1)^2 + z^2$$

$$g(x,y,z) = x+y+z-1 = 0$$

$$\nabla f = \langle 2(x-1), 2(y-1), 2z \rangle$$

$$\nabla g = \langle 1, 1, 1 \rangle$$

$$\Rightarrow 2x-2 = \lambda, \quad 2y-2 = \lambda, \quad 2z = \lambda$$

$\Rightarrow \quad 2x-2 = 2y-2 \quad \Rightarrow \quad x=y$

$\Rightarrow \quad 2x-2 = 2z \quad \Rightarrow \quad x-1 = z \quad \Rightarrow \quad x=y=z+1$

$g(x,x,x-1) = 0$

$\hookrightarrow \quad x+x+x-1-1 = 0$

$\Rightarrow \quad 3x = 2 \quad \Rightarrow \quad x = \frac{2}{3} \quad \rightarrow \left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$

$f\left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right) = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}$

$d = \sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{3}$

Double integrals :

a) Over rectangular regions :

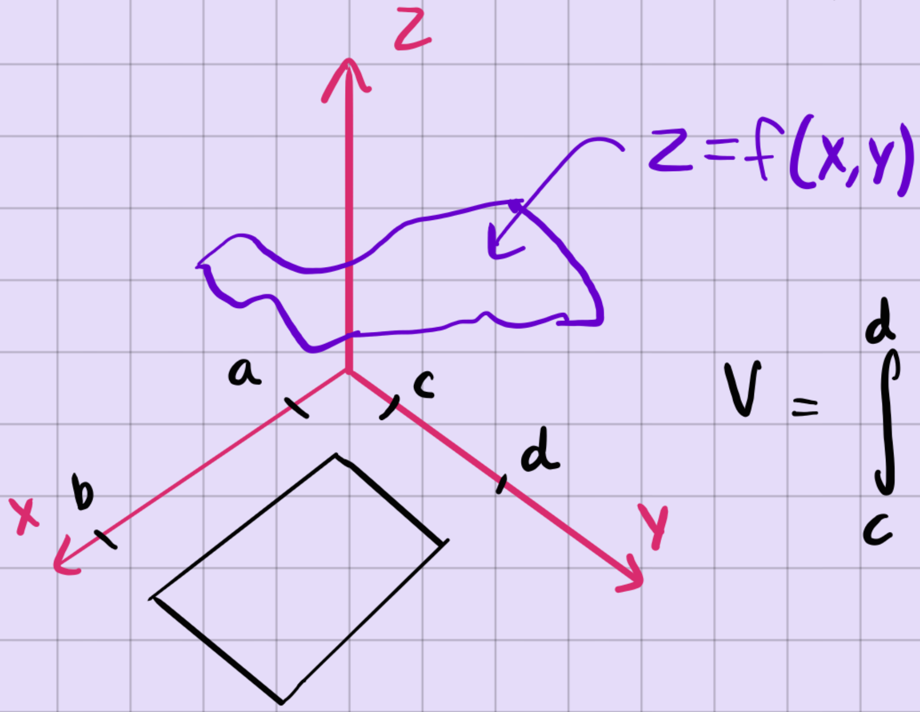
Given a surface $z(x,y)$, we can find the volume under the surface from $a \leq x \leq b$ and $c \leq y \leq d$ with a double integral.

$\hookrightarrow \quad [a,b] \times [c,d]$

$\int^{b} \int^{d}$

$$V = \int_a \int_c f(x,y)\, dy\, dx$$

$$dA$$



$$V = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

In the rectangular region, the order of integration does not matter. Just make sure that the bounds match the variable of integration.

1) $f(x,y) = 3-x-y$, $\underset{(x)}{[0,1]} \times \underset{(y)}{[0,2]}$

$$\Rightarrow V = \int_0^1 \int_0^2 3-x-y \; dy\, dx$$

$$= \int_0^1 \left(3y - xy - \frac{y^2}{2}\right)\Big|_0^2 \; dx$$

$$= \int_0^1 6-2x-2 \; dx$$

$$= \left(\left(6x - x^2 - 2x\right)\right)\Big|_0^1 = 6-1-2 = 3 \text{ units}^3$$

$$= (6x - x^3 - 2x)\Big|_0^1 = 6 - 1 - 2 = \underline{\underline{3 \text{ Units}}}$$

2) Compute $\iint_R \cos(x+y)\, dA$, where $R \in [0,\pi] \times [0,\pi]$

$$\rightarrow \int_0^{\pi} \int_0^{\pi} \cos(x+y)\, dx\, dy$$

$$\Rightarrow \int_0^{\pi} \left(\sin(x+y) \Big|_0^{\pi} \right) dy$$

$\sin(\pi + y) - \sin(y)$

$$\Rightarrow \int_0^{\pi} \sin(\pi+y) - \sin(y)\, dy$$

$$\Rightarrow \left(-\cos(\pi+y) + \cos(y) \right) \Big|_0^{\pi} = \cos(\pi) - \cos(2\pi) - \cos(0) + \cos(\pi)$$

$$= -1 - 1 - 1 - 1 \Rightarrow \underline{\underline{-4}}$$

3) $\iint_R \frac{2xy^2}{x^2+1}\, dA$, $R \in [0,1] \times [0,2]$

$$\int_0^2 \int_0^1 \frac{2xy^2}{x^2+1}\, dx\, dy$$

$u = x^2 + 1$

$du = 2x\, dx \Rightarrow dx = \frac{du}{2x}$

$\cancel{2x}y^2 \cdot du$

$$u \qquad 2x$$

$$\Rightarrow \int_0^2 \int_1^2 y^2 u^{-1} \, du \, dy = \int_0^2 y^2 (\ln 2 - \cancel{\ln 1})^0 \, dx$$

$$= \ln(2) \left. \frac{y^3}{3} \right|_0^2 = \frac{8 \ln 2}{3}$$

Average value :

$$f_{avg} = \frac{1}{A} \iint_R f(x,y) \, dA$$

area of R

4) Find average value of $f(x,y) = x^2 y$ over $(-1,0), (1,0), (1,5), (-1,5)$

$\hookrightarrow R \in [-1,1] \times [0,5] \Rightarrow A = 10$

$$\frac{1}{10} \int_{-1}^{1} \int_0^5 x^2 y \, dy \, dx$$

$$\Rightarrow \frac{1}{10} \int_{-1}^{1} \left(x^2 \frac{y^2}{2} \right) \Big|_0^5 dx \Rightarrow \frac{1}{10} \int_{-1}^{1} \frac{25}{2} x^2 \, dx$$

$$\Rightarrow \frac{25}{20} \left. \frac{x^3}{3} \right|^1$$

$$\Rightarrow \frac{25}{20}\left(\frac{1}{3}+\frac{1}{3}\right)$$

$$\Rightarrow \frac{\cancel{2}^{1}}{3} \times \frac{\cancel{25}^{5}}{\cancel{20}\,\cancel{10}_{2}} = \frac{5}{6}$$

5) $\iint_R \frac{x^2 y}{2+x^3}\, dA$, $R \in [1,2] \times [0,4]$

$$\int_0^4 \int_1^2 \frac{x^2 y}{2+x^3}\, dx\, dx$$

$u = x^3 + 2$
$du = 3x^2\, dx$

$\Rightarrow dx = \frac{du}{3x^2}$

$$\int_3^{10} \frac{\cancel{x^2} y}{u} \frac{du}{3\cancel{x^2}}$$

$$\Rightarrow y \frac{\ln u}{3}\Big|_3^{10} = \frac{y}{3}(\ln 10 - \ln 3)$$

$$\frac{y^2}{2} \frac{(\ln 10 - \ln 3)}{3}\Big|_0^4 = \frac{8}{3}\left(\ln \frac{10}{3}\right)$$

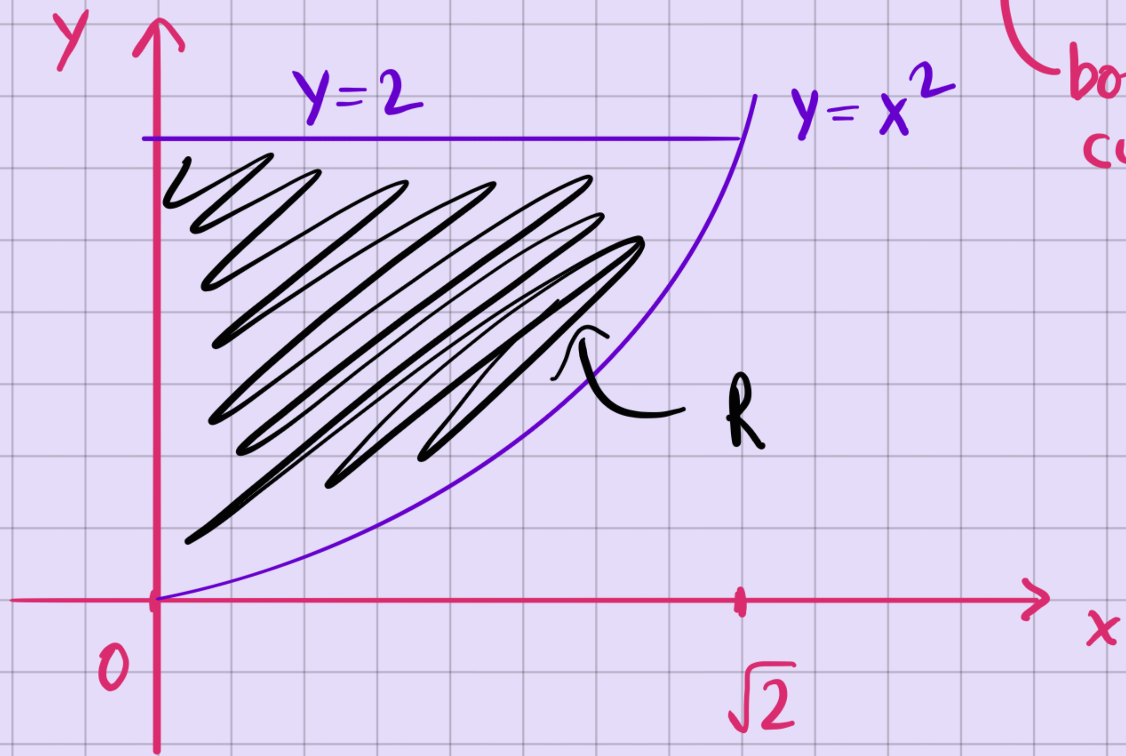
b) Over general regions :

$\hookrightarrow$ say $x^2 \leq y \leq 2$ and $0 \leq x \leq \sqrt{2}$

top curve

$$f(x,y) = xy^2 \qquad R = [\,0 \le x \le \sqrt{2}\ ,\ x^2 \le y \le 2\,]$$

bottom curve



Integrate the constant bounded variable last.

$$V = \int_0^{\sqrt{2}} \int_{x^2}^{2} xy^2 \; dy \; dx$$

$$= \int_0^{\sqrt{2}} \left. \frac{x\,y^3}{3} \right|_{x^2}^{2} dx$$

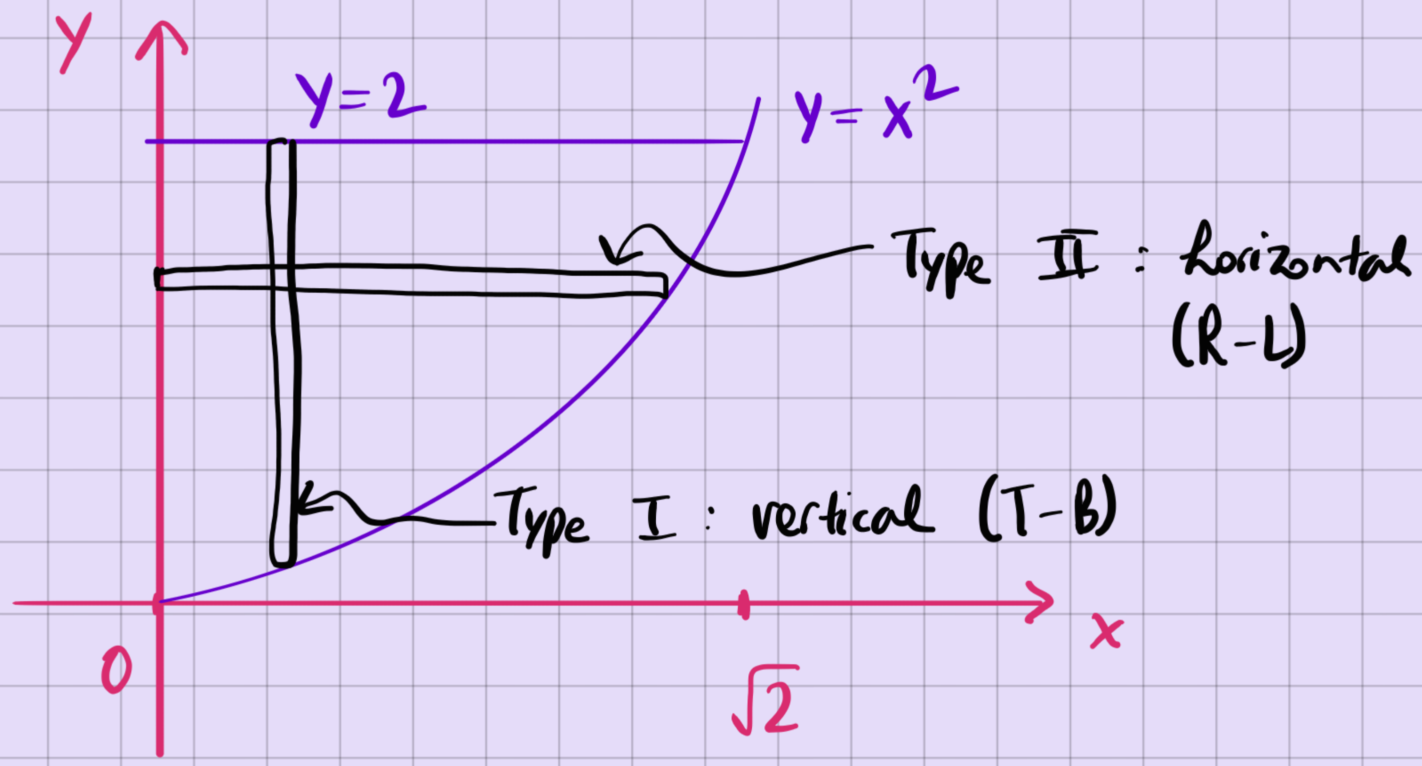
$$= \int_0^{\sqrt{2}} \frac{8}{3}x - \frac{x^7}{3} \; dx$$

$$= \frac{4}{3}x^2 - \frac{x^6}{24}\Big|_0 \qquad = \frac{8}{3} - \frac{\cancel{16}^{\,2}}{\cancel{24}_{\,3}}$$

$$= \frac{6}{3} = 2$$

To switch the order :

$$f(x,y) = xy^2 , \quad R = \left[0 \le x \le \sqrt{2} ,\ x^2 \le y \le 2\right]$$



$$y = x^2 \Rightarrow x = \pm\sqrt{y} \Rightarrow x = \sqrt{y}$$

$$0 \le x \le \sqrt{y} , \qquad 0 \le y \le 2$$

$$V = \int^{2}\int^{\sqrt{y}} xy^2 \, dx \, dy$$

$$= \int_0^2 \frac{x^2 y^2}{2} \Big|_0^{\sqrt{y}} \, dy$$

$$= \int_0^2 \frac{y^3}{2} \, dy = \frac{y^4}{8} \Big|_0^2 = \frac{16}{8} = 2$$

Strategy to switch order of integration :

1) Plot the general region $\mathbb{R}$.

2) Understand the type of the region

↳ Type I if x is bound by constant.

↳ Type II if y is bound by constant.

3) Switch.

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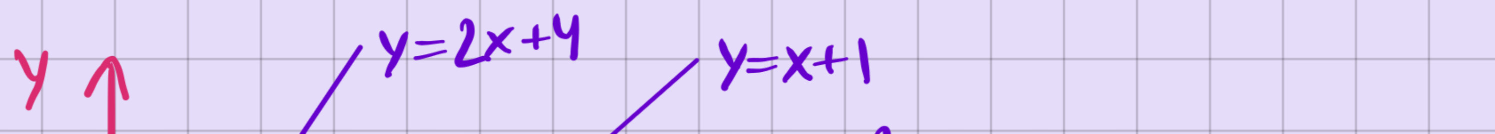
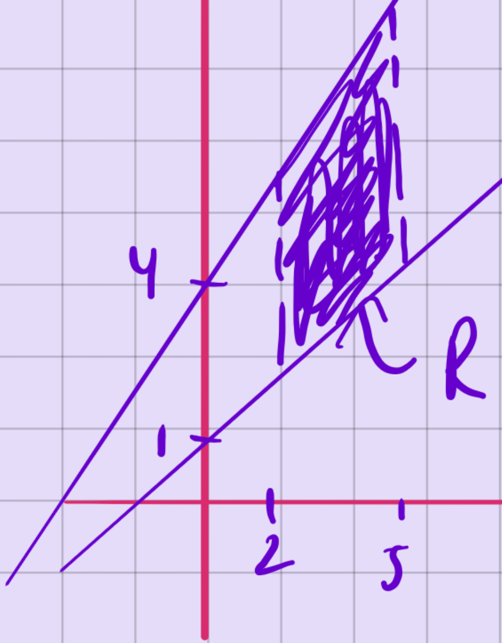
1) Given the region $R: 2 \leq x \leq 5, \; x+1 \leq y \leq 2x+4$, find the double integral to find the volume under $f(x,y)$.



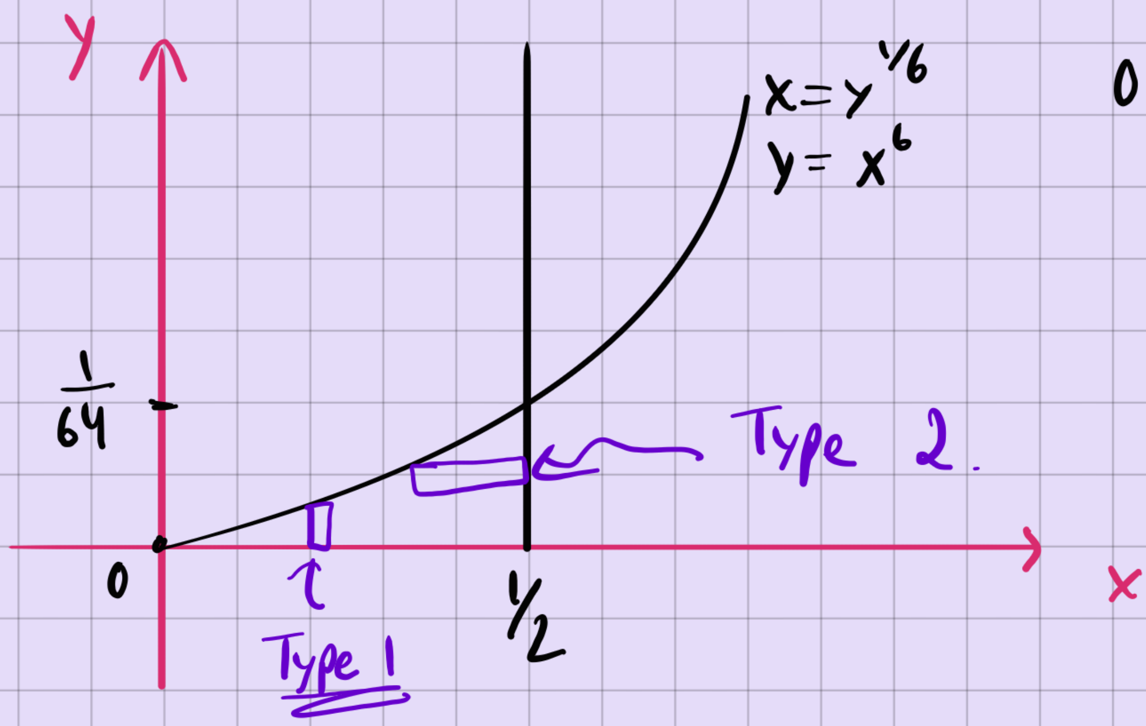


$$\int_2^5 \int_{x+1}^{2x+4} f(x,y)\, dy\, dx$$

2) $$\int_0^{1/64} \int_{y^{1/6}}^{1/2} \cos\left(64\pi x^7\right)\, dx\, dy$$

$y^{1/6} \le x \le 1/2$

$0 \le y \le 1/64$



$0 \le y \le x^6$, $0 \le x \le 1/2$

$$\int^{1/2} \int^{x^6} \cos\left(64\pi x^7\right)\, dy\, dx$$

$$\int_0 \int_0 \cos(64\pi x^7)\, dy\, dx$$

$$= \int_0^{1/2} \cos(64\pi x^7)\, x^6 \, dx$$

$$u = 64\pi x^7$$

$$du = 7 \cdot 64 \pi x^6 \, dx$$

$$dx = \frac{du}{448\pi x^6}$$

$$= \int \frac{\cos(u)}{448\pi} du = \frac{\sin(u)}{448\pi}$$

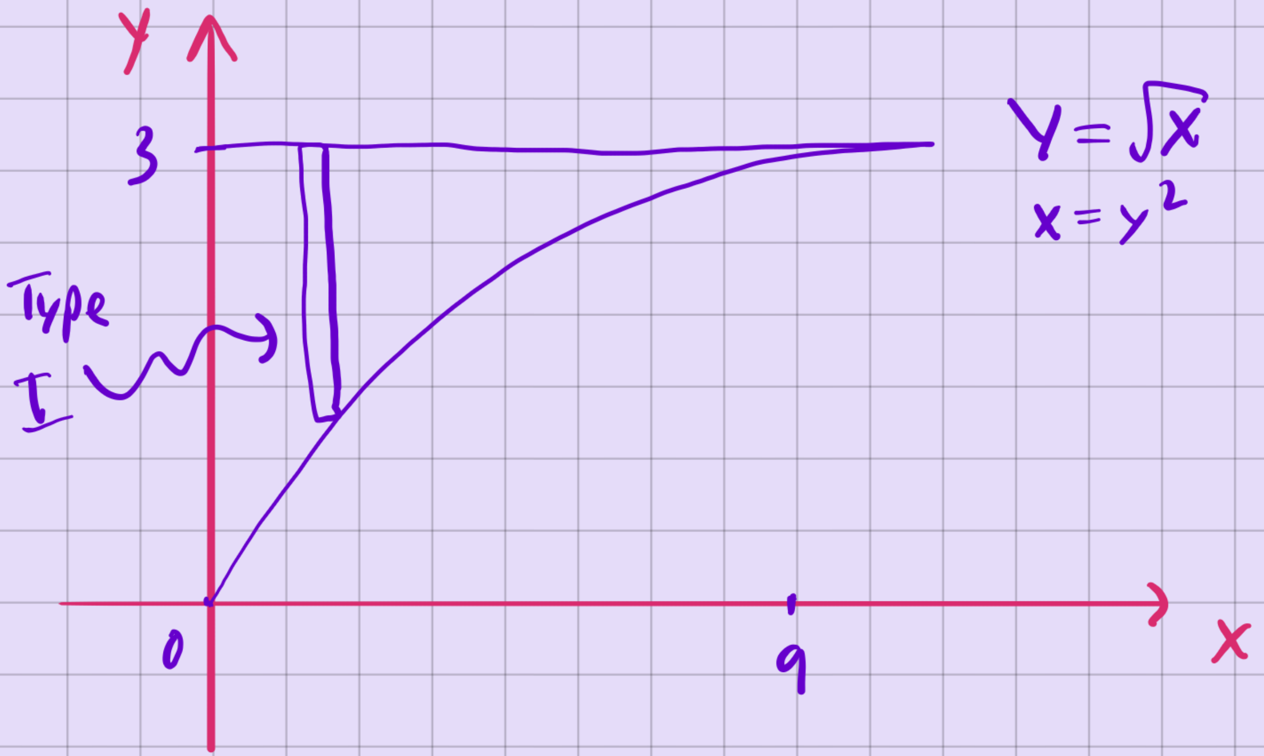
$$\Rightarrow \left. \frac{\sin(64\pi x^7)}{448\pi} \right|_0^{1/2} = \frac{\sin(\pi/2)}{448\pi}$$

$$= \frac{1}{448\pi}$$

3) $$\int_0^9 \int_{\sqrt{x}}^3 \frac{x}{y^5+1}\, dy\, dx$$

$\sqrt{x} \le y \le 3$

$0 \le x \le 9$



$$0 \le x \le y^2 \quad , \quad 0 \le y \le 3$$

$$\int_0^3 \int_0^{y^2} \frac{x}{y^5+1} \, dx \, dy$$

$$\int_0^3 \frac{x^2}{2y^5+2} \Bigg|_0^{y^2} dy$$

$$\int_0^3 \frac{y^4}{2y^5+2} \, dy$$

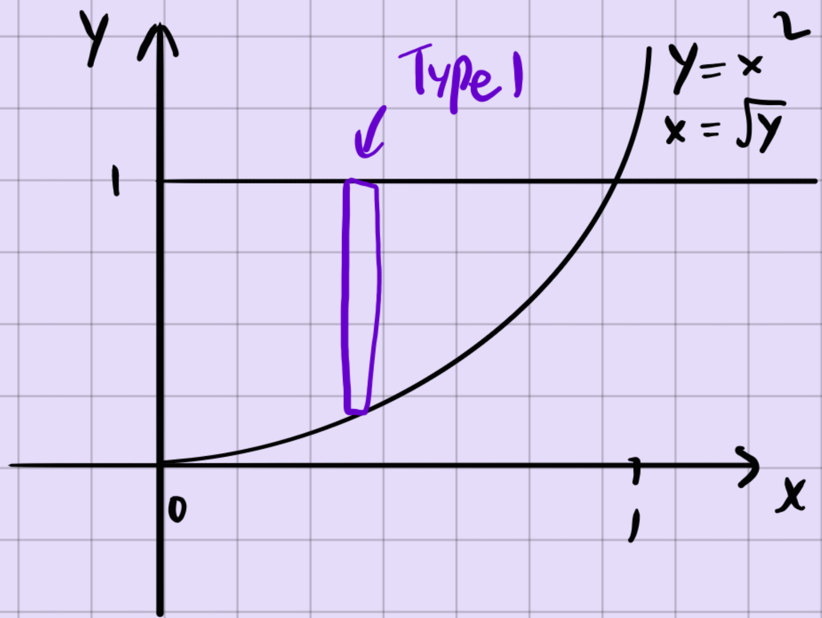
$u = 2y^5 + 2 \Rightarrow du = 10y^4\, dy$

$dy = \dfrac{du}{10y^4}$

$\dfrac{1}{10}\displaystyle\int \frac{1}{u}\, du = \frac{1}{10} \ln(u)$

$\Rightarrow \dfrac{1}{10} \ln(2y^5+2)\Big|_0^3 = \dfrac{\ln(488)}{10}$

4) $\displaystyle\int_0^1 \int_{x^2}^1 6\sqrt{y} \cos y^2 \, dy \, dx$



$0 \le x \le \sqrt{y}$

$0 \le y \le 1$

$\displaystyle\int_0^1 \int_0^{\sqrt{y}} 6\sqrt{y} \cos(y^2) \, dx \, dy$

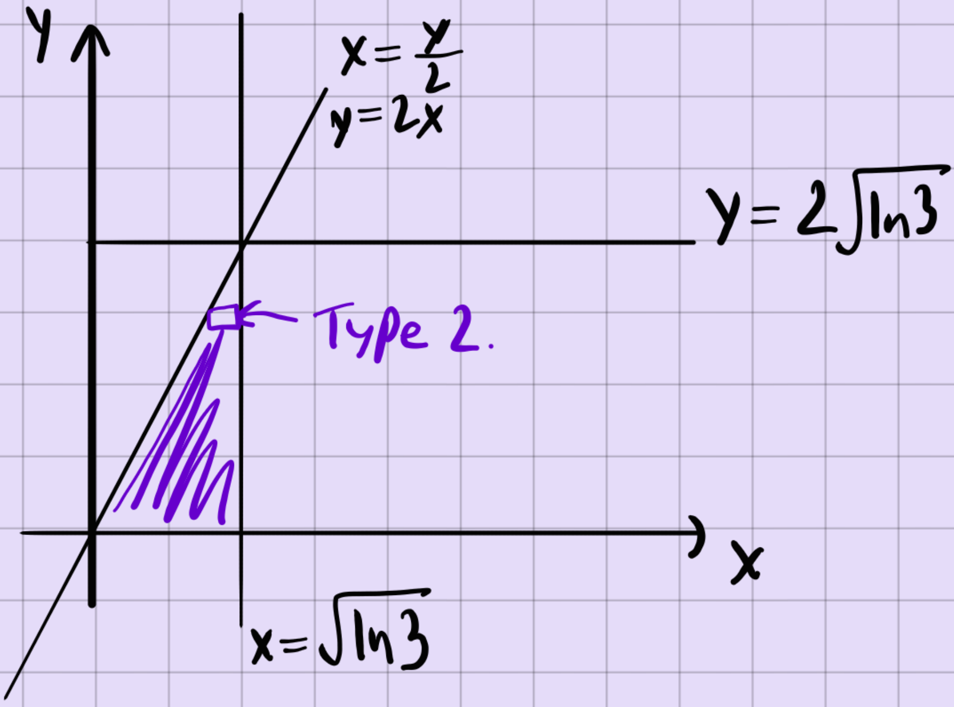
$$\int_0^1 6y \cos(y^2) \, dy$$

$$u = y^2 \Rightarrow du = 2y \, dy \Rightarrow dy = \frac{du}{2y}$$

$$\int 3 \cos(u) \, du = 3 \sin(u) \Rightarrow 3 \sin(y^2)$$

$$\Rightarrow 3 \sin(y^2) \Big|_0^1 = 3 \sin(1)$$

5) $$\int_0^{2\sqrt{\ln 3}} \int_{\frac{y}{2}}^{\sqrt{\ln 3}} e^{x^2} \, dx \, dy$$



$0 \leq y \leq 2x$

$0 \leq x \leq \sqrt{\ln 3}$

$$\int_0^{\sqrt{\ln 3}} \int_0^{2x} e^{x^2} \, dy \, dx$$

0 0

$$\int_0^{\sqrt{\ln 3}} e^{x^2} \, y \Big|_0^{2x} \, dx = e^{x^2}(2x)$$

$$\int_0^{\sqrt{\ln 3}} e^{x^2}(2x) \, dx$$

$u = x^2$

$du = 2x \, dx$

$$\int e^u \, du = e^u \Rightarrow e^{x^2}\Big|_0^{\sqrt{\ln 3}}$$

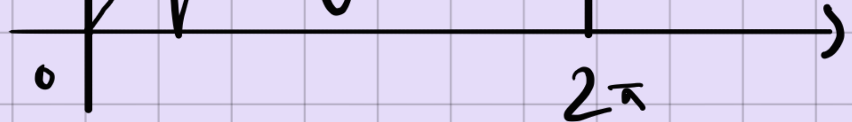
$$= e^{\ln 3} - e^0$$

$$= 3 - 1 = 2$$

6) $I = \int_0^{\sqrt{2\pi}} \int_{y^2}^{2\pi} y \cos(x^2) \, dx \, dy$

$0 \le y \le \sqrt{x}$

$0 \le x \le 2\pi$



$$\int_0^{2\pi}\int_0^{\sqrt{x}} y \cos(x^2) \, dy \, dx$$

$$\int_0^{2\pi} \frac{x}{2} \cos(x^2) \, dx$$

$u = x^2 \Rightarrow du = 2x\,dx$

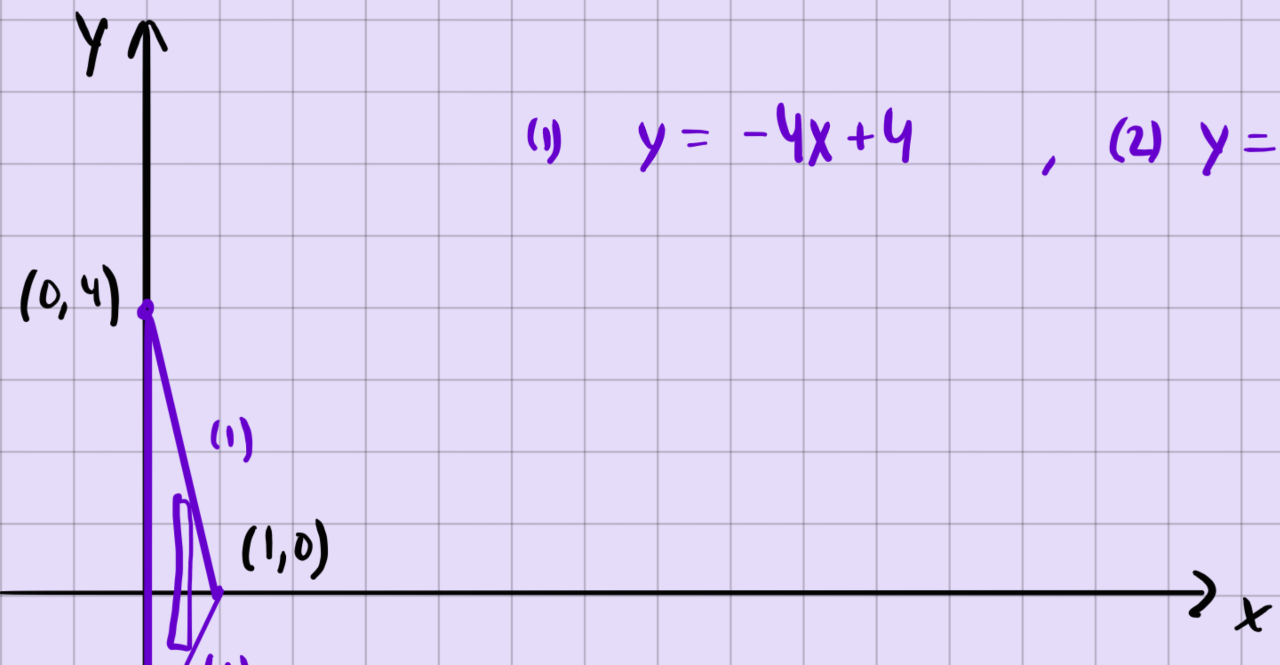
$\Rightarrow dx = \dfrac{du}{2x}$

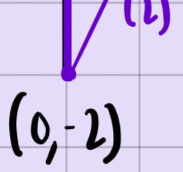
$$\frac{1}{4}\int \cos u \, du = \frac{1}{4} \sin(x^2) \Big|_0^{2\pi}$$

$$= \frac{1}{4} \sin(4\pi^2)$$

7) D is the triangle $(0,4)$, $(1,0)$, $(0,-2)$.

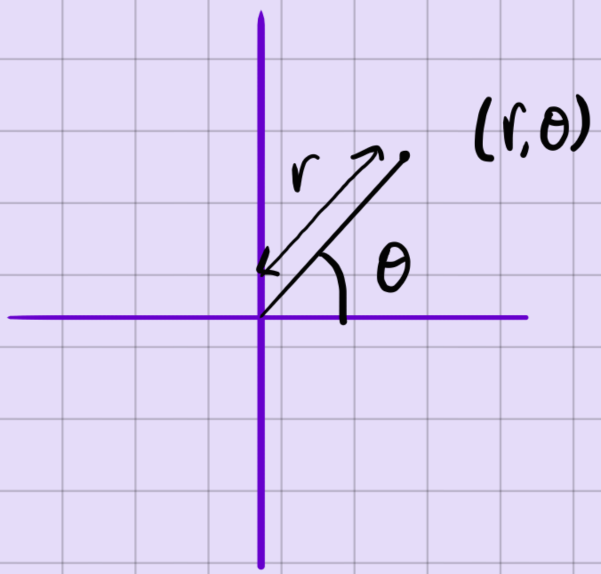
(1) $y = -4x + 4$, (2) $y = 2x - 2$

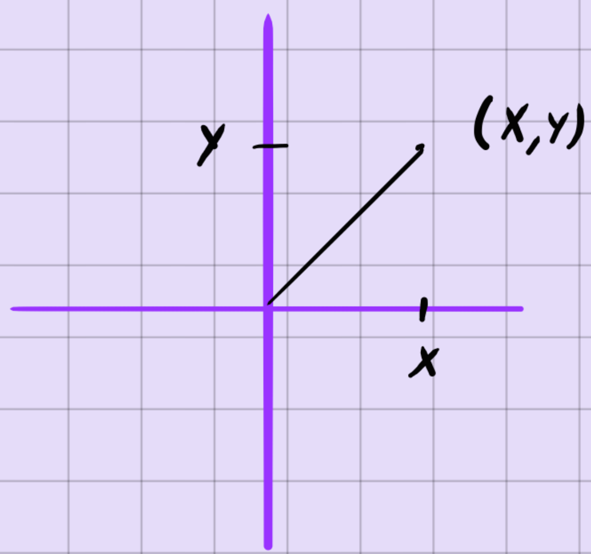




$$\int_0^1 \int_{2x-2}^{-4x+4} f(x,y)\;dy\;dx$$

Polar coordinates and double integrals.

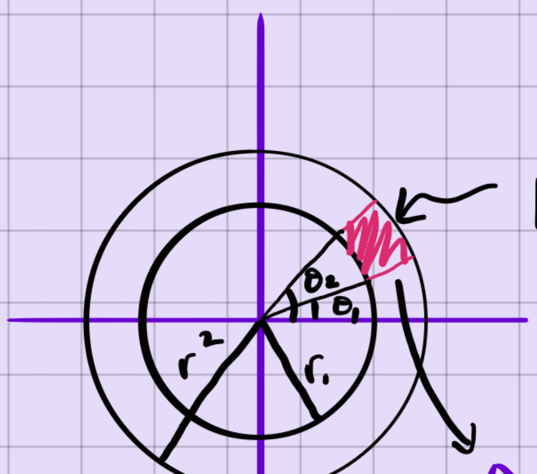




Note : $r = \sqrt{x^2+y^2} \Rightarrow r^2 = x^2+y^2$

Also : $\theta = \tan^{-1}(y/x)$

Also : $y = r\sin\theta \,,\quad x = r\cos\theta$



Bounded by r and θ.

$r_1 \le r \le r_2$
$\theta_1 \le \theta \le \theta_2$

area of a small patch that comes from dr and $d\theta$

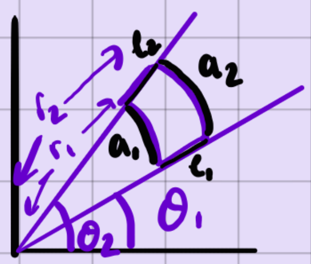
In cartesian, $dA = dx\,dy$ or $dy\,dx$ (rectangle)

$$\iint_R f(r,\theta)\, dA$$

$r\,dr\,d\theta$

or

$r\,d\theta\,dr$



$a_1 = r_1(\theta_2 - \theta_1) \Rightarrow r_1\,d\theta$

$a_2 = r_2(\theta_2 - \theta_1) \Rightarrow r_2\,d\theta$

$d\theta$

$l_1 = l_2 = dr$

Since dr and $d\theta$ are small,

$A = r_1 = r_2 = r. \Rightarrow dA = r\,dr\,d\theta$

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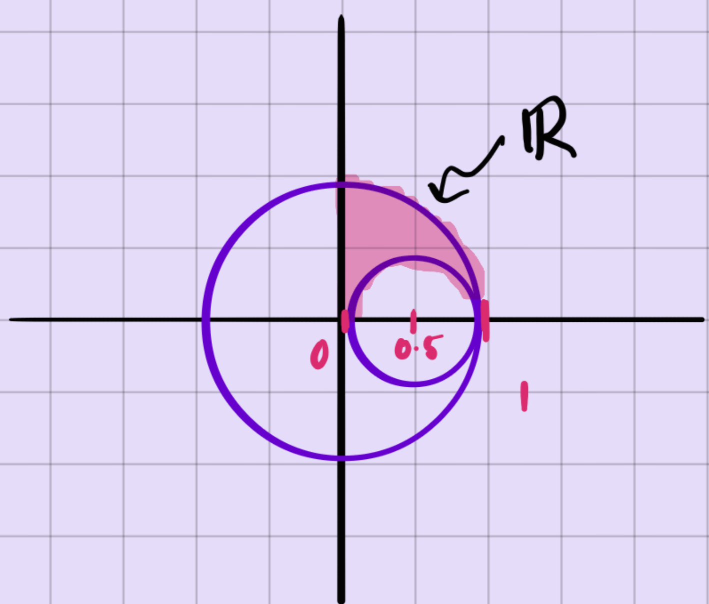
1) $$\int_0^1 \int_{\sqrt{x-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx$$

$\Rightarrow 0 \le x \le 1$, $\sqrt{x-x^2} \le y \le \sqrt{1-x^2}$

$y = \sqrt{x-x^2}$

$\Rightarrow y^2 + x^2 - x = 0$

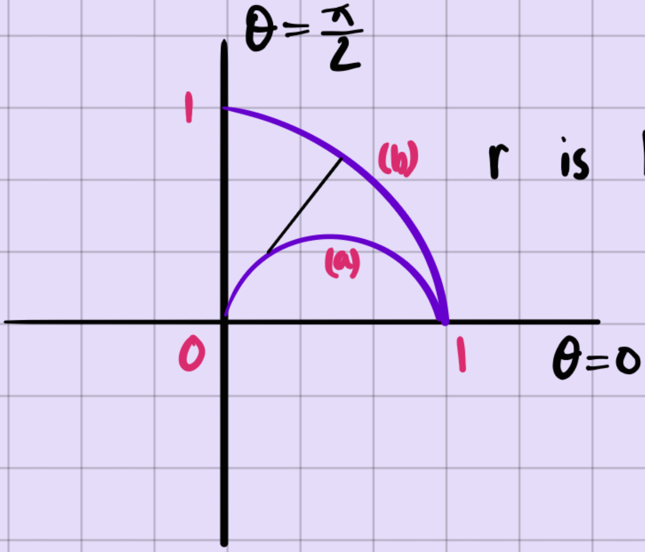
$y^2 + (x-\tfrac{1}{2})^2 = \left(\tfrac{1}{2}\right)^2$



$y = \sqrt{1-x^2}$

$\Rightarrow y^2 + x^2 = 1$

Converting into polar :



r is bounded by (a), (b)

(a) : $y^2 + x^2 - x = 0$

$r^2 - r\cos\theta = 0$

$r(r - \cos\theta) = 0$

$\Rightarrow r = 0$ or $\underline{r = \cos\theta}$

(b) : $y^2 + x^2 = 1$

$r^2 = 1$

$\Rightarrow \underline{r = 1}$ or $\cancel{r=-1}$

$$\int_0^{\frac{\pi}{2}} \int_{\cos\theta}^{1} r^3 \; r \, dr \, d\theta$$

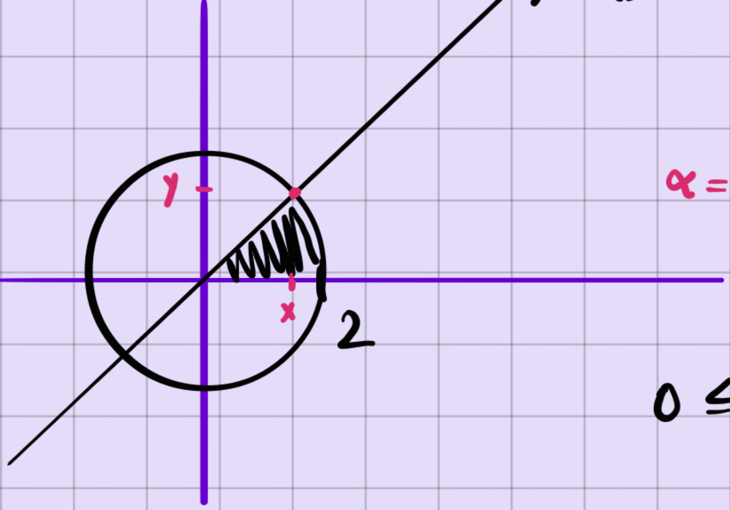
$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{r^5}{5} \Big|_{\cos\theta}^{1} d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{5} - \frac{\cos^5\theta}{5} \, d\theta$$

$$\boxed{= \frac{\pi}{10} - \frac{8}{75}}$$

2) Let $\mathbb{R}$ be the region in the first quadrant between the lines $y=0$, $y = \sqrt{3}x$, and inside the circle $x^2 + y^2 = 4$.

Evaluate $\iint\limits_R xy \; dA$.

$y = \sqrt{3}\,x$



$\alpha = \tan^{-1}\left(\frac{y}{x}\right)$, $x^2 + 3x^2 = 2$

$\Rightarrow x^2 = 2/4$

$\Rightarrow x = \pm\sqrt{\frac{1}{2}}$, $x = \frac{\sqrt{2}}{2}$

$0 \le r \le 2$

$y = \frac{\sqrt{3}\sqrt{2}}{2} = \frac{\sqrt{6}}{2}$

$0 \le \theta \le \alpha$

$\frac{\pi}{3}$

$\frac{y}{x} = \frac{\sqrt{6}}{\cancel{2}} \times \frac{\cancel{2}}{\sqrt{2}}$

$= \sqrt{3}$

$$\int_0^{\frac{\pi}{3}} \int_0^2 r \cdot r\cos\theta \cdot r\sin\theta \; dr\, d\theta$$

$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$$\Rightarrow \int_0^{\frac{\pi}{3}} \frac{r^4}{4} \cos\theta\sin\theta \Big|_{r=0}^{r=2} d\theta$$

$$\Rightarrow 2\int_0^{\pi/3} \sin 2\theta \; d\theta \Rightarrow \cancel{2} \cdot -\frac{\cos 2\theta}{\cancel{2}} \Big|_{\theta=0}^{\theta=\frac{\pi}{3}}$$

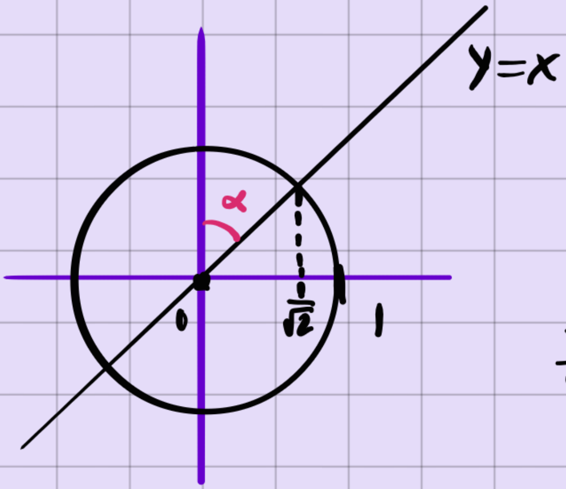
$$\Rightarrow -\left(-\frac{1}{2} - 1\right) = \frac{3}{2}$$

3) $$\int_0^{\frac{1}{\sqrt{2}}} \int_x^{\sqrt{1-x^2}} 3\sqrt{x^2+y^2} \; dy\, dx$$

$x \le y \le \sqrt{1-x^2}$ $\qquad 0 \le x \le$

$x \le y \le \sqrt{1-x^2} \quad , \quad 0 \le x \le \frac{1}{\sqrt{2}}$

$y = x \quad , \quad x^2 + y^2 = 1 \qquad x = 0 , \quad x = \frac{1}{\sqrt{2}}$



$\alpha = \pi/4 = 45°$

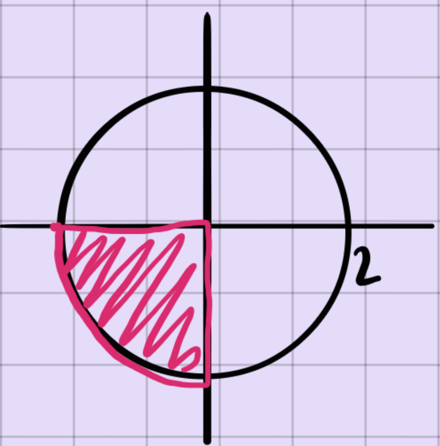
$\frac{\pi}{4} \le \theta \le \frac{\pi}{2} \quad , \quad 0 \le r \le 1$

$$\int_0^{\frac{\pi}{4}} \int_0^1 r \cdot 3 \cdot r \quad dr \quad d\theta$$

$$\Rightarrow \int_0^{\pi/4} r^3 \Big|_{r=0}^{r=\frac{1}{\sqrt{2}}} d\theta = 1 \cdot \frac{\pi}{4} = \frac{\pi}{4}$$

4) $\iint_R \cos(x^2+y^2)\, dA$

R: Circle of radius 2 centered at (0,0), in 3rd quadrant.



$0 \le r \le 2 \quad , \quad \pi \le \theta \le \frac{3\pi}{2}$

$2 \quad \frac{3\pi}{2}$

$$\Rightarrow \int_{0} \int_{\pi} r\cos(r^2)\, d\theta\, dr$$

$$\Rightarrow \int_0^2 r\cos(r^2) \cdot \frac{\pi}{2}\, dr = \frac{\pi}{2}\int_0^2 r\cos(r^2)\, dr$$

$$u = r^2 \Rightarrow du = 2r\,dr$$

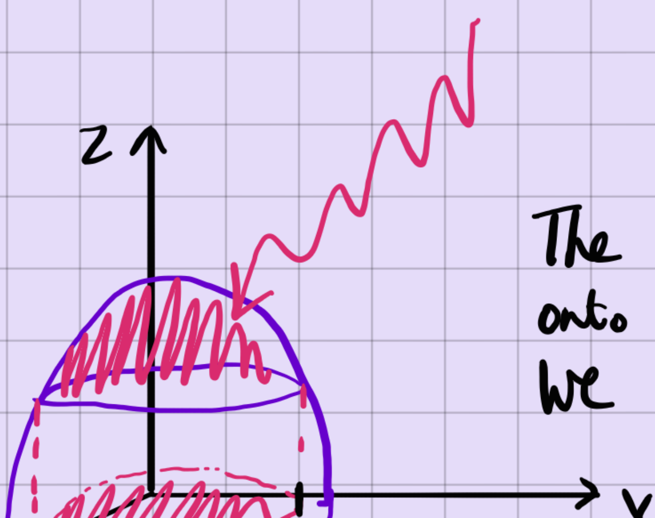
$$\Rightarrow \frac{\pi}{4}\int_0^4 2r\cos(u)\, dr = \frac{\pi}{4}\int_0^4 \cos(u)\, du$$

$$= \frac{\pi}{4}\sin(4) - \frac{\pi}{4}\sin(0)$$

$$= \frac{\pi}{4}\sin(4)$$

5) Find the volume of the solid bounded above by $z = 4 - x^2 - y^2$ and below by $z = 3$.

$$\Rightarrow \quad 3 \leq z \leq 4 - x^2 - y^2$$



The region projects a shadow onto the xy plane as a circle. We integrate above that region.

$$3 = 4 - x^2 - y^2$$

$$\Rightarrow x^2 + y^2 = 1$$

$$\hookrightarrow \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$$

Now integrate the top - bottom of $f(x,y)$ over $\mathbb{R}$

$$\Rightarrow \int_0^{2\pi} \int_0^1 r \cdot (4 - \underbrace{x^2 - y^2}_{r^2} - 3) \, dr \, d\theta$$

$$\Rightarrow \int_0^{2\pi} \int_0^1 \underbrace{r \cdot (1 - r^2)}_{r - r^3} \, dr \, d\theta$$

$$\Rightarrow \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^4}{4} \right) \Big|_0^1 \, d\theta$$

$$\Rightarrow \int_0^{2\pi} \frac{1}{4} \, d\theta \Rightarrow \frac{\pi}{2}$$

6) Find the volume of the solid bounded by $z = f(x,y)$ and the xy plane. $z = 29 - \sqrt{x^2 + y^2}$

On the xy plane, $z = 0$.

$$29 - \sqrt{x^2 + y^2} = 0 \Rightarrow x^2 + y^2 = 29^2 \Rightarrow r = 29$$

$0 \le r \le 29$, $0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^{29} (29-r)r \, dr \, d\theta = \int_0^{2\pi}\int_0^{29} 29r - r^2 \, dr \, d\theta$$

$$\Rightarrow \int_0^{2\pi} \left(\frac{29r^2}{2} - \frac{r^3}{3}\right)\Big|_0^{29} d\theta$$

$$\Rightarrow V = 2\pi \cdot \frac{24389}{6} \Rightarrow V = \frac{24389\pi}{3}$$

7) $z = \dfrac{68}{1+x^2+y^2} - 4$. Find volume between $z=0$ and the function $f(x,y) = z$.

$z = 0$: $x^2+y^2 = 16 \Rightarrow r = 4$

$0 \le r \le 4$, $0 \le \theta \le 2\pi$

$$\int_0^4\int_0^{2\pi} r \cdot \left(\frac{68}{1+r^2} - 4\right) d\theta \, dr$$

$$\Rightarrow 2\pi \int_0^4 \frac{34 \cdot 2r}{1+r^2} dr - 2\pi \int_0^4 4r \, dr$$

$u = 1+r^2 \Rightarrow du = 2r\,dr$

$$2\pi \int_1^{17} \frac{34}{u}\, du - 2\pi \left[2r^2\right]_0^4$$

$$2\pi \left[34 \ln(u)\right]_1^{17} - 2\pi \left[2(4^2) - 0\right]$$

$$2\pi \left[34 \ln(17)\right] - 2\pi \left[32\right] \Rightarrow V = 2\pi \left(34 \ln(17) - 32\right)$$

8) Volume of the solid bounded by $z = 32 - 8x^2 - 8y^2$ and $z = 0$.

$$32 - 8x^2 - 8y^2 = 0$$

$$32 = 8(x^2 + y^2) \Rightarrow x^2 + y^2 = r^2 = 4$$

$$\Rightarrow \underline{\underline{r = 2}}$$

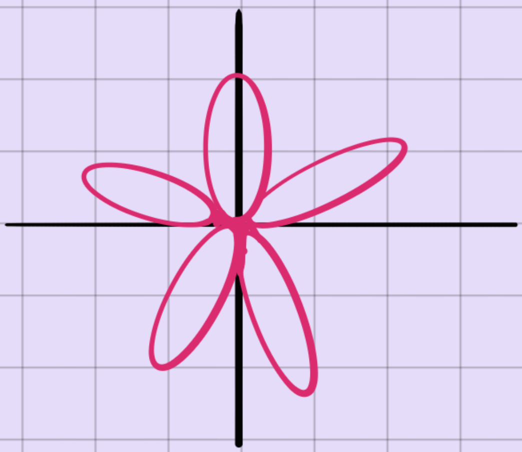
$$0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi$$

$$\int_0^{2\pi} \int_0^{2} r \cdot (32 - 8r^2)\, dr\, d\theta$$

$$\Rightarrow \int_0^{2\pi} \left(16r^2 - 2r^4\right)\Big|_0^2 \, d\theta \Rightarrow 32 \cdot 2\pi \Rightarrow V = 64\pi$$

Some other notes about polar curves:

Say $r = \sin(5\theta)$, for $\theta \in [0, 2\pi]$.



Given $\sin(a\theta)$ or $\cos(a\theta)$:

if a is odd, then the number of petals is a.

if a is even, then the number of petals is $2a$.

If it is $r = \cos(a\theta)$, the graph is symmetrical about x-axis.
If it is $r = \sin(a\theta)$, the graph is symmetrical about y-axis.

To find the area of a single petal with double integrals:

$r_1 = a\sin(k\theta)$ or $r_1 = a\cos(k\theta)$

$\Rightarrow 0 \le \theta \le \frac{\pi}{ak}$, $0 \le r \le r_1$

To find total area:

$r_1 = a\sin(k\theta)$ or $r_1 = a\cos(k\theta)$

$k \in 2b$ (even), multiply area of one petal by $2k$.

$k \in 2b+1$ (odd), multiply area of one petal by k.

---

9) Use a double integral to find the area of the region bounded by $r = \sin(5\theta)$.

$0 \le r \le \sin(5\theta)$, $0 \le \theta \le \frac{\pi}{5}$ [For $\frac{1}{5}^{th}$ A]

$$\iint dA \Rightarrow \int_0^{\frac{\pi}{5}} \int_0^{\sin(5\theta)} r \; dr \; d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{5}} \frac{\sin^2(5\theta)}{2} \; d\theta$$

$\cos(2x) = 1 - 2\sin^2(x)$

$\sin^2(x) = \frac{1-\cos(2x)}{2}$

$$\Rightarrow \int_0^{\frac{\pi}{5}} \frac{1-\cos(10\theta)}{4} \; d\theta$$

$$\Rightarrow \frac{1}{4}\left[\theta - \frac{\sin(10\theta)}{10}\right]_0^{\pi/5}$$

$$\Rightarrow \frac{1}{4}\left[\frac{\pi}{5} - \frac{\sin(2\pi)}{10} - (0-0)\right]$$

$$\Rightarrow \frac{1}{4}\left[\frac{\pi}{5}\right] \Rightarrow \frac{\pi}{20} \quad \text{(Area of one petal)}$$

$\hookrightarrow$ there are 5.

$$\text{Total area} = \frac{\pi}{20} \cdot 5 = \frac{\pi}{4}$$

10) $$\iint_R \frac{1}{4+\sqrt{x^2+y^2}} \, dA \quad , \quad R : (r,\theta) \in [0,2] \times [\tfrac{\pi}{2}, \pi]$$

$$\int_{\frac{\pi}{2}}^{\pi} \int_0^2 \frac{r}{4+r} \, dr \, d\theta \quad \Rightarrow I = \int_0^2 \frac{r}{4+r} \, dr$$

$$u = r \qquad dv = \frac{1}{4+r}$$
$$du = 1 \qquad v = \ln(4+r)$$

$$I = r\ln(4+r)\Big|_0^2 - \int_0^2 \ln(4+r) \, dr$$

$$= 2\ln(6) - \Big[(4+r)\ln(4+r) - (4+r)\Big]_0^2$$

$$= 2\ln(6) - \big[6\ln(6) - 6 - (4\ln(4) - 4)\big]$$

$$= 2\ln(6) - \big[6\ln(6) - 4\ln(4) - 2\big]$$

$$= -4\ln(6) + 4\ln(4) + 2$$

$$= 4\big[\ln(4) - \ln(6)\big] + 2$$

$$I = 4\ln\left(\frac{2}{3}\right) + 2$$

$$\int_{\frac{\pi}{2}}^{\pi} I \, d\theta \Rightarrow \frac{\pi}{2} I \Rightarrow 2\ln\left(\frac{2}{3}\right)\pi + \pi$$

11) $$\iint_R \frac{dA}{(1+x^2+y^2)^2} \qquad R : (r,\theta) \in [1,4] \times \left[\frac{\pi}{2}, \pi\right]$$

$$\int_{\frac{\pi}{2}}^{\pi} \int_1^4 \frac{r}{(1+r^2)^2} \, dr \, d\theta$$

$$I = \frac{1}{2}\int_1^4 \frac{2r}{(1+r^2)^2} \, dr \qquad u = 1+r^2, \; du = 2r\,dr$$

$$= \frac{1}{2}\int_2^{17} \frac{du}{u^2} = \left.\frac{-1}{u}\right|_2^{17} \cdot \frac{1}{2}$$

$$= \left(\frac{-1}{17} + \frac{1}{2}\right)\frac{1}{2}$$

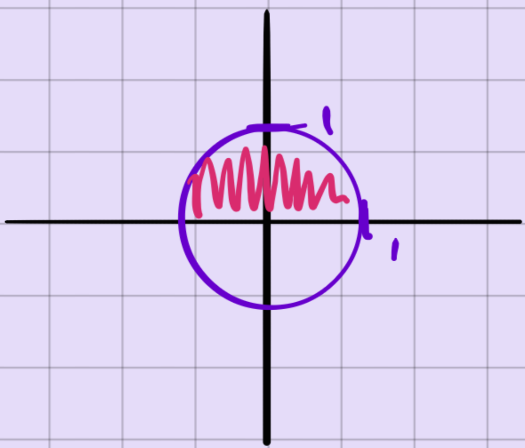
$$= \frac{15}{68}$$

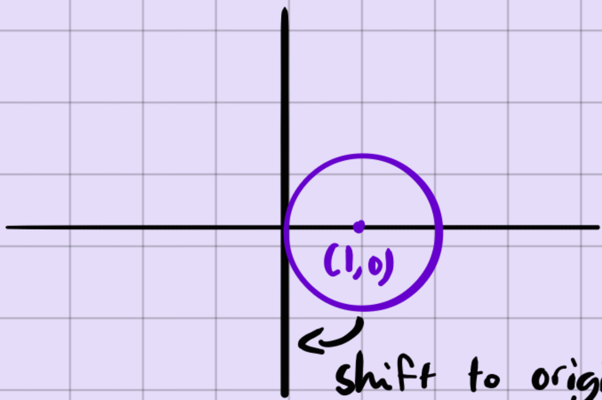
$$\int_{\frac{\pi}{2}}^{\pi} \frac{15}{68} \, d\theta = \frac{15}{68} \cdot \frac{\pi}{2} = \frac{15}{136}\pi$$

$$\int \quad 68 \qquad\qquad 68 \quad 2 \qquad\qquad 136$$
$\frac{\pi}{2}$

12) $\displaystyle\iint_D y \, dA$ where D is inside the disk $x^2+y^2=2x$ and above x-axis.

$\hookrightarrow \quad x^2-2x+y^2=0$
$\Rightarrow (x-1)^2+y^2=1$



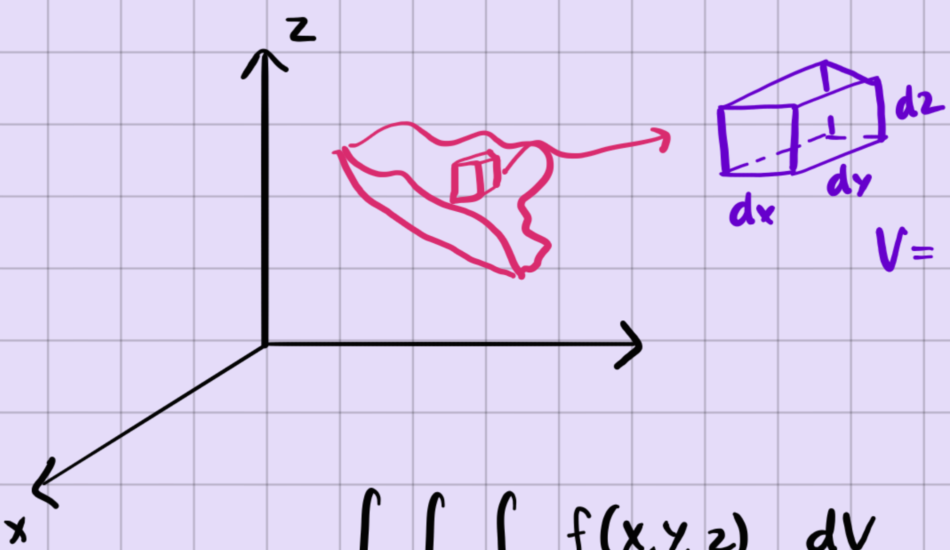


$0 \le \theta \le \pi$
$0 \le r \le 1$

$$\int_0^{\pi}\int_0^{1} r^2 \sin\theta \, dr \, d\theta$$

$$\Rightarrow \quad -\frac{\cos\theta}{3}\Big|_0^{\pi} = \frac{1}{3}+\frac{1}{3}$$
$$= \frac{2}{3}$$

Triple integrals :



$V = dx\,dy\,dz$ (any order)
↳ 6 perms.

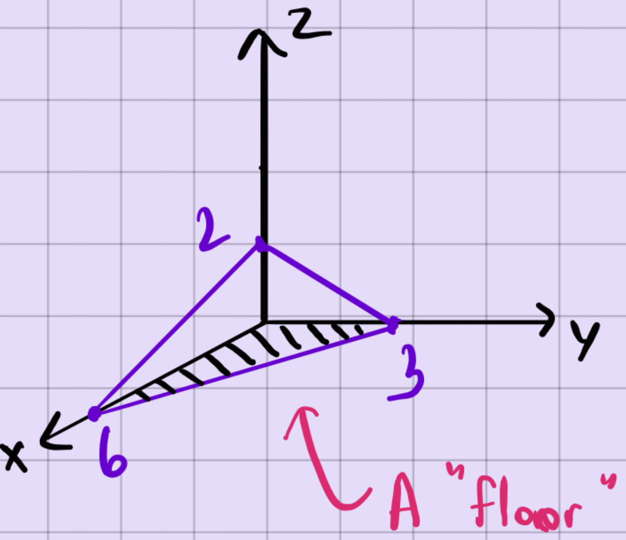
$$\iiint f(x,y,z)\ dV$$

1) Use a triple integral to calculate the volume of the solid under $x+2y+3z=6$ in the first octant.

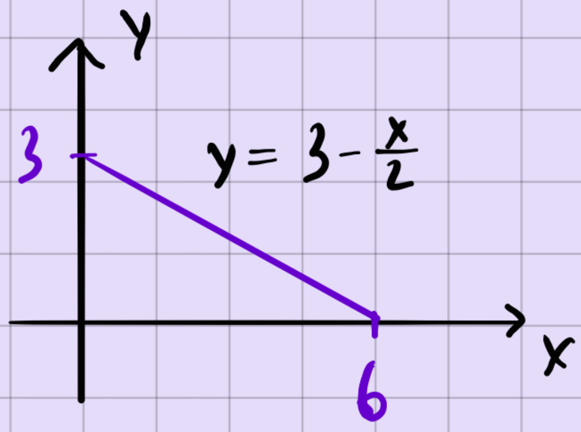
$f(x,y,z) = x+2y+3z-6$

In the first octant, $x \geq 0$, $y \geq 0$, $z \geq 0$.

$$\left.\begin{aligned} &\Rightarrow x = y = 0 : z = 2 \\ &\Rightarrow x = z = 0 : y = 3 \\ &\Rightarrow y = z = 0 : x = 6 \end{aligned}\right] \text{Intercepts}$$



$\Longrightarrow$



So :

$$0 \le x \le 6, \quad 0 \le y \le 3 - \frac{x}{2}, \quad 0 \le z \le 2 - \frac{x}{3} - \frac{2y}{3}$$

$$V = \int_0^6 \int_0^{3-\frac{x}{2}} \int_0^{2-\frac{x}{3}-\frac{2y}{3}} 1 \, dz\, dy\, dx$$

$$= \int_0^6 \int_0^{3-\frac{x}{2}} 2 - \frac{x}{3} - \frac{2y}{3} \, dy\, dx$$

$$= \int_0^6 2 - \frac{x}{3} - \frac{y^2}{3} \Bigg|_{y=0}^{y=3-\frac{x}{2}} dx$$

$$= \int_0^6 2 - \frac{x}{3} - \left(3 - \frac{x}{2}\right)^2 \cdot \frac{1}{3} \, dx \quad = \underline{\underline{6}}$$

2) Given the integral for volume

$$V = \int_0^6 \int_0^{3-\frac{x}{2}} \int_0^{2-\frac{x}{3}-\frac{2y}{3}} dz\, dy\, dx,$$

Convert it into $dx\, dy\, dz$.

Given :

$0 \le x \le 6$, $0 \le y \le 3 - \frac{x}{2}$, $0 \le z \le 2 - \frac{x}{3} - \frac{2y}{3}$

Least complex ones $\Rightarrow$ used xy plane as "floor".

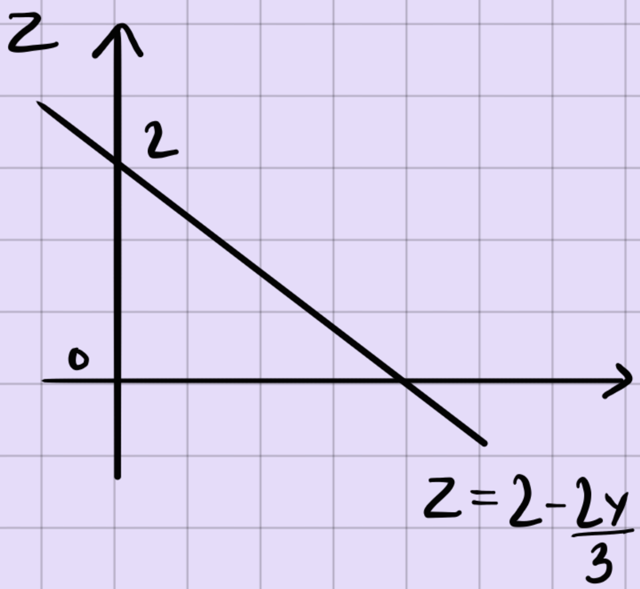
We need in $dx\, \underbrace{dy\ dz}$

↳ Use yz plane

↳ $\underline{\underline{x = 0}}$

$$z = 2 - \frac{x}{3} - \frac{2y}{3}$$

$$\Rightarrow z = 2 - \frac{2y}{3} \Rightarrow y = 3 - \frac{3z}{2}$$



$0 \le z \le 2$

$0 \le y \le 3 - \frac{3z}{2}$

$$z = \frac{6 - x - 2y}{3} \Rightarrow x = 6 - 3z - 2y$$

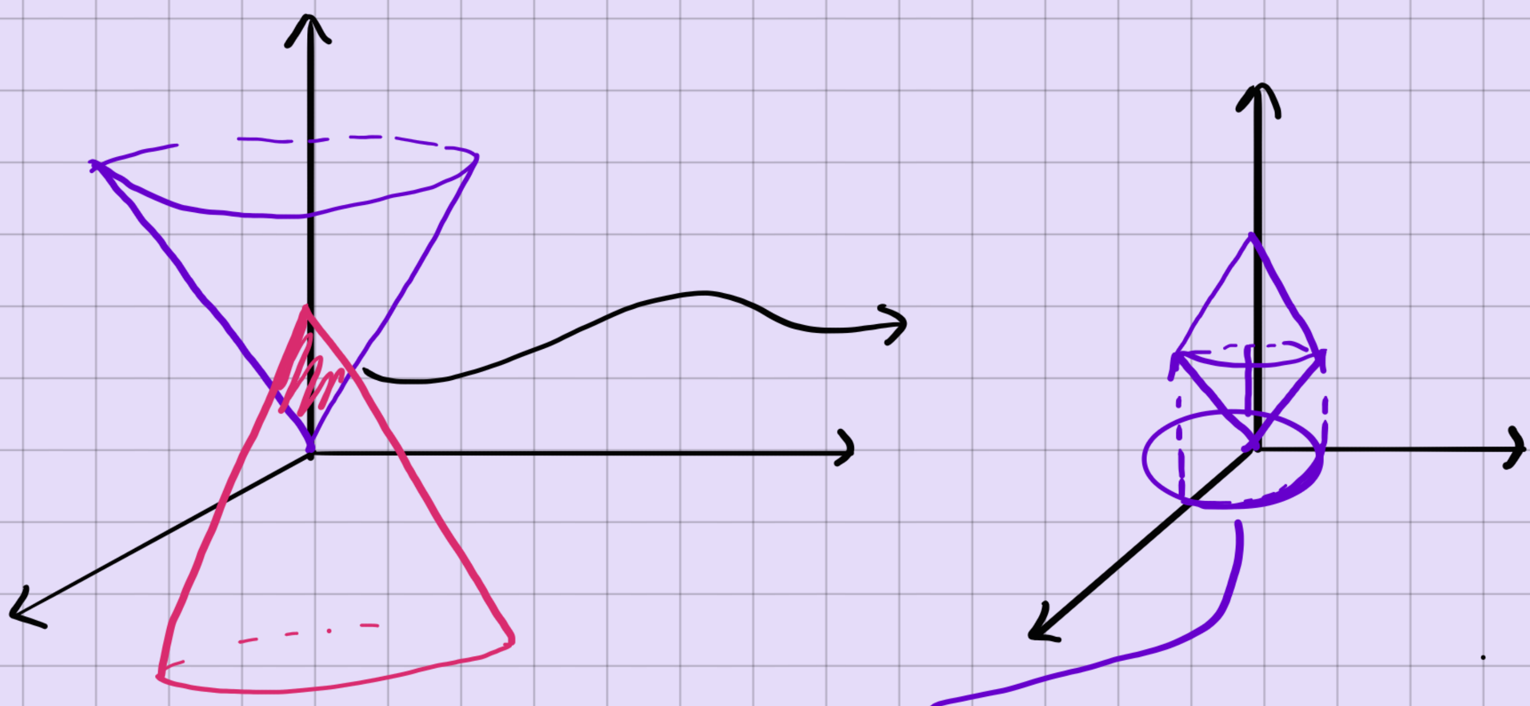
$0 \le x \le 6 - 3z - 2y$

Overall :

$$\int_0^2 \int_0^{3-\frac{3z}{2}} \int_0^{6-3z-2y} dx\, dy\, dz$$

---

1) Find volume above $z = \sqrt{x^2+y^2}$ and below $z = 2 - \sqrt{x^2+y^2}$.



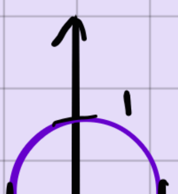
$$2 - \sqrt{x^2+y^2} = \sqrt{x^2+y^2}$$

$$x^2+y^2 = 1$$

$\hookrightarrow r = 1$

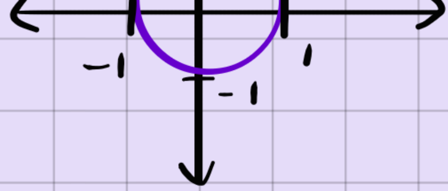
$-1 \leq x \leq 1$

$-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$



$\sqrt{x^2+y^2} \leq z \leq 2 - \sqrt{x^2+y^2}$

$$\sqrt{x^2+y^2} \le z \le 2-\sqrt{x^2+y^2}$$

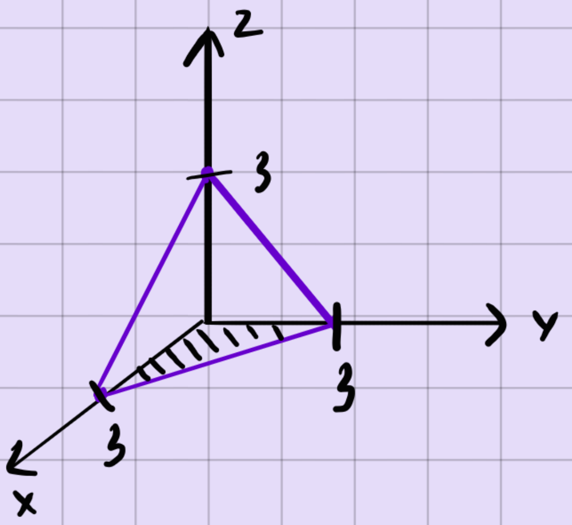


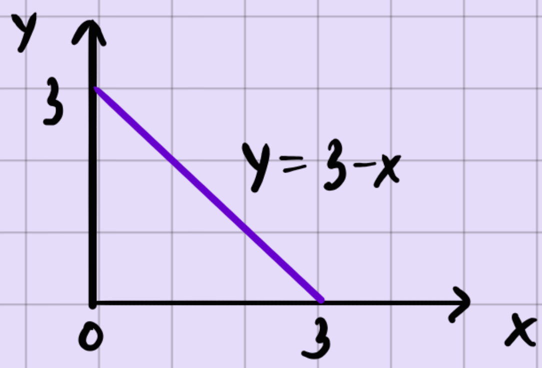
$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{2-\sqrt{x^2+y^2}} dz\ dy\ dx$$

2) Find volume of the region in first octet bounded by the plane $9x+9y+9z=27$.

$$9x+9y+9z = 27$$

$\hookrightarrow$ $x = y = z = 3$.





$$0 \le x \le 3$$
$$0 \le y \le 3-x$$
$$0 \le z \le 3-x-y$$

$$V = \int_0^3 \int_0^{3-x} \int_0^{3-x-y} dz\ dy\ dx = \frac{9}{2}$$

3) Find volume of the solid bounded by surfaces $z = e^y$ and $z = 1$ over $(x,y) \in [0,1] \times [0, \ln 2]$

$0 \le x \le 1 \quad , \quad 0 \le y \le \ln 2 \quad , \quad 1 \le z \le e^y$

$$V = \int_0^1 \int_0^{\ln 2} \int_1^{e^y} dz \; dy \; dx$$

$$V = \int_0^1 \int_0^{\ln 2} e^y - 1 \quad dy \; dx$$

$$= \int_0^1 e^y - y \Big|_{y=0}^{y=\ln 2}$$

$$= \int_0^1 2 - \ln 2 - (1-0) \; dx \; = \int_0^1 1 - \ln 2 \; dx$$

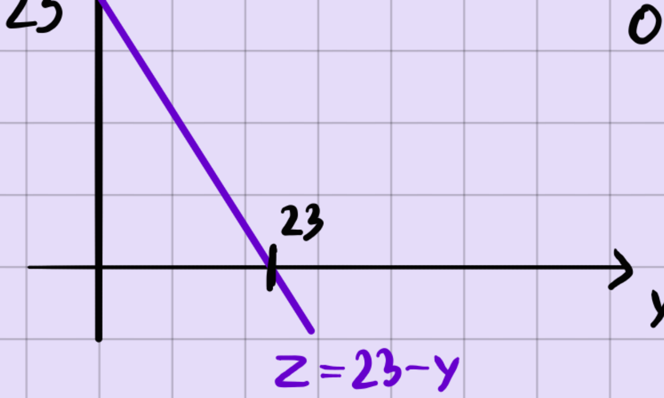
$$V = 1 - \ln 2$$

4) Volume bounded by $\underbrace{y = x^2}$ and $z = 23 - y$ and $z = 0$.

↳ Parabolic cylinder

$z = 23 - y$

$0 \le y \le 23$



$0 \le z \le 23 - y$

$y = x^2 \Rightarrow x = \pm\sqrt{y}$

$\Rightarrow -\sqrt{y} \le x \le \sqrt{y}$

$$V = \int_0^{23} \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{23-y} dz \; dx \; dy$$

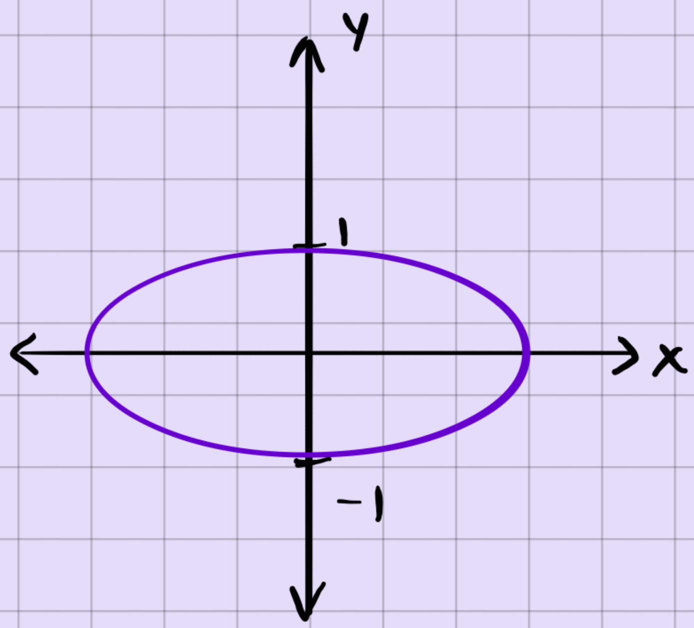
$$= \int_0^{23} \int_{-\sqrt{y}}^{\sqrt{y}} 23 - y \; dx \; dy$$

$$= \int_0^{23} 2\sqrt{y}\,(23-y)\; dy$$

$$= \int_0^{23} 46\, y^{1/2} - 2y^{3/2} \; dy = \frac{4232\sqrt{23}}{15}$$

5) Volume of the solid bounded by the cylinder $x^2 + 25y^2 = 25$, and the planes $z = 4 - x$ and $z = x - 4$.

$$x^2 + (5y)^2 = 5^2 \Rightarrow x = \pm\sqrt{25 - 25y^2}$$



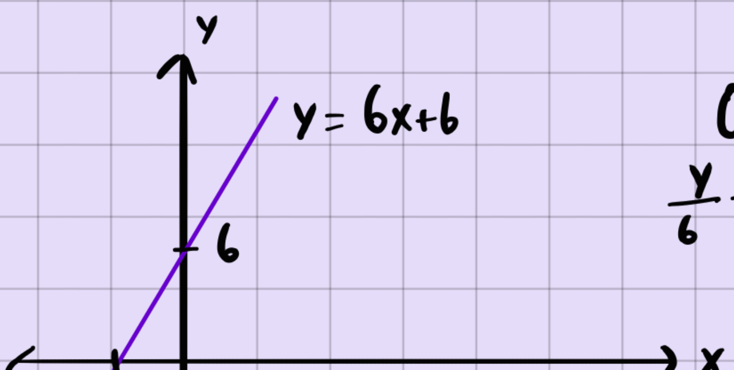
$$-1 \le y \le 1$$
$$y - x \le z \le x - y$$

$$V = \int_{-1}^{1} \int_{-\sqrt{25-25y^2}}^{\sqrt{25-25y^2}} \int_{y-x}^{x-y} dz\, dx\, dy = 40\pi$$

6) Rewrite the following

$$\int_0^7 \int_{-1}^{0} \int_0^{6x+6} dy\, dx\, dz \quad \text{into} \quad dz\, dx\, dy.$$

$0 \le z \le 7$, $-1 \le x \le 0$, $0 \le y \le 6x+6$



$$0 \le y \le 6$$
$$\frac{y}{6} - 1 \le x \le 0$$

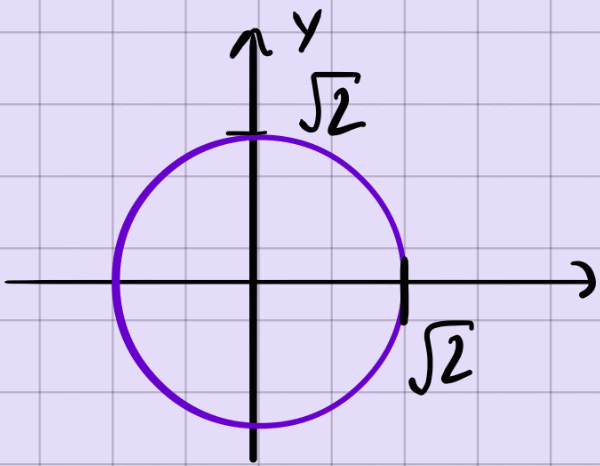
$$0 \le z \le 7$$

$$V = \int_0^6 \int_{\frac{y}{6}-1}^{0} \int_0^7 dz\, dx\, dy$$

7) Volume between $z = 4 - x^2 - y^2$ and $z = x^2 + y^2$.

$$z = z :$$

$$\Rightarrow 4 - x^2 - y^2 = x^2 + y^2 \Rightarrow x^2 + y^2 = (\sqrt{2})^2$$



$$-\sqrt{2} \le x \le \sqrt{2}$$

$$-\sqrt{2 - x^2} \le y \le \sqrt{2 - x^2}$$

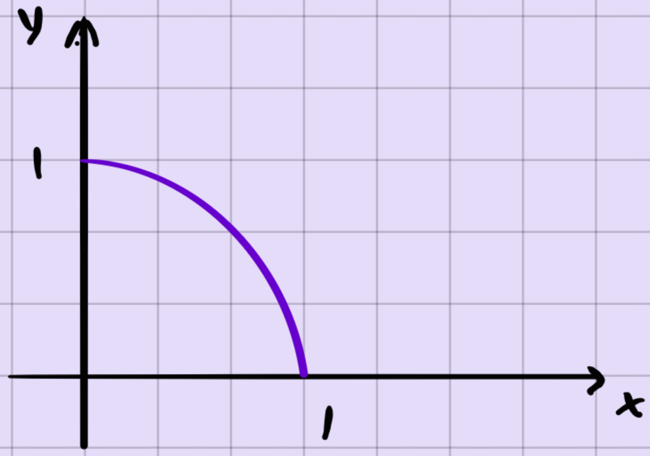
$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} dz\, dy\, dx$$

$$= 4\pi$$

8) Evaluate $\iiint_E z \, dV$ where E is the

region in the first octet bounded by $x^2+y^2+z^2=1$.

Take xy trace, $z=0$:



$0 \le x \le 1$

$0 \le y \le \sqrt{1-x^2}$

$0 \le z \le \sqrt{1-x^2-y^2}$

$$V = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} z \; dz \; dy \; dx$$

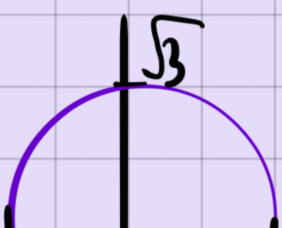
$$= \frac{\pi}{16}.$$

9) V is the volume bounded by the surfaces $z=6-x^2-y^2$ and $z=x^2+y^2$. Find the triple integral in form dz dy dx.

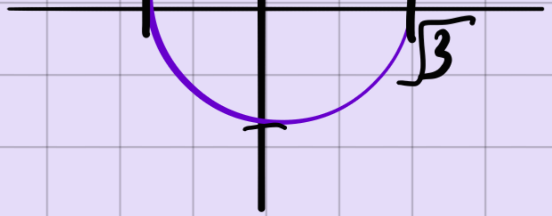
$z = z$:

$6-x^2-y^2 = x^2+y^2$

$\Rightarrow x^2+y^2=3$



$-\sqrt{3} \le x \le \sqrt{3}$



$$-\sqrt{3-x^2} \le y \le \sqrt{3-x^2}$$

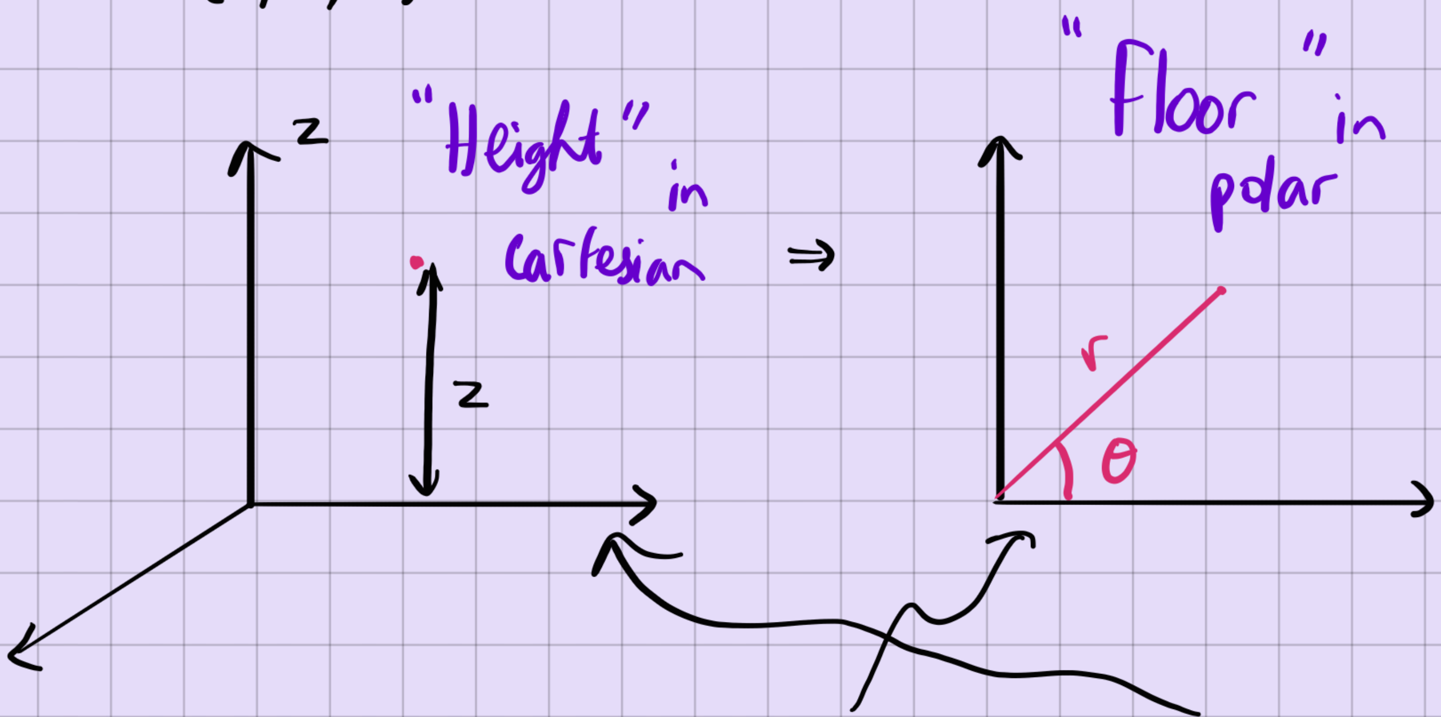
$$x^2+y^2 \le z \le 6-x^2-y^2$$

$$V = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{x^2+y^2}^{6-x^2-y^2} dz\, dy\, dx$$

Cylindrical triple integrals :

Triple integrals for questions including $\sqrt{x^2+y^2}$ type-shi

↳ $f(r, \theta, z)$



⇒ Hybrid of polar and cartesian.

Conversion between cylindrical (r, θ, z) and cartesian (x, y, z).

↳ $x = r\cos\theta$, $y = r\sin\theta$, $\frac{y}{x} = \tan\theta$

$$r^2 = x^2 + y^2$$

1) Evaluate

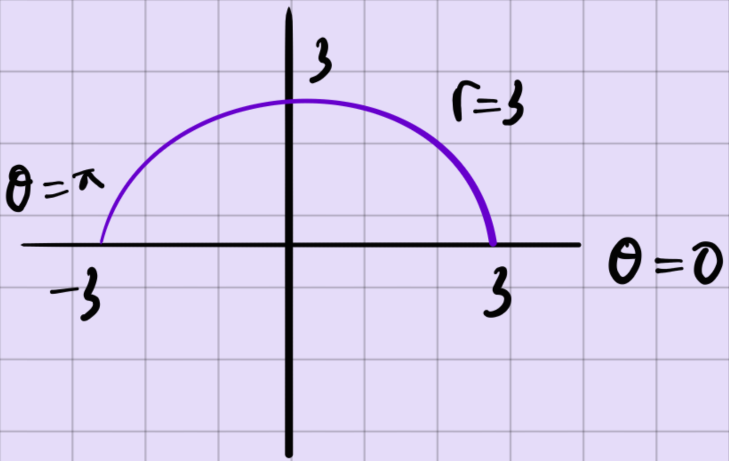
$$\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\, dz\, dy\, dx$$

$$r\, dV = r \cdot r\, \underline{dr\, d\theta\, dz}$$

Given :

$$-3 \le x \le 3 , \quad 0 \le y \le \sqrt{9-x^2}, \quad 0 \le z \le 9-x^2-y^2$$

$$9-r^2$$



$$0 \le r \le 3$$
$$0 \le \theta \le \pi$$
$$0 \le z \le 9-r^2$$

$$\int_{0}^{3} \int_{0}^{\pi} \int_{0}^{9-r^2} r^2\, dz\, d\theta\, dr$$

$$\int_{0}^{3} \int_{0}^{\pi} r^2 (9-r^2)\, d\theta\, dr$$

$$\int_0^3 \pi\,(9r^2 - r^4)\,dr = 3r^3 - \frac{r^5}{5}\Big|_0^3 \cdot \pi$$

$$= 81 - \frac{243}{5} \cdot \pi$$

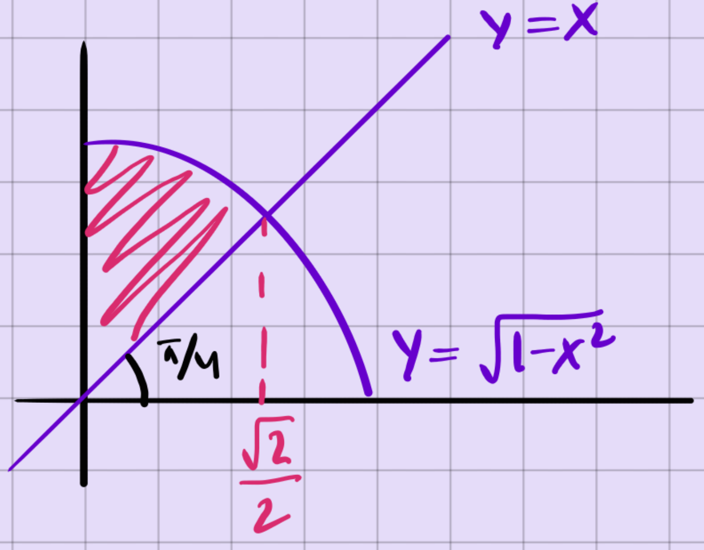
$$= \frac{162\pi}{5}$$

2) Evaluate :

$$\int_0^4 \int_0^{\frac{1}{\sqrt{2}}} \int_x^{\sqrt{1-x^2}} e^{-x^2-y^2}\,dy\,dx\,dz$$

Given :

$0 \le x \le \frac{\sqrt{2}}{2}$, $x \le y \le \sqrt{1-x^2}$, $0 \le z \le 4$



$\frac{\pi}{4} \le \theta \le \pi/2$

$0 \le r \le 1$

$$\int^1 \int^{\frac{\pi}{2}} \int^4 e^{-r^2}\,r\,dz\,d\theta\,dr$$

$$\int_0 \int_{\frac{\pi}{4}} \int_0$$

$$\Rightarrow \int_0^1 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4e^{-r^2} r \, d\theta \, dr$$

$$\Rightarrow \frac{1}{2}\int_0^1 -\pi e^{-r^2} 2r \, dr \qquad u = -r^2$$
$$du = -2r \, dr$$

$$\Rightarrow \frac{\pi}{2} \int_{r=0}^{r=1} -e^u \, du \qquad \Rightarrow -\frac{\pi}{2} e^{-r^2}\Big|_0^1$$

$$\Rightarrow \frac{-\pi}{2e} + \frac{\pi}{2}$$

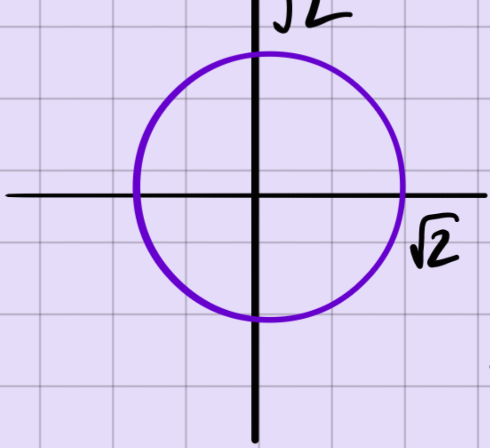
$$\Rightarrow \frac{\pi}{2} - \frac{\pi}{2e}$$

3) Find the mass of the solid bounded above by $x^2+y^2+z^2=4$ and below by $z=\sqrt{x^2+y^2}$, with density $\rho(x,y,z)=z$.

$$\Rightarrow x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2 = 4$$

$$\Rightarrow x^2+y^2 = 2$$

$$\int^{\sqrt{2}}$$



$$0 \le r \le \sqrt{2}$$

$$0 \le \theta \le 2\pi$$

$$\sqrt{x^2+y^2} \le z \le \sqrt{4-x^2-y^2}$$

$$V = \int_0^{\sqrt{2}} \int_0^{2\pi} \int_r^{\sqrt{4-r^2}} z\,r \; dz \; d\theta \; dr$$

accumulate density · dV = mass

$$= \int_0^{\sqrt{2}} \int_0^{2\pi} \frac{z^2}{2} r \Big|_{z=r}^{z=\sqrt{4-r^2}} \; d\theta \; dr$$

$$= \int_0^{\sqrt{2}} \int_0^{2\pi} 2r - r^3 \; d\theta \; dr$$

$$= 2\pi \int_0^{\sqrt{2}} 2r - r^3 \; dr$$

$$= 2\pi \left[r^2 - \frac{r^4}{4} \right]_0^{\sqrt{2}} = 2\pi \left[2 - 1 \right] = \underline{\underline{2\pi}}$$

General overview:

1) Converting from cartesian to cylindrical or other way around.

a) Cartesian to cylindrical:

$$f(x,y,z) \rightarrow f(r,\theta,z)$$

Plot the xy axis and convert into (r,θ).

b) Cylindrical to Cartesian

$$f(r,\theta,z) \rightarrow f(x,y,z)$$

Plot the r,θ and convert to (x,y)

2) Finding mass given density function.

↳ Integrate with $\rho(x,y,z)$ or $\rho(r,\theta,z)$ as the integrand.

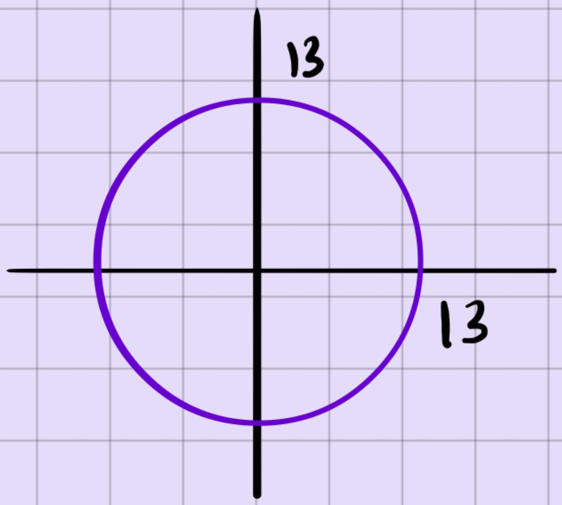
$$\iiint_{\mathbb{R}} \rho(x,y,z)\, dx\, dy\, dz \quad \text{or} \quad \iiint_{\mathbb{R}} r\cdot\rho(r,\theta,z)\, dr\, d\theta\, dz$$

---

1) Volume bounded below by $z=\sqrt{x^2+y^2}$ and above by $x^2+y^2+z^2=338$.

$\Rightarrow x^2 + y^2 + \left(\sqrt{x^2+y^2}\right)^2 = 338$

$\Rightarrow x^2+y^2 = 169$



$0 \le r \le 13$

$0 \le \theta \le 2\pi$

$\sqrt{x^2+y^2} \le z \le \sqrt{338-x^2-y^2}$

$$V = \int_0^{13} \int_0^{2\pi} \int_r^{\sqrt{338-r^2}} r \, dz \, d\theta \, dr$$

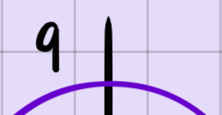
$$= 338^{3/2} \cdot \frac{2\pi}{3}\left[1 - \frac{\sqrt{2}}{2}\right]$$

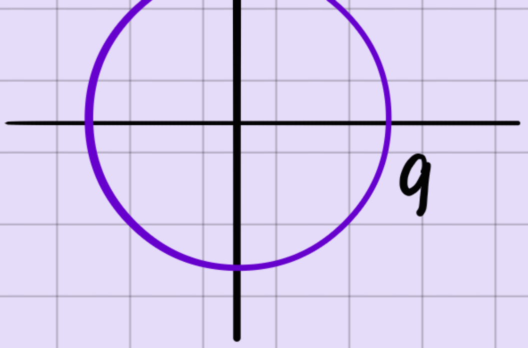
2) Volume bounded by $z=0$ and $z = \sqrt{82} - \sqrt{1+x^2+y^2}$

$z = 0$

$\Rightarrow \sqrt{82} = \sqrt{1+x^2+y^2}$

$\Rightarrow x^2+y^2 = 81$





$$0 \le r \le 9$$
$$0 \le \theta \le 2\pi$$

$$0 \le z \le \sqrt{82} - \sqrt{1+r^2}$$

$$\int_0^9 \int_0^{2\pi} \int_0^{\sqrt{82}-\sqrt{1+r^2}} r \; dz \; d\theta \; dr$$

$$\Rightarrow \int_0^9 \int_0^{2\pi} \sqrt{82}\, r - r\sqrt{1+r^2} \; d\theta \; dr$$

$$\Rightarrow \int_0^9 2\pi\sqrt{82}\, r - 2\pi r\sqrt{1+r^2} \; dr$$

$$\Rightarrow \left.\frac{2\pi\sqrt{82}\, r^2}{2}\right|_0^9 - \pi \int_0^9 2r\sqrt{1+r^2} \; dr$$

$$\Rightarrow \pi\sqrt{82} \cdot 81 - \pi \int_1^{82} u^{1/2} \; du$$

$$\Rightarrow 81\pi\sqrt{82} - \pi\left[\frac{2}{3} u^{3/2}\right]_1^{82}$$

$$\Rightarrow 81\pi\sqrt{82} - \pi\left[\frac{2}{3}\left[82^{1.5} - 1\right]\right]$$

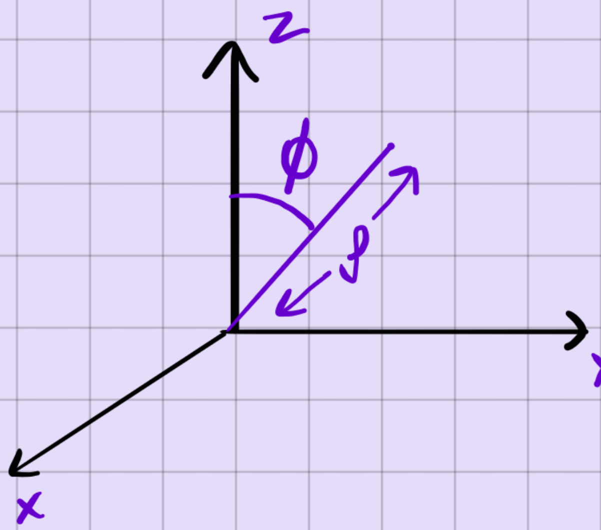
Spherical coordinates and triple integrals :

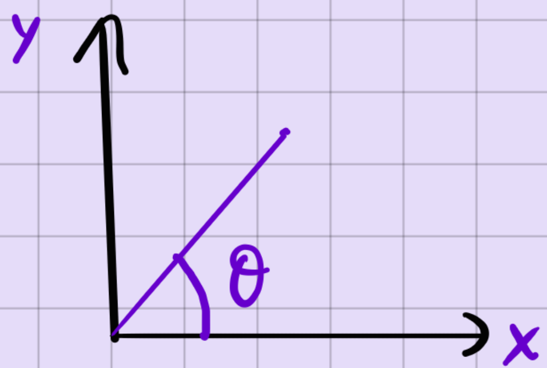
⇒ $f(\rho, \phi, \theta)$

⇒ ρ is the distance from origin to point

⇒ ϕ is the angle from the z-axis down to the line connecting the point and the origin.

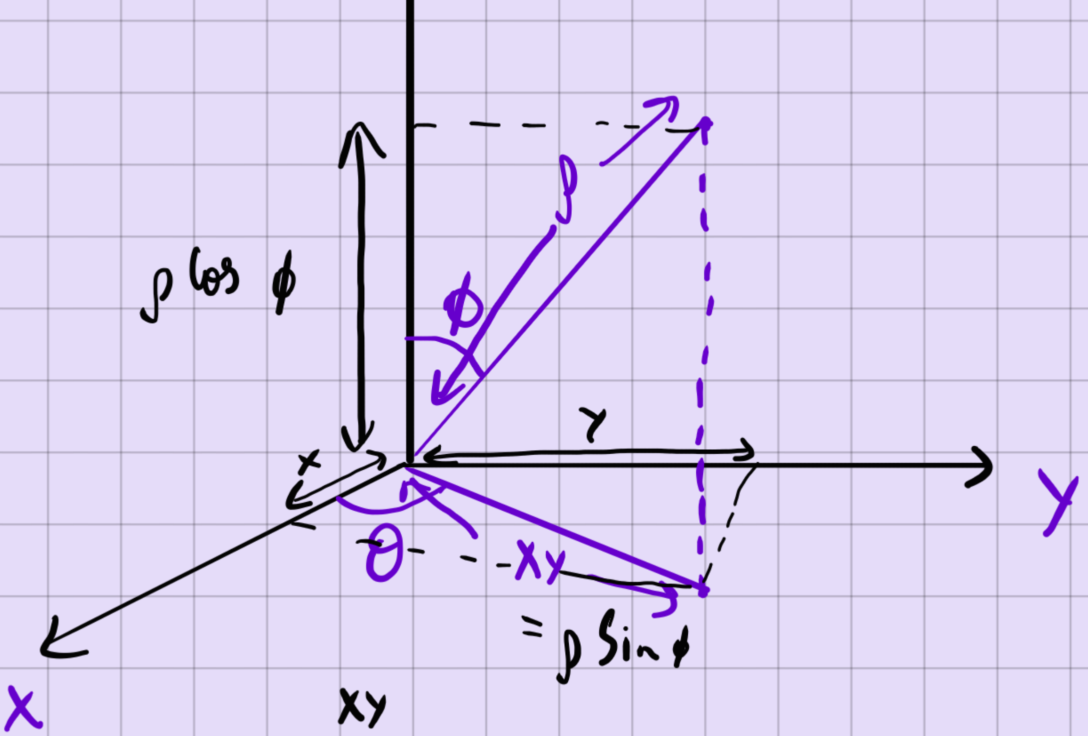
⇒ θ is the angle from the x-axis to the projection of the point on the xy-plane.

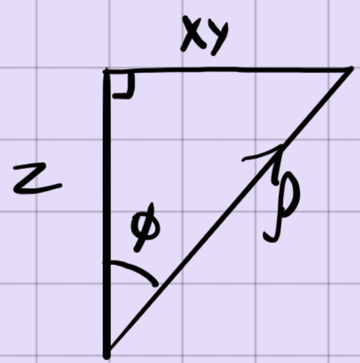




$$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi$$

⇒ Conversion between spherical and Cartesian :





$\Rightarrow \quad \sin\phi = \dfrac{xy}{\rho}$

$\Rightarrow xy = \rho \sin\phi$

$\Rightarrow \cos\phi = \dfrac{z}{\rho}$

$$y = xy \sin\theta$$
$$= \rho \sin\phi \sin\theta$$

$\Rightarrow z = \rho\cos\phi$

$$x = xy \cos\theta$$
$$= \rho \sin\phi \cos\theta$$

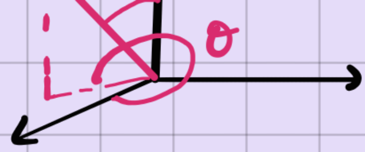
$$\boxed{\therefore \; z = \rho\cos\phi \;, \quad y = \rho\sin\phi\sin\theta, \quad x = \rho\sin\phi\cos\theta}$$

Also: $\quad x^2 + y^2 + z^2 = \rho^2$

---

1) Convert cartesian to spherical:

$(x, y, z) = (1, -1, \sqrt{2})$



$$\rho = \sqrt{1^2 + (-1)^2 + (\sqrt{2})^2} = 2$$

$$\sqrt{2} = 2\cos\phi \Rightarrow \cos\phi = \frac{\sqrt{2}}{2} \Rightarrow \phi = \frac{\pi}{4}$$

$$1 = 2 \cdot \frac{\sqrt{2}}{2}\cos\theta \Rightarrow \cos\theta = \frac{\sqrt{2}}{2}$$

$$-1 = 2 \cdot \frac{\sqrt{2}}{2}\sin\theta \Rightarrow \sin\theta = -\frac{\sqrt{2}}{2}$$

$$\Rightarrow \tan\theta = -1 \Rightarrow \theta = \frac{7\pi}{4}$$

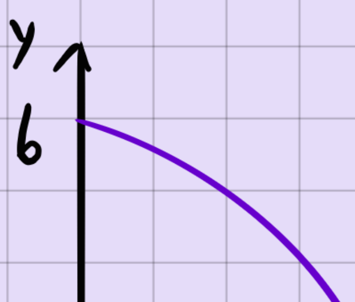
$$\left(2, \frac{\pi}{4}, \frac{7\pi}{4}\right)$$

In spherical triple integration,

$$dV = \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$

---

1) $$\int_0^6 \int_0^{\sqrt{36-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{72-x^2-y^2}} dz\, dy\, dx$$



$$0 \le \theta \le \pi/2$$

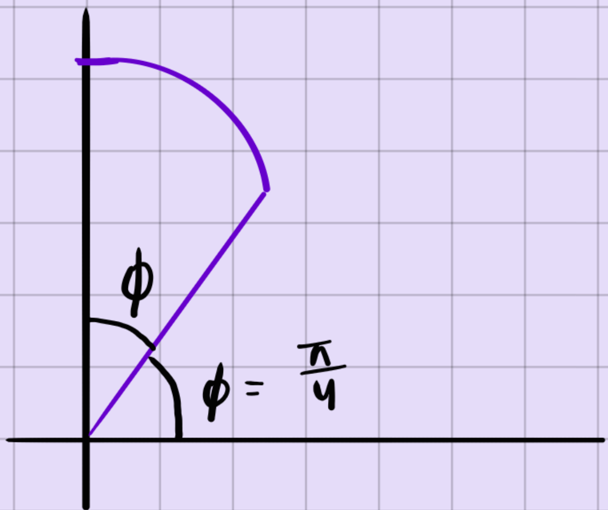
$z \geq \sqrt{x^2+y^2}$ (cone pointing upwards from origin)

$z \leq \sqrt{72-x^2-y^2}$ (sphere of radius $\sqrt{82}$ centered at origin)

↳ distance from center to origin ρ :

$$0 \leq \rho \leq \sqrt{72}$$

yz - trace :



$z = \sqrt{x^2+y^2}$, with $x = 0$

$\Rightarrow z = y$

$0 \leq \phi \leq \pi/4$

$$V = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{72}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

2) Volume of solid outside $\rho = 1$ and inside $\rho = 2\cos\phi$

inside

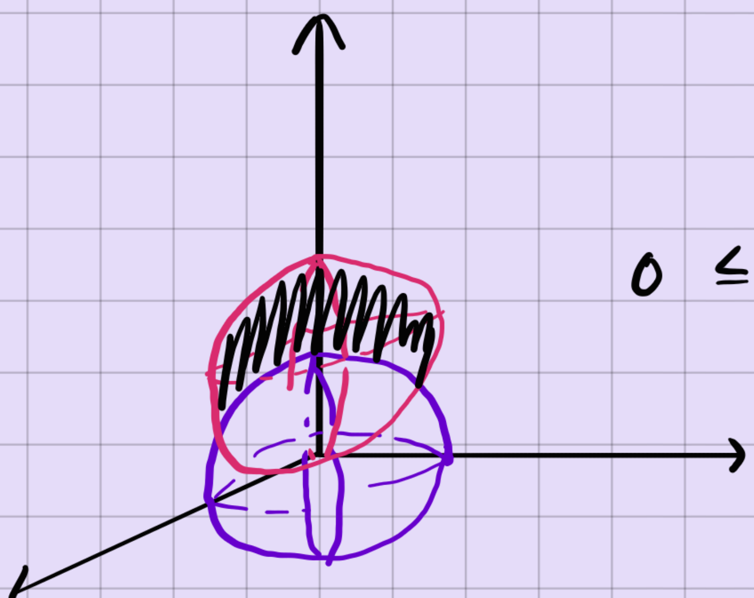
$\rho^2 = 1^2 \Rightarrow x^2+y^2+z^2=1$ (sphere centered at $(0,0,0)$, $r=1$)

$\rho = 2\cos\phi$

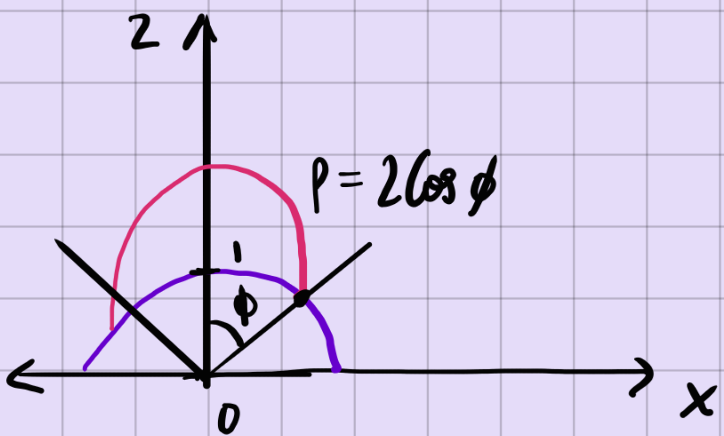
$\Rightarrow \sqrt{x^2+y^2+z^2} = \dfrac{2\cos\phi\,\rho}{\rho} = \dfrac{2z}{\rho}$

$\Rightarrow x^2+y^2+z^2-2z = 0$

$\Rightarrow x^2+y^2+(z-1)^2 = 1$ (sphere centered at $(0,0,1)$, $r=1$)



$0 \le \theta \le 2\pi$



$1 \le \rho \le 2\cos\phi$

$0 \le \phi \le \pi/3$

$2\cos\phi = 1$

$\cos\phi = 1/2 \Rightarrow \phi = \pi/3$

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_1^{2\cos\phi} r^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

3) $\iiint_E x^2+y^2 \, dV$ where E is outside $x^2+y^2+z^2=1$ and inside $x^2+y^2+z^2=4$, in the first octant.

$1 \le \rho \le 2$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{2}$

$x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$

$x^2+y^2 = \rho^2 \sin^2\phi$

$$\iiint_E \rho^4 \sin^3 \circ \quad dV$$

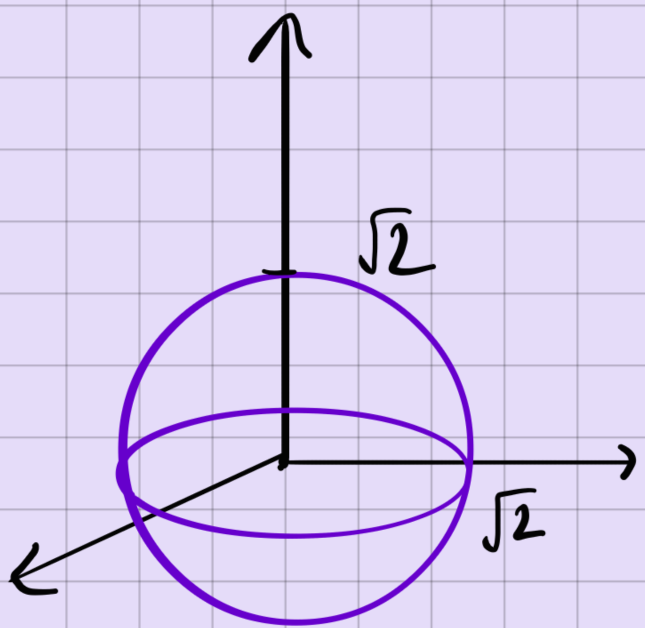
$$\Rightarrow V = \int_1^2 \int_0^{\pi/2} \int_0^{\pi/2} \rho^4 \sin^3\phi \quad d\theta \, d\phi \, d\rho$$

$= \dfrac{31\pi}{15}$

4) $\displaystyle\iiint_E 6e^{(x^2+y^2+z^2)^{3/2}}\,dV$

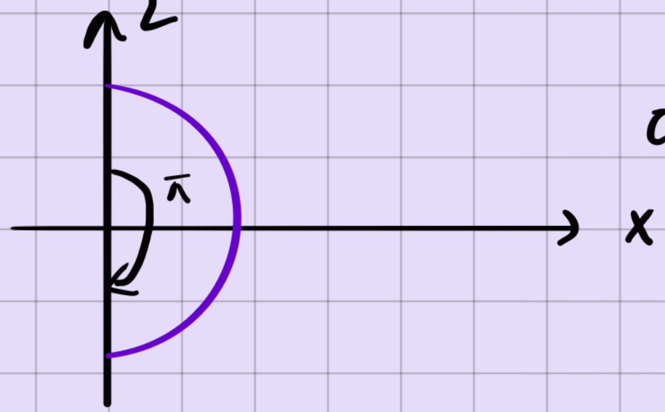
where E is bounded by the sphere $x^2+y^2+z^2=2$.

Integrand : $6e^{\rho^3}\cdot\rho^2\sin\phi$



$0 \le \rho \le \sqrt{2}$

$0 \le \theta \le 2\pi$

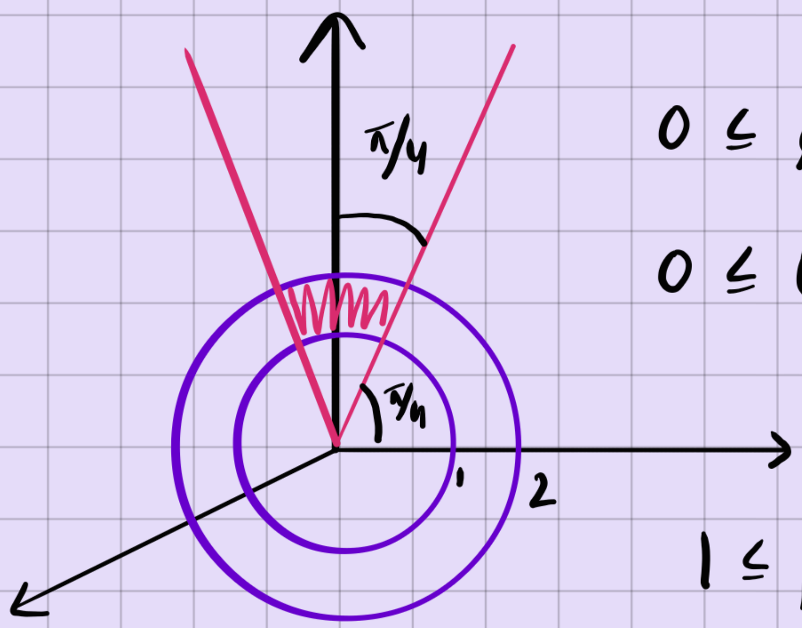


$0 \le \phi \le \pi$

$\displaystyle\int^{\sqrt{2}}\int^{2\pi}\int^{\pi}$

$$V = \int_0 \int_0 \int_0 6e^{\rho^3} \cdot \rho^2 \sin\phi \; d\phi \; d\theta \; d\rho$$

5) Volume of the solid enclosed by $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ and $z=\sqrt{x^2+y^2}$.



$0 \le \phi \le \pi/4$

$0 \le \theta \le 2\pi$

$1 \le \rho \le 2$

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_1^{2} \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$

$$\Rightarrow V = \int_0^{2\pi} \int_0^{\pi/4} \frac{7}{3} \sin\phi \; d\phi \; d\theta$$

$$\Rightarrow V = \int_0^{2\pi} \left(1 - \frac{\sqrt{2}}{2}\right)\left(\frac{7}{3}\right) d\theta$$

$$= \frac{14\pi}{3}\left(1 - \frac{\sqrt{2}}{2}\right)$$

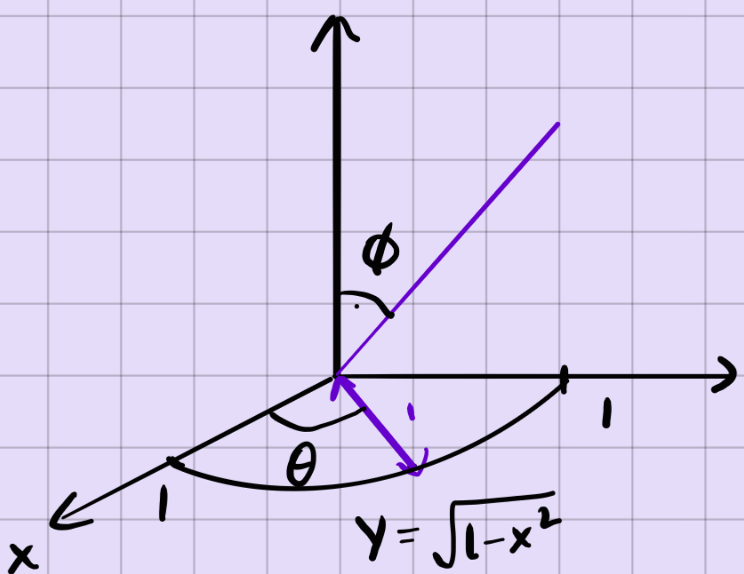
6) Convert the given spherical integral to a cartesian one :

$$\int_0^{\pi/2} \int_{\pi/4}^{\pi/2} \int_0^{\csc\phi} \rho^2 \sin\phi \; d\rho \, d\phi \, d\theta$$

$0 \le \theta \le \pi/2$, $\frac{\pi}{4} \le \phi \le \pi/2$, $0 \le \rho \le \frac{1}{\sin\phi}$

$\rho = \frac{1}{\sin\phi} \Rightarrow \rho \sin\phi = 1$

↳ trace on xy axis = 1.



$0 \le x \le 1$
$0 \le y \le \sqrt{1-x^2}$

$\phi \ge \pi/4$

$\Rightarrow \rho \cos(\pi/4) = \frac{\rho}{\sqrt{2}}$

$\phi \le \pi/2$

$$\Rightarrow \rho \cos(\pi/2) = 0$$

$$\Rightarrow 0 \le z \le \frac{\rho}{\sqrt{2}}$$

$$\Rightarrow z\sqrt{2} = \sqrt{x^2+y^2+z^2}$$

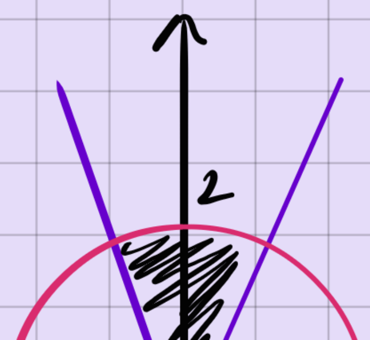
$$\Rightarrow 2z^2 = x^2+y^2+z^2$$

$$\Rightarrow x^2+y^2 = z^2 \Rightarrow z = \sqrt{x^2+y^2}$$

$$0 \le x \le 1, \quad 0 \le y \le \sqrt{1-x^2}, \quad 0 \le z \le \sqrt{x^2+y^2}$$

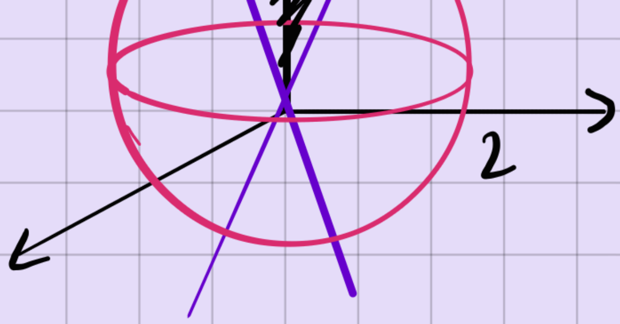
7) $$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{3(x^2+y^2)}}^{\sqrt{4-x^2-y^2}} dz \; dy \; dx$$

$$-1 \le x \le 1, \quad -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, \quad \sqrt{3(x^2+y^2)} \le z \le \sqrt{4-x^2-y^2}$$



$$z \le \sqrt{4-x^2-y^2}$$

$$\Rightarrow x^2+y^2+z^2 = 2^2$$



(sphere)

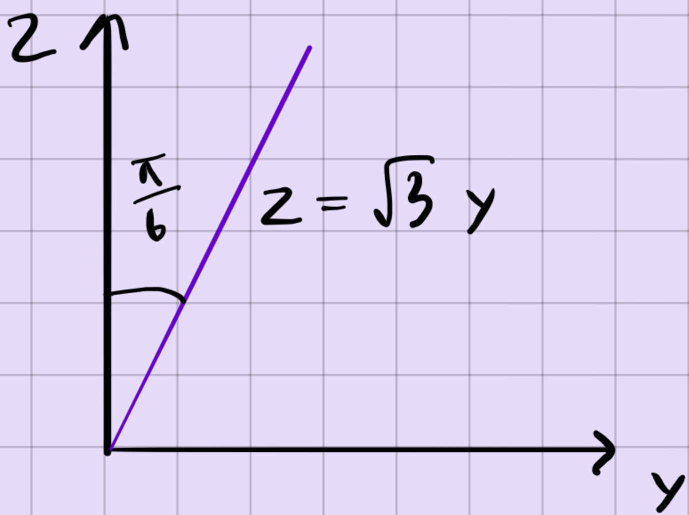
$z \geq \sqrt{3(x^2+y^2)}$

$\Rightarrow z^2 = 3x^2+3y^2$
(cone)

$0 \leq \theta \leq 2\pi$

$0 \leq \rho \leq 2$

yz :



$0 \leq \phi \leq \pi/6$

$$\Rightarrow V = \int_0^2 \int_0^{2\pi} \int_0^{\pi/6} \rho^2 \sin\phi \; d\phi \; d\theta \; d\rho$$

8) Volume bounded by $\rho = 36 \cos\phi$ and $\rho = 18$, $z \geq 0$.

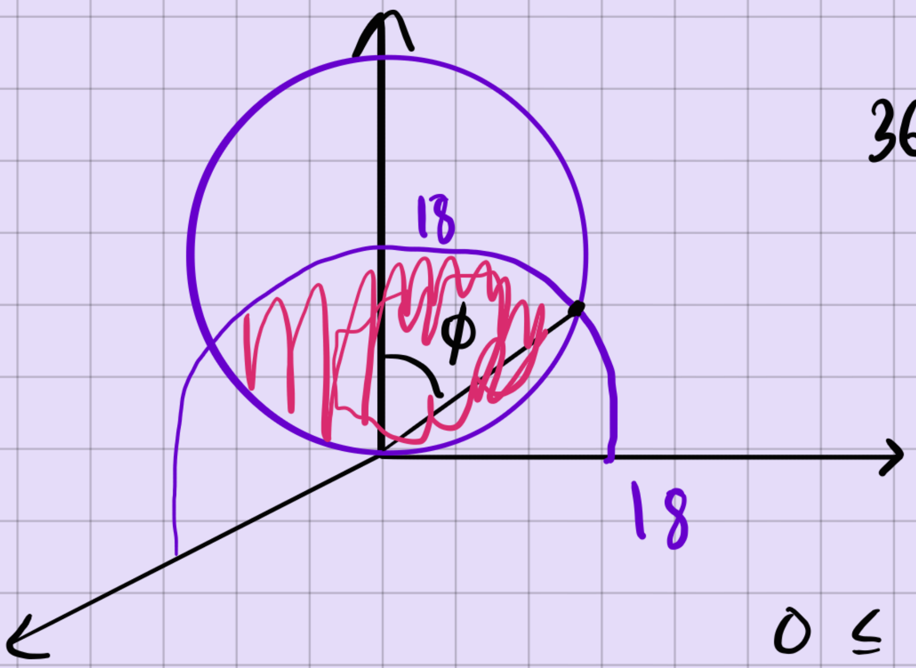
$\rho = 36 \cos\phi$

$\rho = 18$

$\rho^2 = 36\, \rho \cos\phi$

$x^2+y^2+z^2 = 324$

$$x^2+y^2+z^2 = 36z \Rightarrow x^2+y^2+(z-18)^2 = 324$$



$$36\cos\phi = 18$$

$$\cos\phi = \tfrac{1}{2}$$

$$\phi = \pi/3$$

$$0 \le \phi \le \pi/3$$
$$0 \le \theta \le 2\pi$$
$$18 \le \rho \le 36\cos\phi$$

$$\int_0^{2\pi}\int_0^{\pi/3}\int_{18}^{36\cos\phi} \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

Overall :

$$x = \rho \sin\phi \cos\theta \qquad \rho^2 = x^2+y^2+z^2$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

Approach :

1) Plot the graphs and find the needed volume.

2) Draw the xy trace to find θ.

3) Draw the yz (or xz) trace to find ϕ.

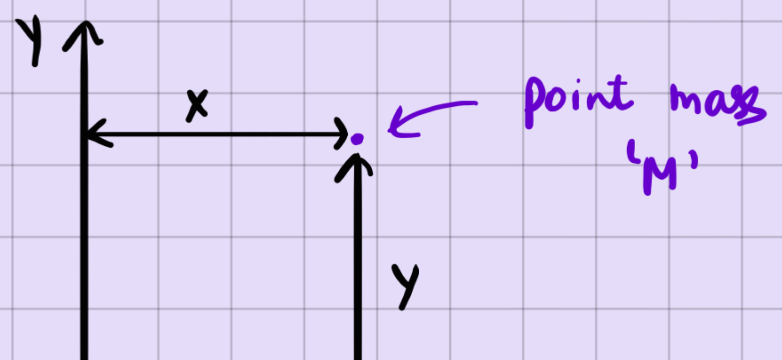
4) Calculate the lowest and highest point in the shaded region for ρ.

Integrals for mass calculations

The accumulation of density over the region through double/triple integrals gives the mass.

↳ $m = \iint_A \rho(x,y)\, dA$ or $m = \iiint_A \rho(x,y,z)\, dA$

To find the centroid, we first have to find the mass moment.



My is the moment about y-axis

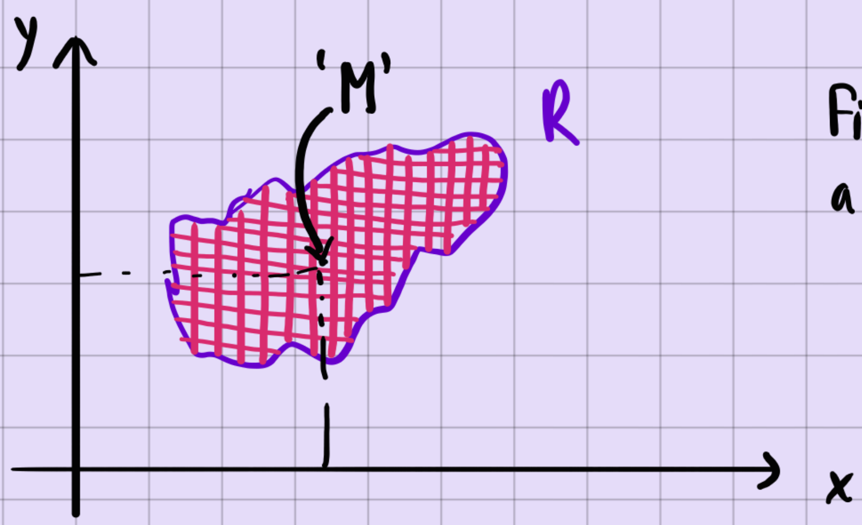
about y-axis.

$\Rightarrow M_y = M_x$

M_x is the moment about x-axis.

$\Rightarrow M_x = M_y$

Over a general region, say a 2D plate of a given shape, we integrate.



Find M_y and M_x for a single piece:

Given mass density $\rho(x,y)$, and dimensions dy and dx,

$M = \rho(x,y)\, dy\, dx$

$M = \rho(x,y)\, dA$

dy □ dx

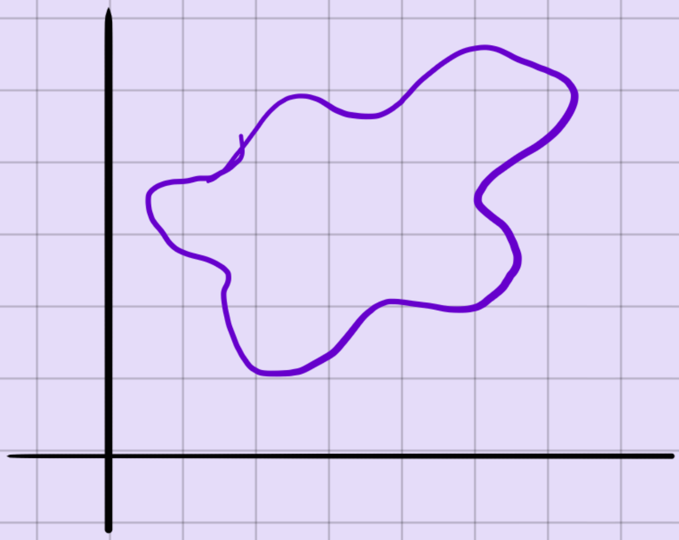
$M = dy\, dx \cdot \rho(x,y)$

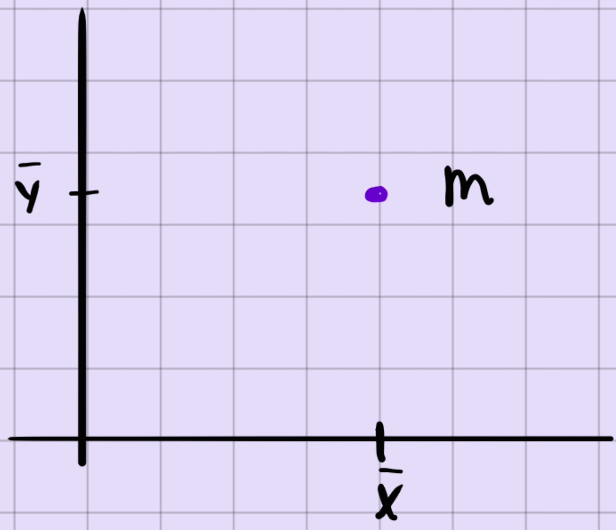
$M_y = \rho(x,y) \cdot x \quad dA \Rightarrow \boxed{M_y = \iint_{\mathbb{R}} x \cdot \rho(x,y) \quad dA}$

$M_x = \rho(x,y) \cdot y \quad dA$

$\Rightarrow \boxed{M_x = \iint_{\mathbb{R}} \rho(x,y) \cdot y \quad dA}$

Now to find the center of mass, the point where the weight is acting from, $\bar{x}, \bar{y}$:





$$M_y = \iint_{R} \rho(x,y) \cdot x \, dA \longleftrightarrow M_y = m\bar{x}$$

$$M_x = \iint_{R} \rho(x,y) \cdot y \, dA \longleftrightarrow M_x = m\bar{y}$$

We get :

$$\boxed{\begin{aligned} \bar{x} &= \frac{1}{m} \iint_{R} \rho(x,y) \cdot x \, dA \\ \bar{y} &= \frac{1}{m} \iint_{R} \rho(x,y) \cdot y \, dA \\ m &= \iint_{R} \rho(x,y) \, dA \end{aligned}}$$

1) $R: (x,y) \in [0,1] \times [0,5]$

$\rho(x,y) = 2e^{-y/2}$

$$m = \int_0^5 \int_0^1 2e^{-\frac{y}{2}} \, dx \, dy$$

$$= \int_0^5 2e^{-\frac{y}{2}} \, dy = 2e^{-\frac{y}{2}} \div \frac{-1}{2} \Big|_0^5$$

$$= -4e^{-\frac{y}{2}} \Big|_0^5$$

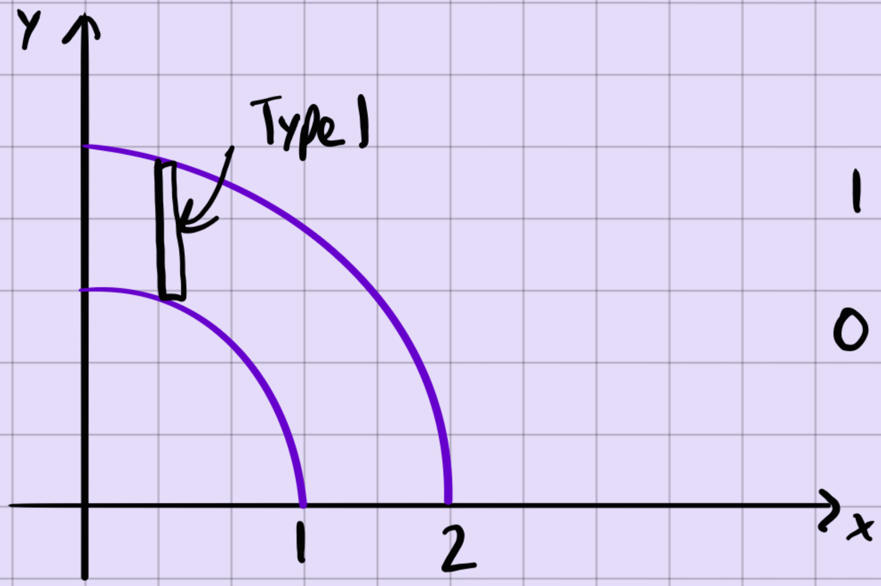
$$= -4e^{-5/2} + 4e^0$$

$$= 4 - 4e^{-5/2}$$

$$\bar{x} = \frac{1}{m} \int_0^5 \int_0^1 x \cdot 2e^{-\frac{y}{2}} \, dx \, dy = \frac{1}{2}$$

$$\bar{y} = \frac{1}{m} \int_0^5 \int_0^1 y \cdot 2e^{-y/2} \, dx \, dy = \frac{4-14e^{-5/2}}{2(1-e^{-5/2})}$$

2) R is the region between circles centered at $(0,0)$ of radii 1 and 2 in first quadrant with $\rho(x,y) = \sqrt{x^2+y^2}$



$$1 \le r \le 2$$

$$0 \le \theta \le \pi/2$$

$$m = \int_0^{\pi/2}\int_1^2 r^2 \; dr\, d\theta$$

$$= \int_0^{\pi/2} \frac{8}{3} - \frac{1}{3} \; d\theta = \frac{\pi}{2}\cdot\frac{7}{3} = \frac{7\pi}{6}$$

$$\bar{x} = \frac{1}{m}\int_0^{\pi/2}\int_1^2 r^2 \cdot r\cos\theta \; dr\, d\theta$$

$$= \frac{6}{7\pi}\int_0^{\pi/2}\left(4 - \frac{1}{4}\right)\cos\theta \; d\theta$$

$$= \frac{15}{\cancel{4}\,2}\cdot\frac{\cancel{6}\,3}{7\pi}\Big[\sin\theta\Big]_0^{\pi/2}$$

$$= \frac{15}{2}\cdot\frac{3}{7\pi}\Big[1\Big] = \frac{45}{14\pi}$$

$$\bar{y} = \frac{1}{m} \int_0^{\pi/2} \int_1^2 r^2 \cdot r \sin\theta \; dr \; d\theta$$

$$= \frac{6}{7\pi} \int_0^{\pi/2} \left(4 - \frac{1}{4}\right) \sin\theta \; d\theta$$

$$= \frac{15}{2} \cdot \frac{3}{7\pi} \Big[-\cos\theta \Big]_0^{\pi/2} = \frac{45}{14\pi}$$

In triple integrals:

$$m = \iiint_{\mathbb{R}} \rho(x,y,z) \; dV$$

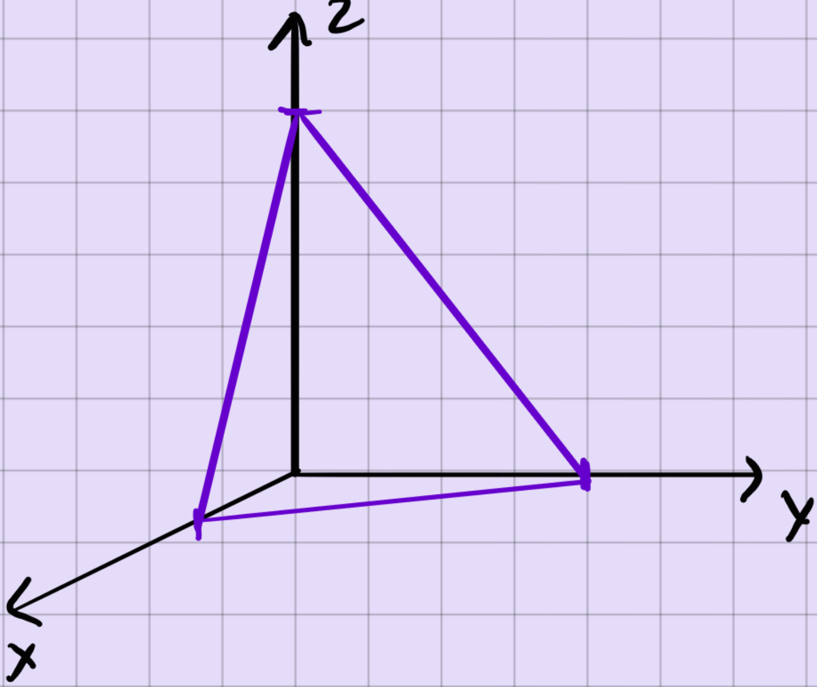
$$\bar{x} = \frac{1}{m} \iiint_{\mathbb{R}} x \cdot \rho(x,y,z) \; dV$$

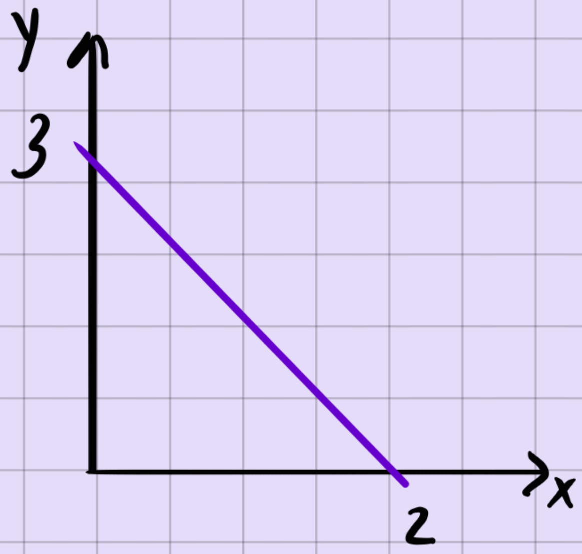
$$\bar{y} = \frac{1}{m} \iiint_{\mathbb{R}} y \cdot \rho(x,y,z) \; dV$$

$$\bar{z} = \frac{1}{m} \iiint z \cdot \rho(x,y,z) \; dV$$

$\mathbb{R}$

1) Center of mass of the solid bounded by
$3x+2y+z=6$ and $\rho(x,y,z) = 1+x$.





$$y = -\frac{3}{2}x + 3$$

$$0 \le x \le 2, \quad 0 \le y \le 3 - \tfrac{3}{2}x, \quad 0 \le z \le 6-3x-2y$$

$$m = \int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} 1+x \; dz\,dy\,dx = 9$$

$$\bar{x} = \frac{1}{9} \int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} (1+x)x \; dz\,dy\,dx = \frac{3}{5}$$

$$\bar{y} = \frac{1}{9}\int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} (1+x)y \, dz\, dy\, dx = \frac{7}{10}$$

$$\bar{z} = \frac{1}{9}\int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} (1+x)z \, dz\, dy\, dx = \frac{7}{5}$$

2) $r(t) = \frac{1}{\sqrt{2}} \langle \cos(t), \sin(t), t \rangle$, $0 \le t \le 6\pi$.

Mass density $\rho(x,y,z) = 1+z$.

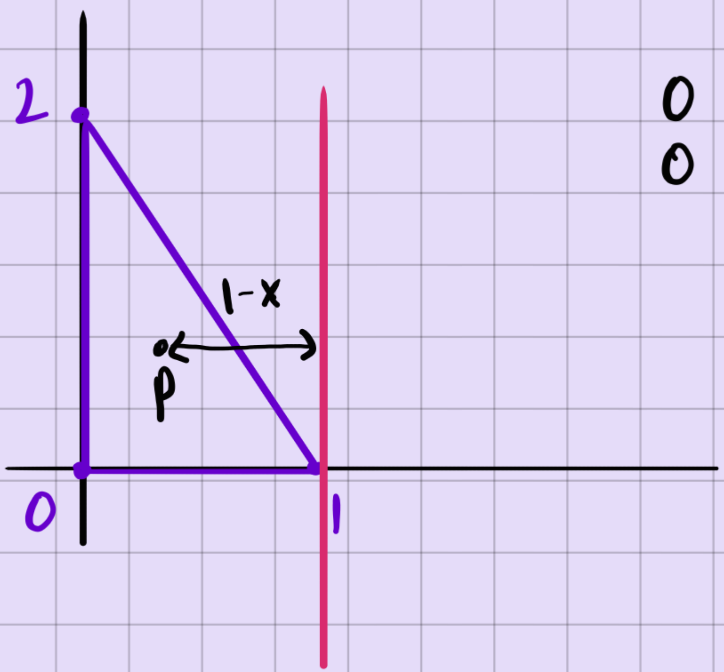
$$m = \int_0^{6\pi} |r'(t)| \cdot \rho(t) \, dt$$

$$r'(t) = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle$$

$$|r'(t)| = \frac{1}{\sqrt{2}} \sqrt{2} = 1$$

$$\int_0^{6\pi} 1 \cdot \left(1 + \frac{t}{\sqrt{2}}\right) dt = 6\pi + \frac{18\pi^2}{\sqrt{2}}$$

3) What is the mass of a lamina in the shape of a triangle with vertices (0,0), (1,0) and (0,2) if the material density at a point is equal to ½ the distance from the line $x=1$?



$0 \leq x \leq 1$
$0 \leq y \leq 2-2x$

$\rho = \frac{1}{2}(1-x)$

$$m = \int_0^1 \int_0^{2-2x} \frac{1}{2}(1-x) \, dy \, dx$$

$$\frac{1}{2} \cdot \int_0^1 (1-x)(2-2x) \, dx$$

$2-2x-2x+2x^2$

$$\int_0^1 1-2x+x^2 \, dx = \left[x - x^2 + \frac{x^3}{3}\right]_0^1$$

$$= \cancel{x} - \cancel{x} + \frac{1}{3} \Rightarrow m = 1/3$$

Vector fields :

→ Function can yield one value or multiple.

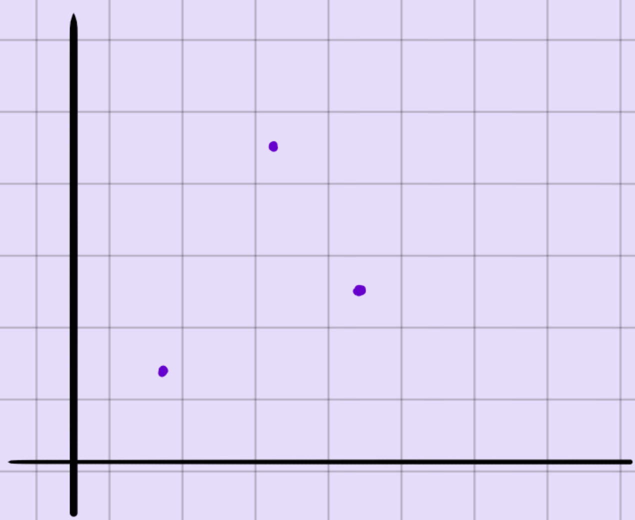
↳ $f(x,y) = x + y$ yields a single value (z).

↳ Scalar field

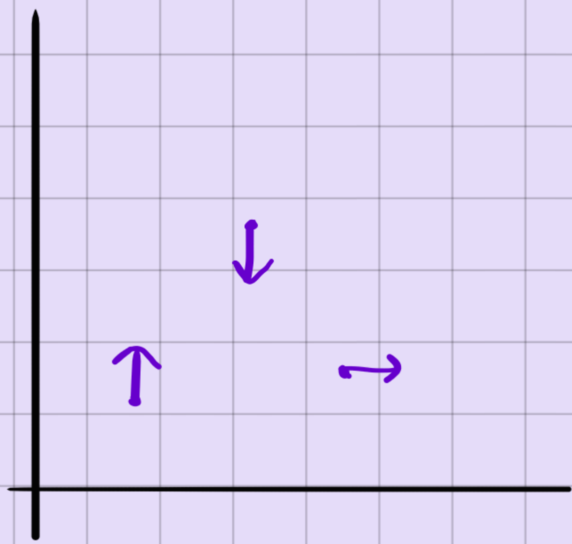
↳ $\vec{F}(x,y) = \langle x, y \rangle$ yields multiple values.

↳ vector field

in form of a vector



Scalar field



Vector field

1) $\vec{F}(x,y) = \left\langle \dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}} \right\rangle$

To sketch :

$$\vec{F}(x,y) = \frac{1}{\sqrt{x^2+y^2}} \langle x, y \rangle$$

magnitude

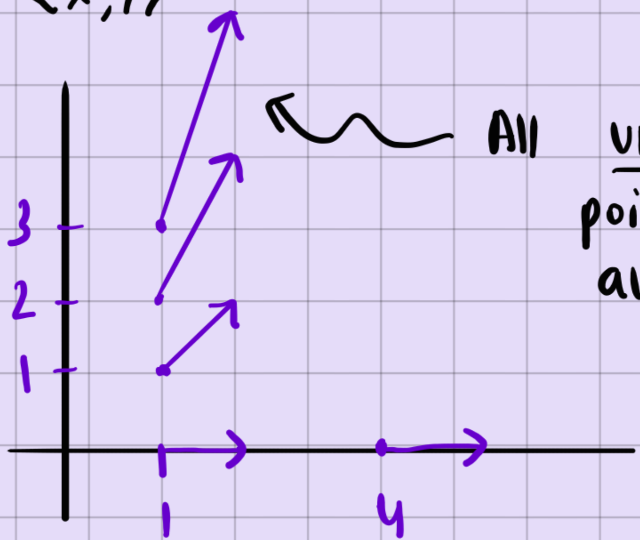
direction

a) Find magnitude :

$$\left|\vec{F}(x,y)\right| = \frac{1}{\cancel{\sqrt{x^2+y^2}}} \cdot \left[\cancel{\sqrt{x^2+y^2}}\right] = 1$$

b) Then find direction :

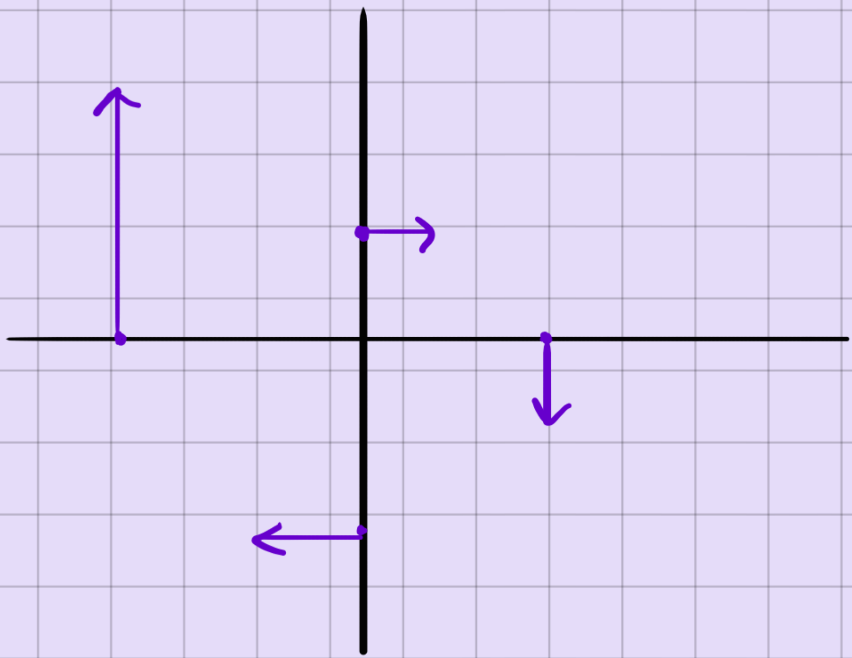
$\langle x, y \rangle$



All $\underline{\text{unit vectors}}$ pointing radially away from origin

2) $\vec{F}(x,y) = \langle y, -x \rangle$

$\left|\vec{F}(x,y)\right| = \sqrt{x^2+y^2}$ ⇒ Magnitude increases as we go away from the origin.



If the vector field $\vec{F}(x,y)$ is the gradient of a scalar function U, then U is called the potential function of $\vec{F}(x,y)$.

$$U = \frac{1}{\sqrt{x^2+y^2}} \qquad \nabla U = \langle U_x , U_y \rangle$$

$$\Rightarrow \quad U_x = \frac{\partial}{\partial x}\left(x^2+y^2\right)^{-1/2} = \frac{-1}{\cancel{2}}\left(x^2+y^2\right)^{-3/2} \cdot \cancel{2}x$$

$$U_x = \frac{-x}{\left(\sqrt{x^2+y^2}\right)^3}$$

$$\Rightarrow \quad U_y = \frac{-y}{\left(\sqrt{x^2+y^2}\right)^3}$$

$$\therefore \quad \nabla U = \left\langle \frac{-x}{(x^2+y^2)^{3/2}}, \frac{-y}{(x^2+y^2)^{3/2}} \right\rangle$$

So U, the scalar function is a potential function of ∇U, the vector gradient function. Additionally, the gradient is always perpendicular to the level curve, so the vector field of ∇U is perpendicular to the level curves of U.

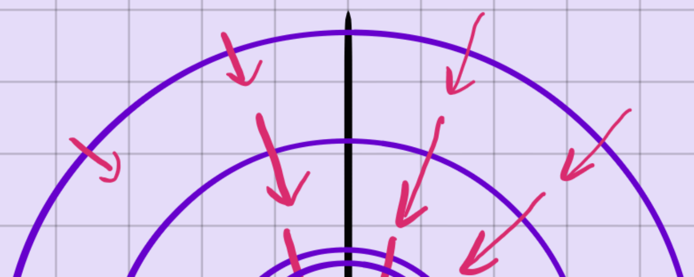
$$U = z = \frac{1}{\sqrt{x^2+y^2}}$$

Set $z = k$:

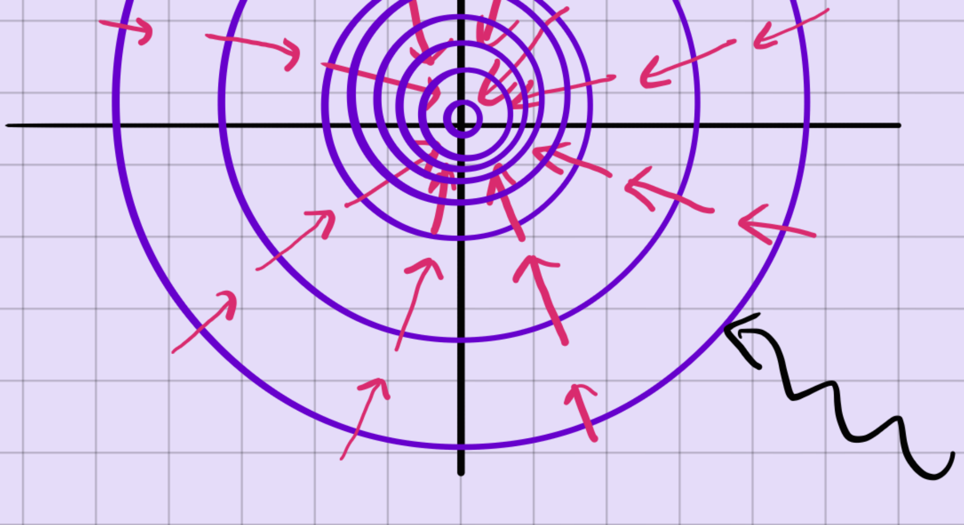
level curves

$$\sqrt{x^2+y^2} = \frac{1}{k} \quad \Rightarrow \quad x^2+y^2 = \frac{1}{k^2}$$

The vector field $\vec{\nabla} U$ is $\perp$ to the level curves.



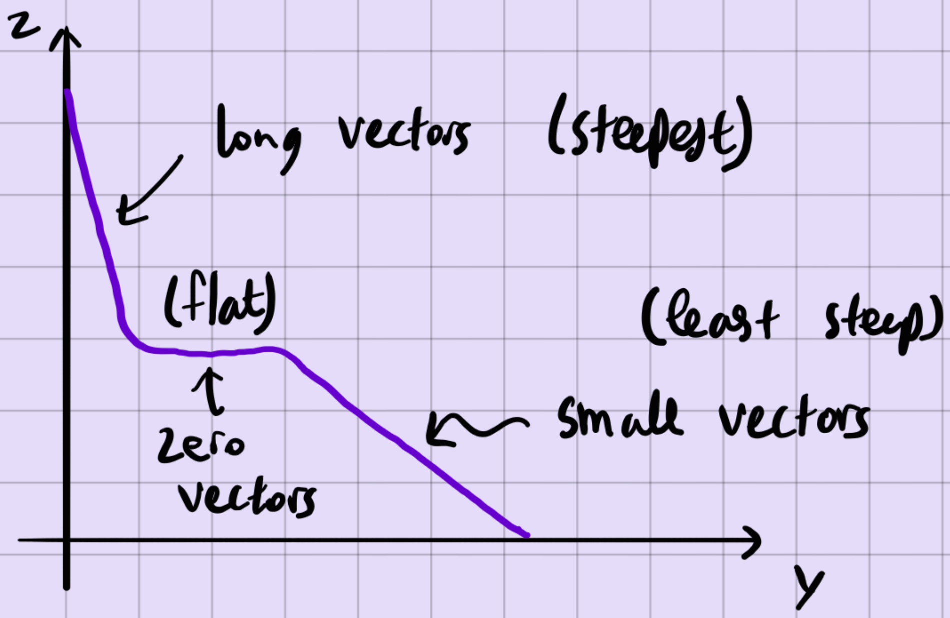
The length of the vectors tells us how fast the level curves are changi



curves are changing.

↳ The faster the ascent/descent, the longer the vectors.

Using the analogy of a mountain:



If a vector field has a potential function meaning that there is a scalar function U with $\vec{\nabla} U = \vec{F}$, then, the vector field is conservative.

↳ In a conservative force field, the work done is independent of path.

$$\Rightarrow \quad \vec{F}(x, y) = -\nabla U$$

Vector

Scalar field

vector field

a) Line integrals of Functions

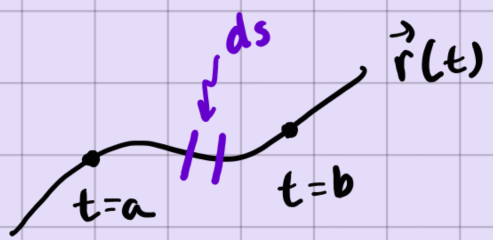
accumulation of a function along a curve.

$$\int_C f(x,y)\ \underline{ds}$$

curve

length of a small segment of curve

Remember



$$\text{length} = \int_{t=a}^{t=b} \underbrace{|\vec{r}\,'(t)|\ dt}_{ds}$$

So a line integral for $f(x,y) = 1$ is just the length of the curve within the given bounds.

---

1) $\displaystyle\int_C xe^y\ ds$, with C: line segment from (0,0) to (4,-1)

(0,0) → (4,-1) parametrization of c.

$(0,0) \to (4,-1)$

$$\Rightarrow \vec{r}(t) = \langle 4,-1\rangle t = \langle 4t, -t\rangle \quad \text{for } 0 \le t \le 1$$

$$ds = |r'(t)|\, dt$$

$$= |\langle 4,-1\rangle|\, dt$$

$$= \sqrt{17}\, dt$$

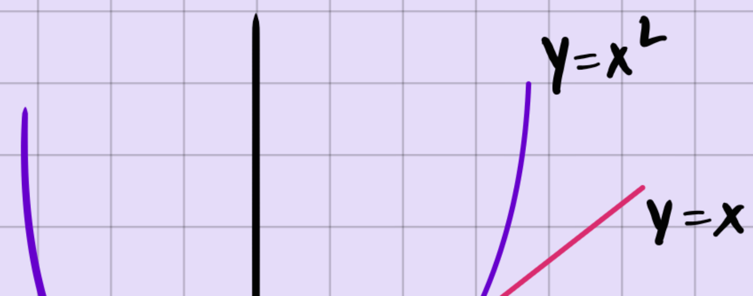
$$\int_C x e^y \, ds = \int_0^1 4t\, e^{-t} \cdot \sqrt{17}\, dt$$

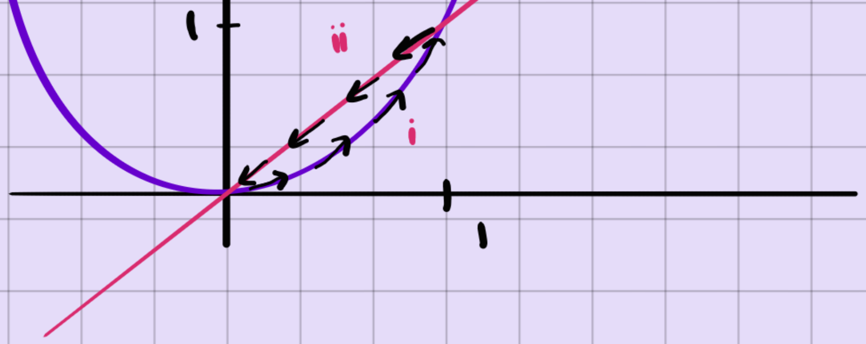
$$= 4\sqrt{17} \int_0^1 t\, e^{-t}\, dt$$

Note: There are many possible parametrizations of a given 'c'. However, the line integral is <u>independent</u> of the "path" taken.

↳ Always choose the simplest path

2) $\displaystyle\int_C (x + \sqrt{y})\, ds$, C: from $(0,0)$ to $(1,1)$ along the curve $y = x^2$, then from $(1,1)$ to $(0,0)$ along $y = x$.





i) $y = x^2 \Rightarrow r_1(t) = \langle t, t^2 \rangle \quad 0 \le t \le 1$

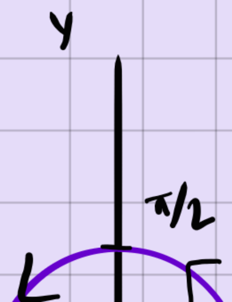
$$ds = |r_1'(t)| = |\langle 1, 2t \rangle|$$
$$= \sqrt{1 + 4t^2}\ dt$$

ii) $y = x \Rightarrow r_2(t) = \langle 1-t, 1-t \rangle \quad 0 \le t \le 1$

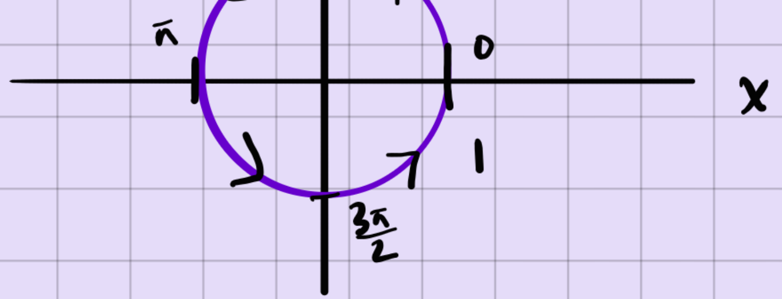
$$ds = |r_2'(t)| = |\langle -1, -1 \rangle|$$
$$= \sqrt{2}\ dt$$

$$\int_0^1 \left(t + \sqrt{t^2}\right)\sqrt{1+4t^2}\ dt + \sqrt{2}\int_0^1 \left((1-t) + \sqrt{(1-t)}\right) dt$$

3) $\int_C (x+y+z)\ ds$, C: intersection of $z = x^2+y^2$ and $x^2+y^2 = 1$ going counter-clockwise when viewed from above.



(At $z = 1$)



$\vec{r}(t) = \langle \cos(t), \sin(t), 1 \rangle$

<u>intersection</u>

$0 \le t \le 2\pi$

$$\int_0^{2\pi} \left(\cos(t) + \sin(t) + 1\right) \cdot \quad dt$$

$$= \int_0^{2\pi} \left(\cos(t) + \sin(t) + 1\right) \; dt$$

$$= \Big[\sin(t) - \cos(t) + t \Big]_0^{2\pi}$$

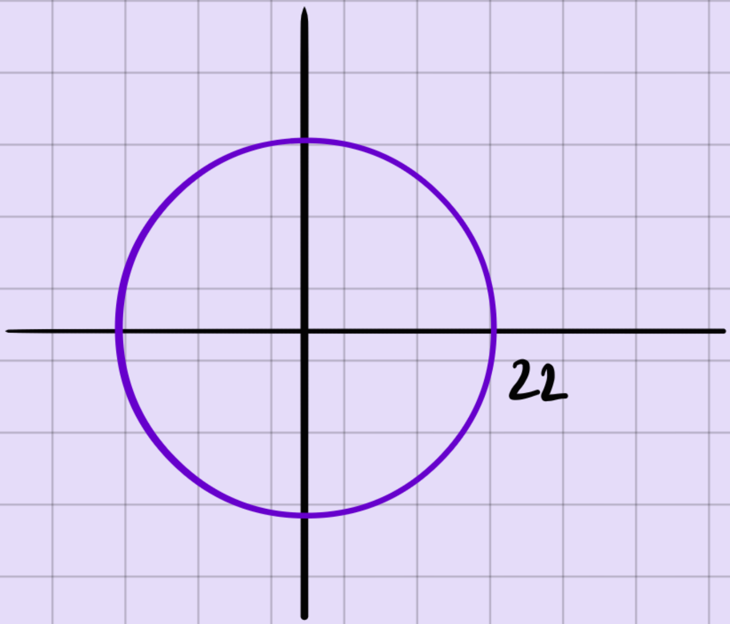
$$= \Big[\cancel{(0-0)} - \cancel{(1-1)} + (2\pi - 0) \Big] = 2\pi$$

Given

$$\int_C f(x,y,z) \; ds$$

, if $f(x,y,z)$ is the density function, we can evaluate the mass with the line integral.

4) $\int_C x^2+y^2 \, ds$, where c is the circle of $r=22$ centered at $(0,0)$.



$\vec{r}(t) = \langle 22\cos(t), 22\sin(t) \rangle$

$0 \leq t \leq 2\pi$

$\vec{r}\,'(t) = \langle -22\sin(t), 22\cos(t) \rangle$

$|\vec{r}\,'(t)| = 22$

$$\int_0^{2\pi} \left(22^2 \cos^2 t + 22^2 \sin^2 t\right) 22 \, dt$$

$$= 22^3 \cdot 2\pi$$

5) $\int_C (x+y+z) \, ds$, where c is part of $\vec{r}(t) = \langle 4\cos t, 0, 4\sin t \rangle$ for $0 \leq t \leq 3\pi/2$

$\vec{r}(t) = \langle 4\cos t, 0, 4\sin t \rangle \Rightarrow ds = 4 \, dt$

$$4 \int_0^{3\pi/2} 4\cos t + 4\sin t \quad dt \quad = 0$$

6) $\vec{r}(t) = \langle 20 \sin t/4,\ 20 \cos t/4,\ t/2 \rangle \qquad 0 \le t \le 2$

Find length.

$\vec{r}\,'(t) = \langle 5 \cos t/4,\ -5 \sin t/4,\ 1/2 \rangle$

$|\vec{r}\,'(t)| = \sqrt{1/4 + 25} = \dfrac{\sqrt{101}}{2}$

$\displaystyle\int_0^2 \frac{\sqrt{101}}{2}\, dt = \frac{2}{2}\sqrt{101} = \sqrt{101}$

7) $\displaystyle\int_c x+y+z \; ds$ for c from $(5,2,1)$ to $(4,-2,-2)$

$(5,2,1) \rightarrow (4,-2,-2)$

$\vec{r}(t) = \langle 5,2,1 \rangle + \langle -1,-4,-3 \rangle t \qquad 0 \le t \le 1$

$= \langle 5-t,\ 2-4t,\ 1-3t \rangle$

$\vec{r}\,'(t) = \langle -1,-4,-3 \rangle \Rightarrow ds = \sqrt{26}\, dt$

$\sqrt{26} \displaystyle\int_0^1 5-t+2-4t+1-3t \; dt = 4\sqrt{26}$

0

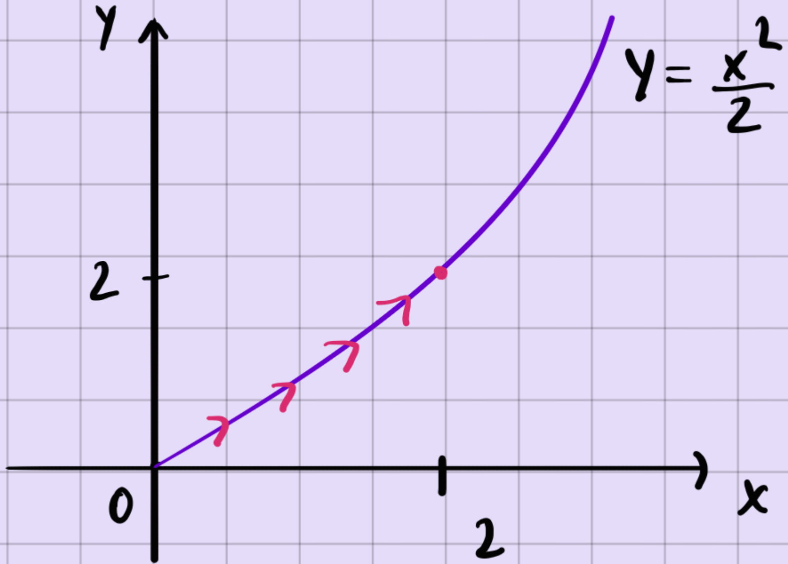
8) $\displaystyle\int_C y e^{x^2}\, ds$, along C : $\vec{r}(t) = \langle 5t, -12t \rangle$
$-2 \le t \le 2$

$\vec{r}\,(t) = \langle 5t, -12t \rangle$

$\vec{r}\,'(t) = \langle 5, -12 \rangle \Rightarrow ds = 13\, dt$

$$13 \int_{-2}^{2} (-12t)\, e^{25t^2}\, dt = 0$$

9) $f(x,y) = x^3/y$, C : $y = \dfrac{x^2}{2}$, $0 \le x \le 2$



$\vec{r}(t) = \langle t, t^2/2 \rangle$

$0 \le t \le 2$

$\vec{r}\,'(t) = \langle 1, 2t/2 \rangle = \langle 1, t \rangle$

$ds = \sqrt{1+t^2}\, dt$

$$\int_0^2 \frac{t^3}{t^2/2} \sqrt{1+t^2}\, dt$$

$$= 2 \int_0^2 t \sqrt{1+t^2}\, dt = \frac{2}{3}\left(5\sqrt{5} - 1\right)$$

10) Evaluate the line integral where C is the curve $\vec{r}(t) = \langle t, 2t, t^2 \rangle$ $0 \le t \le 3$.

$$\int_C x^2\, dx + dy + y\, dz$$

$$x^2 dx + dy + y\, dz$$

$$= \left[t^2 \cdot 1 + 2 + (2t)^2\right] dt$$

$$= \left[5t^2 + 2\right] dt$$

$$\Rightarrow \int_0^3 5t^2 + 2\, dt = 51$$

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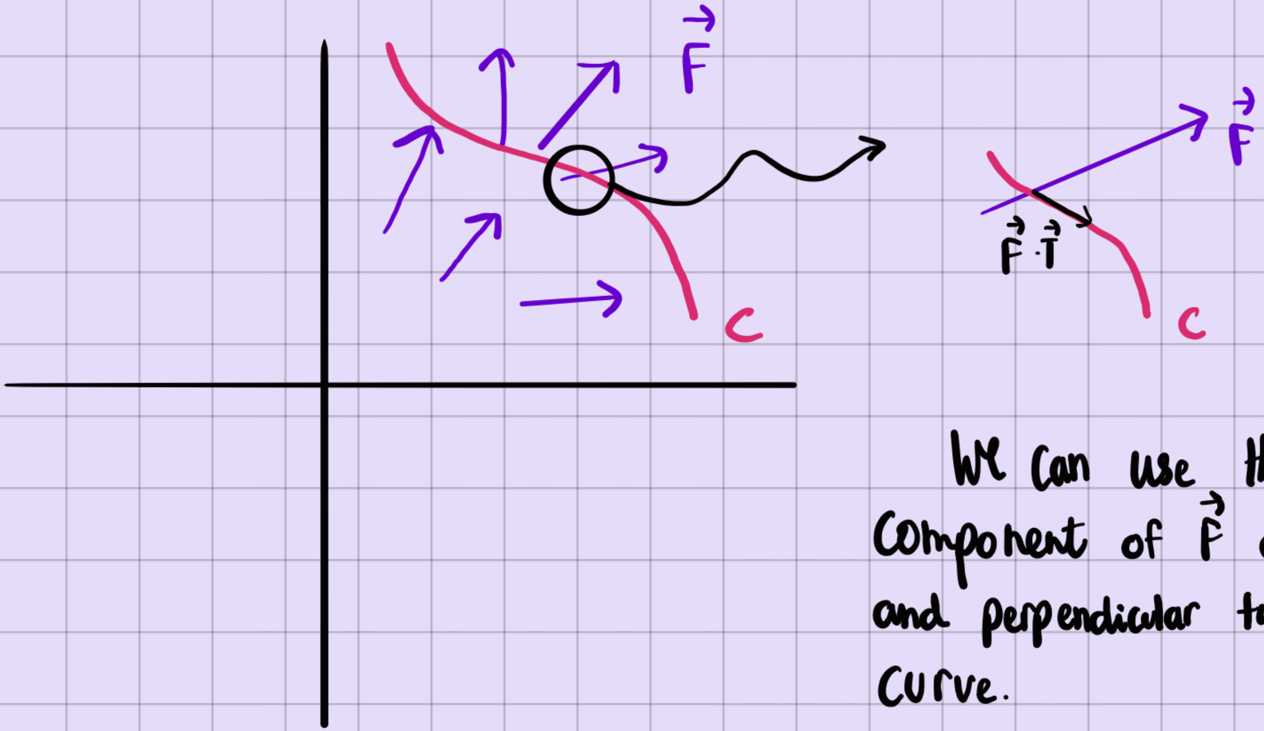
b) Line integrals of Vector Fields :

accumlate some component of the vector field along a path c.



We can use the component of $\vec{F}$ along, and perpendicular to the curve.

unit tangent vector of path, C

along the path: $\vec{F}\cdot\vec{T}$

Say we parametrize C as a vector function $\vec{r}(t)$.

↳ $\vec{T} = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$. Also, $ds = |\vec{r}'(t)|\,dt$

$$\Rightarrow \int_C \vec{F}\cdot\vec{T}\,ds = \int_{t=a}^{t=b} \vec{F}\cdot\vec{r}'(t)\,dt$$

Additionally, $\vec{r}'(t) = \dfrac{d\vec{r}}{dt}$

t=b   t=b

$$\Rightarrow \int_{t=a} \vec{F} \cdot \vec{r}\,'(t)\ dt = \int_{t=a} \vec{F} \cdot d\vec{r}$$

---

1) $\vec{F} = \langle xy,\ y-x \rangle$. C is a line segment from $(0,1)$ to $(2,4)$. Find

$$\int_C \vec{F} \cdot \vec{T}\ ds$$

$C : \langle 0, 1 \rangle + \langle 2, 3 \rangle t$

$\vec{r}(t) = \langle 2t,\ 1+3t \rangle \qquad 0 \leq t \leq 1$

$\vec{r}\,'(t) = \langle 2, 3 \rangle = d\vec{r}$

$$\Rightarrow \int_0^1 \langle (2t)(1+3t),\ 1+3t-2t \rangle \cdot \langle 2, 3 \rangle\ dt$$

$$= \int_0^1 \langle 2t + 6t^2,\ 1+t \rangle \cdot \langle 2, 3 \rangle\ dt$$

$$= \int_0^1 4t + 12t^2 + 3 + 3t\ \ dt$$

$$= \int_0^1 12t^2 + 7t + 3 \ dt = \left(4t^3 + \frac{7t^2}{2} + 3t\right)\Bigg|_{t=0}^{t=1}$$

$$= 4 + \frac{7}{2} + 3 = \frac{21}{2}$$

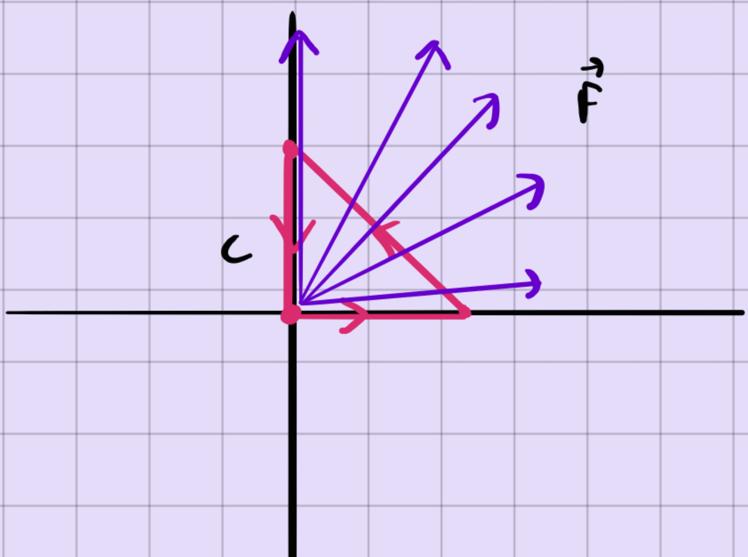
If C, the path, is a closed loop (starting point = ending point), the $\int_C \vec{F} \cdot \vec{T} \, ds = \int_C \vec{F} \cdot \vec{r}'(t) \, dt = \int_C \vec{F} \cdot d\vec{r}$ is called a circulation of $\vec{F}$ on C.

---

2) $\vec{F} = \langle x, y \rangle$, C : $(0,0) \to (1,0)$ along a line.

$(1,0) \to (0,1)$ along a line.

$(0,1) \to (0,0)$ along a line.



Calculate

$$\int_C \vec{F} \cdot \vec{T} \, ds$$

$$\vec{r}_1(t) = \langle 0,0\rangle + \langle 1,0\rangle t \qquad 0 \le t \le 1$$

$$\vec{r}_2(t) = \langle 1,0\rangle + \langle -1,1\rangle t \qquad 0 \le t \le 1$$

$$\vec{r}_3(t) = \langle 0,1\rangle + \langle 0,-1\rangle t \qquad 0 \le t \le 1$$

$$\vec{r}_1 = \langle t,0\rangle \quad \Rightarrow \quad \vec{r}_1{}' = \langle 1,0\rangle$$

$$\vec{r}_2 = \langle 1-t, t\rangle \quad \Rightarrow \quad \vec{r}_2{}' = \langle -1,1\rangle$$

$$\vec{r}_3 = \langle 0,1-t\rangle \quad \Rightarrow \quad \vec{r}_3{}' = \langle 0,-1\rangle$$

$$\Rightarrow \int_0^1 \langle t,0\rangle \cdot \langle 1,0\rangle \, dt + \int_0^1 \langle 1-t,t\rangle \cdot \langle -1,1\rangle \, dt$$

$$+ \int_0^1 \langle 0,1-t\rangle \cdot \langle 0,-1\rangle \, dt$$

$$\Rightarrow \int_0^1 t \, dt + \int_0^1 2t-1 \, dt + \int_0^1 t-1 \, dt$$

$$\Rightarrow \left(\frac{t^2}{2} + t^2 - t + \frac{t^2}{2} - t\right)\Bigg|_{t=0}^{t=1}$$

$$\Rightarrow \frac{1}{2} + \cancel{1-1} + \frac{1}{2} - 1 = 0$$

Another equivalent form for

$$\int_C \vec{F} \cdot \vec{T} \, ds = \int_C \vec{F} \cdot \vec{r}\,'(t) \, dt = \int_C \vec{F} \cdot d\vec{r}$$

$\Rightarrow$ Let $C : \vec{r}(t) = \langle \vec{x}(t), \vec{y}(t) \rangle$

$$\Rightarrow \vec{r}\,'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

$\frac{d\vec{r}}{dt}$

$$\Rightarrow \vec{r}\,'(t) \, dt = \langle dx, dy \rangle$$

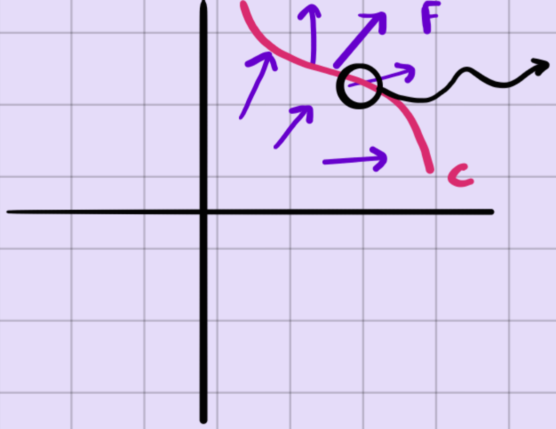
$\Rightarrow$ Let $\vec{F} = \langle f, g \rangle$

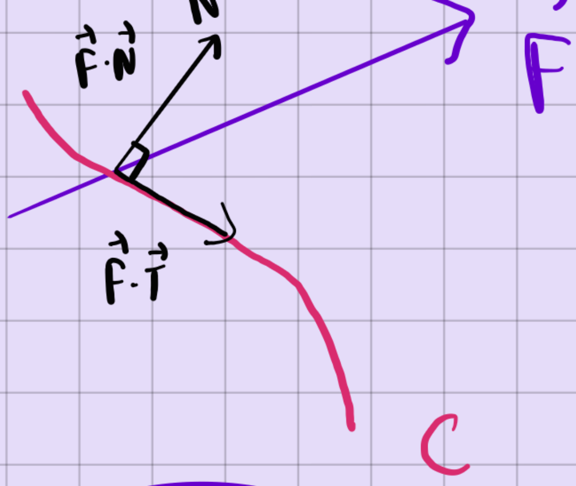
$$\Rightarrow \int_C \vec{F} \cdot \vec{r}\,'(t) \, dt = \boxed{\int_C f\,dx + g\,dy}$$

Note that everything above was for the accumulation of the component of $\vec{F}$ along the path. We can do the same thing for the component perpendicular to the path.

normal unit vector $\perp$ to C.

$\vec{N}$



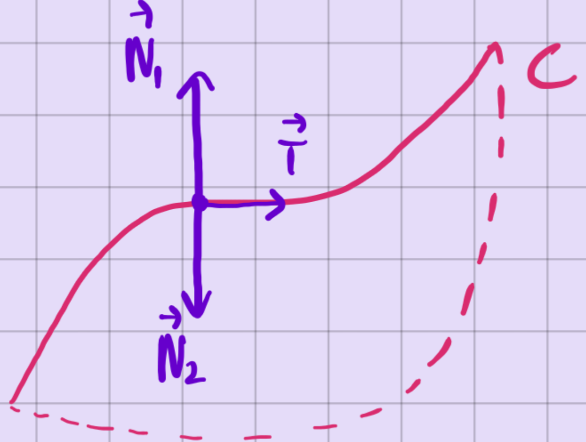


So we now calculate $\int_C \vec{F}\cdot\vec{N}\, ds$

accumulates all flowing through the path, unlike $\int_C \vec{F}\cdot\vec{T}\, ds$ which accumulates along the path.

"Flux integral"

There are 2 possible $\vec{N}$:



(Imagine closed loop)

If C is a closed loop, choose the $\vec{N}$ that points out $\rightarrow \vec{N}_1$.

If C is not closed, choose the $\vec{N}$ that is to the <u>right</u>.

↳ Face C along $\vec{T}$, and choose the $\vec{N}$ on the right.

$\hookrightarrow \vec{N}_2$.

Conservative vector fields & the fundamental theory of line integrals

irrotational

If $\vec{F}$ is conservative, then, there exists a function such that

$$\vec{F} = \nabla \phi.$$

potential function

$\langle \phi_x, \phi_y \rangle$

Say $\vec{F} = \langle f, g \rangle$ is conservative.

$$\Rightarrow \vec{F} = \langle f, g \rangle, \quad \nabla \phi = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right\rangle$$

$$\Rightarrow f = \frac{\partial \phi}{\partial x}, \quad g = \frac{\partial \phi}{\partial y}$$

Additionally, given a function i, $i_{xy} = i_{yx}$.

$\hookrightarrow$ True for continuous functions

$$\Rightarrow f_y = \phi_{xy}, \quad g_x = \phi_{yx}$$

$$\Rightarrow f_y = g_x \text{ since } \phi_{xy} = \phi_{yx}.$$

∘∘ If $\vec{F} = \langle f, g \rangle$ is conservative, then $f_y = g_x$.

↳ If $\vec{F} = \langle f, g \rangle$, and $f_y = g_x$, then $\vec{F}$ is conservative.

---

1) $\vec{F} = \langle x, y \rangle$. Is $\vec{F}$ conservative?

↳ $f = x$, $g = y$

$\Rightarrow f_y = 0$, $g_x = 0$ $\Rightarrow$ Since $f_y = g_x = 0$

↳ $\vec{F}$ is conservative.

2) $\vec{F} = \langle -y, x \rangle$.

↳ $f = -y$, $g = x$

$\Rightarrow f_y = -1$, $g_x = 1$ $\Rightarrow$ $-1 \neq 1$

↳ $\vec{F}$ is not conservative.

If $\vec{F}$ is conservative, we can find the potential function.

---

3) $\vec{F} = \langle x+y, x \rangle$

↳ $f = x+y$, $g = x$

$\Rightarrow f_y = 1$, $g_x = 1$ $\Rightarrow$ conservative

$f_y = 1, \; g_x = 1 \Rightarrow$ conservative

So, since $\vec{F} = \nabla \phi$

$$\Rightarrow \langle x+y, x \rangle = \langle \phi_x, \phi_y \rangle$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = x+y \quad , \quad \frac{\partial \phi}{\partial y} = x$$

$$\phi = \int \phi_x \, dx = \int x+y \, dx$$

$$= \frac{x^2}{2} + xy + a(y)$$

$$\phi_y = x + \frac{da}{dy} = x$$

$$\Rightarrow \frac{da}{dy} = 0 \qquad (a(y) \text{ is a constant} = C)$$

So, $\phi = \dfrac{x^2}{2} + xy + C$

$$\nabla \phi = \langle x+y, x \rangle$$

The same method of checking if $\vec{F}$ is conservative or not and finding ϕ can be used in 3D.

4) $\vec{F} = \langle \sin(y), x\cos(y), 1 \rangle$.

$$\vec{F} = \vec{\nabla}\phi$$

$$\Rightarrow f = \phi_x \quad , \quad g = \phi_y \quad , \quad h = \phi_z$$

$$\rightarrow \phi_x = \sin(y) \qquad \Rightarrow \phi_{xy} = \cos(y) \qquad \Rightarrow \phi_{xz} = 0$$

$$\rightarrow \phi_y = x\cos(y) \qquad \Rightarrow \phi_{yx} = \cos(y) \qquad \Rightarrow \phi_{yz} = 0$$

$$\rightarrow \phi_z = 1 \qquad \Rightarrow \phi_{zx} = 0 \qquad \Rightarrow \phi_{zy} = 0$$

$$\Rightarrow \phi_{xy} = \phi_{yx} \, , \quad \phi_{zx} = \phi_{xz} \, , \quad \phi_{yz} = \phi_{zy} .$$

$$\Rightarrow \phi_x = \sin(y)$$

$$\Rightarrow \phi = \int \sin(y)\, dx = x\sin(y) + a(y,z)$$

$$\Rightarrow \phi_y = x\cos(y) + \frac{da}{dy} = x\cos(y)$$

$$\Rightarrow \frac{da}{dy} = 0 \longrightarrow a(z)$$

$$\Rightarrow \phi = x\sin(y) + a(z)$$

$$\Rightarrow \phi_z = \frac{da}{dz} = 1 \quad \Rightarrow \boxed{a(z) = z + C}$$

$$\Rightarrow \boxed{\phi = x\sin(y) + z + c}$$

$$\vec{\nabla}\phi = \langle \sin(y), x\cos(y), 1\rangle = \vec{F}$$

The potential function can help simplify the circulation.

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{r}'(t)\,dt = \int_C \vec{F}\cdot d\vec{r}$$

If $\vec{F} = \vec{\nabla}\phi$, meaning the vector field $\vec{F}$ is conservative, then the circulation is path independent.

only starting and ending points matter.

If $\vec{F} = \vec{\nabla}\phi = \langle \phi_x, \phi_y\rangle$ and $r(t) = \langle x(t), y(t)\rangle$

$$\Rightarrow \vec{F}\cdot\vec{r}'(t)\,dt = \langle \phi_x, \phi_y\rangle \cdot \langle x'(t), y'(t)\rangle\,dt$$

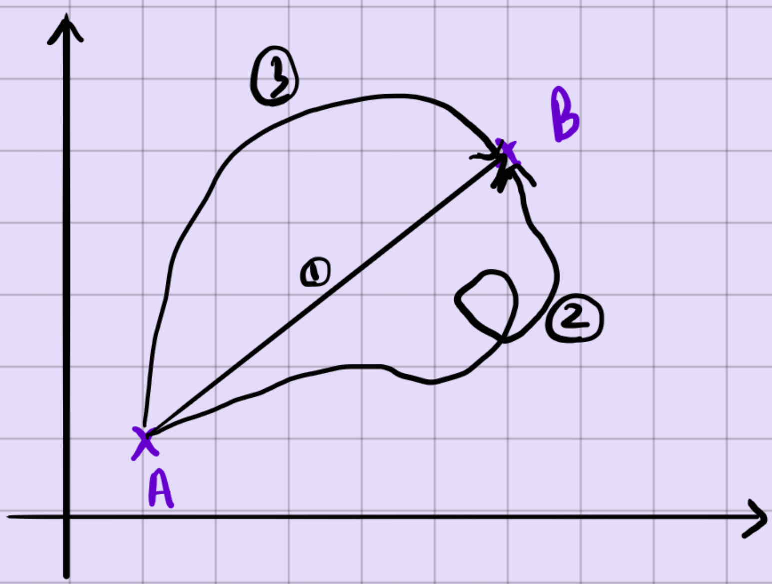
$$= \left[\left(\frac{\partial\phi}{\partial x}\frac{dx}{dt}\right) + \left(\frac{\partial\phi}{\partial y}\frac{dy}{dt}\right)\right] dt$$

$$= \frac{d\phi}{\cancel{dt}} \cancel{dt} = d\phi$$

$$\Rightarrow \int_C \vec{F} \cdot \vec{r}\,'(t)\, dt = \int_C d\phi = \underbrace{\phi(B)}_{\text{Final point}} - \overbrace{\phi(A)}^{\text{initial point}}$$

$\boxed{\text{Fundamental theorem of Line integrals}}$

If $\vec{F}$ is conservative, the circulation $= \phi(B) - \phi(A)$

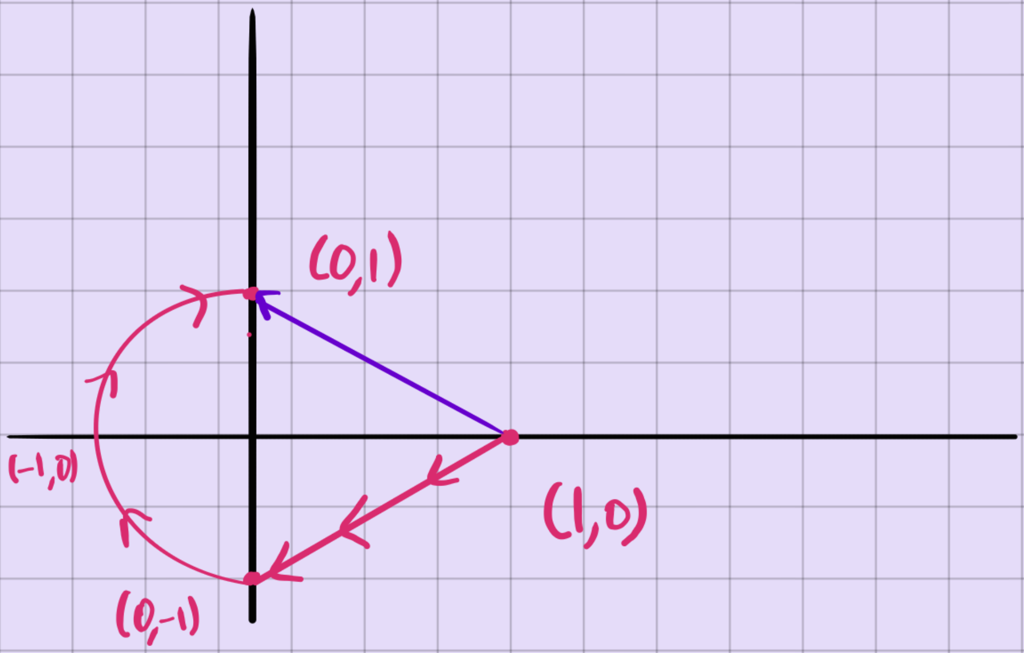


Circulation, $\int_C \vec{F} \cdot \vec{T}\, ds$ is path-independent.

---

5) $\vec{F} = \langle x+y, x \rangle$, C : line from $(1,0) \rightarrow (0,-1)$ then along left half of

$x^2+y^2=1$ to $(0,1)$.



$\vec{F} = \langle x+y, x \rangle \Rightarrow f = x+y, \quad g = x$

$\Rightarrow f_y = 1, \quad g_x = 1$

↳ conservative.

$\phi_x = x+y, \quad \phi_y = x$

$\Rightarrow \phi = \int x+y \, dx = \frac{x^2}{2} + xy + a(y)$

$\phi_y = x + a'(y) = x$

$\Rightarrow a'(x) = 0 \Rightarrow a(x) = C.$

$\Rightarrow \phi = x^2 + xy + C$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \phi(B) - \phi(A), \quad B(0,1),\ A(1,0)$$

$$= 0 - \left(\tfrac{1}{2}\right) = -\frac{1}{2}$$

Green's theorem :

↳ Converting Line integrals → double integrals

Given $\vec{F} = \langle f, g \rangle$, the curl $F = \langle 0, 0, g_x - f_y \rangle$

$\Rightarrow |\text{curl } \vec{F}| = |g_x - f_y|$ (measure of rotation)

$F = \langle -y, x \rangle$

$\Rightarrow \text{curl } \vec{F} = \langle 0, 0, 2 \rangle \Rightarrow |\text{curl } \vec{F}| = 2$

when $\neq 0$, indicates rotation.

Additionally, a vector field $\vec{F} = \langle f, g \rangle$ is conservative if $g_x = f_y$.

$\Rightarrow \quad g_x - f_y = 0$

$\hookrightarrow$ A Conservative vector field will have $|\text{curl } \vec{F}| = 0 \Rightarrow$ no rotation.

irrotational

If $\vec{F} = \langle f, g \rangle$ and C is a simple closed path traversed once in a counter-clockwise direction, then :

closed path

$$\oint_C \vec{F} \cdot \vec{T} \, ds = \oint_C \vec{F} \cdot d\vec{r}$$

$$= \iint_{\mathbb{R}} \underbrace{g_x - f_y}_{\hookrightarrow |\text{curl } \vec{F}|} \, dA$$

$\Rightarrow$ accumulating rotation on $\mathbb{R}$.

region encircled by the closed loop C.

---

1) $\vec{F} = \langle y+2, x^2+1 \rangle$. $C : (-1,1) \rightarrow (1,1)$ along $y = x^2$

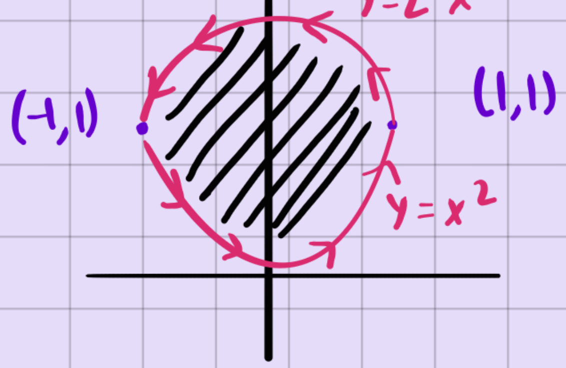
$(1,1) \rightarrow (-1,1)$ along $y = 2 - x^2$

$f_y = 1$, $g_x = 2x \Rightarrow$ function $\vec{F}$ is not conservative.

$\Rightarrow \text{curl } F = g_x - f_y = 2x - 1$

$\Rightarrow |\text{curl } F| = 2x - 1$

$y = 2 - x^2$



$$R: \quad x^2 + (y-1)^2 = 1$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_{\mathbb{R}} 2x - 1 \quad dA$$

$$= \int_{-1}^{1} \int_{x^2}^{2-x^2} 2x - 1 \quad dy \; dx$$

$$= \int_{-1}^{1} 2x\left[2 - x^2 - x^2\right] - 1\left[2 - x^2 - x^2\right] \; dx$$

$$= \int_{-1}^{1} 4x - 4x^3 - 2 + 2x^2 \quad dx$$

$$= \frac{4x^2}{2} - \frac{4x^4}{4} - 2x + \frac{2x^3}{3} \Bigg|_{-1}^{1}$$

$$= 2x^2 - x^4 - 2x + \frac{2}{3}x^3 \Bigg|_{-1}^{1}$$

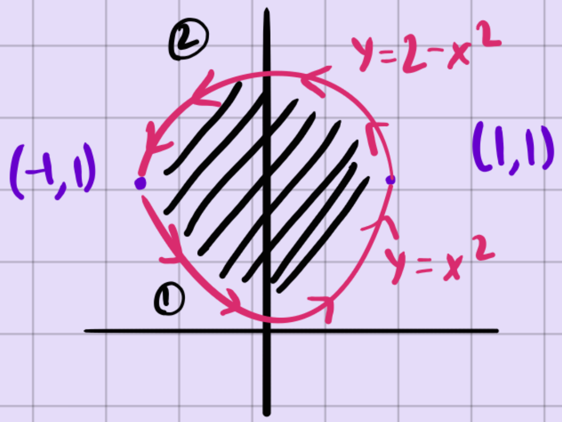
$$= \cancel{2} - \cancel{1} - 2 + 2/3 - \left(\cancel{2} - \cancel{1} + 2 - 2/3\right)$$

$$= -4 + 4/3 = \frac{-8}{3}$$

Otherwise :

$\vec{F} = \langle y+2,\ x^2+1 \rangle$. $C: (-1,1) \to (1,1)$ along $y = x^2$

$(1,1) \to (-1,1)$ along $y = 2-x^2$



$\vec{r}_1(t) = \langle t, t^2 \rangle \quad -1 \le t \le 1$

$\vec{r}_2(t) = \langle -t, 2-t^2 \rangle \quad -1 \le t \le 1$

$\vec{r}_1{}'(t) = \langle 1, 2t \rangle$, $\vec{r}_2{}'(t) = \langle -1, -2t \rangle$

$$\Rightarrow \int_{-1}^{1} \langle t^2+2,\ t^2+1 \rangle \cdot \langle 1, 2t \rangle \, dt$$

$$+ \int_{-1}^{1} \langle 2-t^2+2,\ t^2+1 \rangle \cdot \langle -1, -2t \rangle \, dt$$

$$\Rightarrow \int_{-1}^{1} t^2+2+2t^3+2t \quad dt + \int_{-1}^{1} t^2-4-2t^3-2t \quad dt$$

$$= -\frac{8}{3}$$

$$\vec{F} = \langle f, g \rangle \Rightarrow \boxed{f_y = g_x}$$

Note :

↳ If $\vec{F}$ is conservative, then $\oint_C \vec{F} \cdot \vec{T}\, ds = 0$.

↳ Because end point = start point

$\Rightarrow \phi(B) - \phi(A) = \phi(A) - \phi(A) = 0$

↳ $f_y = g_x \Rightarrow g_x - f_y = 0$

$\Rightarrow |\text{curl } \vec{F}| = 0$

We can also use Green's theorem to find the area of R (area with closed simple loop C) with a line integral.

↳ $$\oint f\, dx + g\, dy = \iint_{\mathbb{R}} g_x - f_y\, dA$$

$\vec{F} = \langle f, g \rangle$, $\vec{r}(t) = \langle x(t), y(t) \rangle$

↳ $d\vec{r} = \langle x'(t), y'(t) \rangle dt = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_C \langle f, g \rangle \cdot \langle dx, dy \rangle$$

$$= \int_C f\,dx + g\,dy$$

If $g_x - f_y = 1$, meaning that the $|\text{curl } F| = 1$, then RHS = area of $\mathbb{R}$.

↳ $\iint_{\mathbb{R}} dA$

So if $\vec{F} = \langle f, g \rangle$ is such that $g_x - f_y = 1$, we can find the area of any C that is simple, closed, doesn't overlap, and follows counter-clockwise motion.

↳ $\oint_C \vec{F} \cdot \vec{T}\, ds = \oint_C \vec{F} \cdot d\vec{r} = \oint_C f\,dx + g\,dy$

---

2) According to Green's theorem, which of the following line integrals is NOT equal to the area of the region enclosed by a simple closed curve C, oriented counter clockwise?

A) $\frac{1}{5} \int_C y\,dx + 6x\,dy$ $\Rightarrow \vec{F} = \langle y, 6x \rangle$

↳ $g_x - f_y = 6 - 1 = 5$

B) $\frac{1}{5} \int_C -3y\,dx + 2x\,dy$ $\Rightarrow \vec{F} = \langle -3y, 2x \rangle$

↳ $g_x - f_y = 2 - -3 = 5$

C) $\frac{1}{5} \int 2y\,dx - 3x\,dy$ $\Rightarrow \vec{F} = \langle 2y, -3x \rangle$

↳ $g_x - f_y = -3 - 2 = \underline{\underline{-5}}$

general form :

$$\Rightarrow \frac{1}{5} \int_C \langle f, g\rangle \cdot \langle dx, dy\rangle$$

↳ if $g_x - f_y = 1$, then $\int_C f\,dx + g\,dy = \iint_R dA$

↳ if $g_x - f_y = F$, then $\int_C f\,dx + g\,dy = F \iint_R dA$

$$\Rightarrow \iint_R dA = \frac{1}{F} \int_C f\,dx + g\,dy$$

So $F = 5$

$$\Rightarrow g_x - f_y = \underline{\underline{5}}$$

Conversely, instead of the circulation, we can use the flux :

↳ $\int_C \vec{F} \cdot \vec{N}\, ds$

$$\Rightarrow \oint f\,dy - g\,dx = \iint_R (f_x + g_y)\, dA$$

Overall :

1) $\vec{F} = \langle f, g\rangle$ is conservative if $f_y = g_x$. If conservative, $|\text{curl } \vec{F}| = |g_x - f_y| = 0$.

2) If $\vec{F} = \langle f, g \rangle$ is conservative, there exists a scalar ϕ such that $\vec{F} = \vec{\nabla}\phi$.

$\hookrightarrow$ $\phi_x = f$, $\phi_y = g$

3) Knowing ϕ, the line integral $\int_C \vec{F} \cdot \vec{T} \, ds = \phi_B - \phi_A$,

where B is the end point of C, and A is the start point. This is because a conservative vector is path independent.

4) If C is a closed simple loop, then B=A, meaning the line integral evaluates to 0.

5) According to Green's theorem, if C is a simple closed loop of counter-clockwise orientation, then circulation:

$$\oint_C f \, dx + g \, dy = \iint_R (g_x - f_y) \, dA$$

$g_x - f_y$ = curl $\vec{F}$. If $g_x - f_y = 0$, the function is conservative.

$\hookrightarrow$ A conservative loop has $\oint_C \vec{F} \cdot \vec{T} \, ds = 0$.

R is the region inside C.

6) If $g_x - f_y = 1$, then $\iint_R g_x - f_y \, dA$ is

the area of R, the region within the loop. This can be evaluated with the circulation.

$\therefore$ If $|g_x - f_y| = F$, then $\iint_R g_x - f_y \, dA$

is $F \iint_R dA = \oint_C f\,dx + g\,dy$

$$\Rightarrow \iint_R dA = \frac{1}{F} \oint_C f\,dx + g\,dy .$$

7) For the flux, we get the following from Green's theorem :

$$\oint_C f\,dy - g\,dx = \iint_R \underbrace{(f_x + g_y)}_{\text{divergence of } \vec{F}.} \, dA$$

$\operatorname{div}(\vec{F})$ for $\vec{F} = \langle f, g \rangle = f_x + g_y$

In a more general sense :

∇ is the "del operator"

↳ $\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

Given $\vec{F} = \langle f, g, h \rangle$

⇒ <u>curl</u> $\vec{F} = \vec{\nabla} \times \vec{F}$ → if $= 0$, then $\vec{F}$ is conservative.

↳ Measure of rotation ⇇

⇒ <u>div</u> $\vec{F} = \vec{\nabla} \cdot \vec{F}$

↳ Measure of rate of spreading out.