(1)

MA 262 - post exam 2 review

sections 5.1 - 7.6

- general form of 2nd order diff eq:

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

cannot contain y → makes it linear

- can be homogeneous OR nonhomogeneous dep. on $f(x)$

HOMOGENEOUS (2nd ORDER) DIFF EQ

form:

$$y'' + p(x)\,y' + q(x)\,y = 0$$

general solution: linear combo of solutions

$$y(x) = c_1 y_1 + c_2 y_2$$

- y_1 and y_2 form basis for solution space

y_1 and y_2 are LINEARLY INDP.

if $c_1 = c_2 = 0$ / $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$ (Wronskian)

* $W = 0$ → lin. dep.

* for an n^{th} order diff eq, n linearly indp. solutions exist

↳ can use REDUCTION OF ORDER for higher-order diff eq where at least one solution is known: $y_2 = v(x)\,y_1$ — plug in known solution

- solve for $y_2 \cdots y_2^{(n)}$ and sub into OG equation to solve for $v(x)$

↳ choose constants so that $v(x) = 0$ OR a constant and sub back into $y_2 = v(x)\,y_1$ w/ known values

$$\ast\; y_2 = y_1 \int \frac{e^{-\int p(t)}}{(y_1)^2}$$

Homogeneous diff eq w/ CONSTANT COEFFICIENTS

- set up characteristic eq. such that $y^{(n)} = r^n$ → solve for r

SOLUTION: $y = e^{rx}$

3 CASES:

- Distinct roots – r's are $\mathbb{R}$, $r_1 \neq r_2 \cdots \neq r_n$

$y_1 = e^{r_1 x},\; y_2 = e^{r_2 x},\; \cdots\; y_n = e^{r_n x}$

- Repeated roots – r's are $\mathbb{R}$, $r_1 = r_2 \cdots = r_n$

$y_1 = e^{rx},\; y_2 = xe^{rx},\; \cdots\; y_n = x^n e^{rx}$

- Complex roots – r's are $\mathbb{C}$, conjugate pairs

- only solve for 1 root – doesn't matter which

NOTE: higher order diff. eqs can have a combination of any of these cases

2^{nd}-order: $2\,\mathbb{R}$, 2 rep $\mathbb{R}$, $\mathbb{C}$ pair

EULER'S FORMULA: $e^{ix} = \underbrace{\cos x}_{real} + i\underbrace{\sin x}_{imaginary}$, $\quad e^{-ix} = \underbrace{\cos x}_{real} - i\underbrace{\sin x}_{imaginary}$

↳ real and imaginary parts become solutions

$y_1 = \cos(x),\; y_2 = \sin(x)$

* $e^{(a+bi)x} = e^{ax} \cdot e^{bix} \rightarrow y = e^{ax}(c_1\cos(bx) + c_2\sin(bx))$

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EIGENVALUE METHOD FOR LINEAR SYSTEMS

- find λ's w/ $\det(A - \lambda I) = 0$ → 3 cases

→ λ's are $\mathbb{R}$, distinct – solution is linear combo of $e^{\lambda_n t} v_n$

→ λ's are $\mathbb{C}$ conjugate pairs – * Euler's formula: $e^{it} = \cos t + i\sin t$

Solutions: $x_1 = \vec{u} + i\vec{v}$, $x_2 = \vec{u} - i\vec{v}$ where $\vec{u}$ is real part, $\vec{v}$ is imaginary part

→ λ's are repeated:

algebraic multiplicity – # times λ is rep.

geometric multiplicity – # (lin indp.) vectors resulting from a λ

dimension of Eigenspace

* matrix is COMPLETE if alg. mult. = geo. mult.

↳ solutions formed normally

* when a $\vec{v}$ is missing (geo < alg), matrix is DEFECTIVE

↳ $\vec{v}_1$ = true ordinary eigen vector (what you get from $(A-\lambda I) = 0$), v_2 = generalized vector

solution: $x = e^{\lambda t}[t\vec{v}_1 + v_2]$

* find generalized vector: $(A - \lambda I)\vec{v}_{n+1} = v_n$ OR $\underbrace{(A - \lambda I)^n}_{\text{always zero matrix when missing } n-1\ \vec{v}\text{'s}} v_n = 0$

- use variations of these to find missing $\vec{v}$

2 methods when mult. $\vec{v}$'s are missing

"BUILD UP" — start w/ v_1 (ord. $\vec{v}$) and use $(A - \lambda I)\vec{v}_{n+1} = v_n$ to find others

same thing!

"BUILD DOWN" — choose v_n to be anything and use $(A - \lambda I)\vec{v}_n = \vec{v}_{n-1}$ to find others — $\vec{v}_1$ MAY NOT MATCH ord. $\vec{v}$ – use it anyway

↳ v_n cannot be zero vector or any mult of true $\vec{v}$

* for multiple missing $\vec{v}$'s for rep. λ's, solution will look like integration:

ex. $x = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t}[t v_1 + v_2] + c_3 e^{\lambda t}[\frac{1}{2} v_1 t^2 + t v_2 + v_3] \cdots$

if you get $v_1 = \vec{0}$ it's bc v_2 is linear combo of true $\vec{v}$'s — set v_1 to one of the true $\vec{v}$'s

↳ don't include v_1 as a repeat (mult. by t)

SOLUTION CURVES OF LINEAR SYSTEMS

"sink" → toward origin → $\lambda < 0$

"source" → away from origin → $\lambda > 0$

"improper" → two asymptotes exist
└ DEFECTIVE matrix
└ defined by $\vec{v}$'s

"proper" → no asymptotes, radial
┌ COMPLETE matrix

"saddle point" → some solutions go away, others towards origin

↳ associated w/ opp. signs in λ's – asymptotes are defined by $\vec{v}$'s

"center" → ellipses around origin → λ is complex

"spiral" → occurs when real part exists and e^t precedes $\vec{v}$'s

rotation: indicated by presence of cos / sin in eigenvectors

MASS-SPRING DAMPER (homogeneous diff-eq application)

$mx'' + cx' + kx = 0$ (spring) $F_s = -kx$ (spring constant) (damper) $F_d = -cx'$ (damper constant)

solution – alt. form:

$x(t) = C\cos(\omega t - \alpha)$

- will get complex roots
- A and B are coeff. of cos and sin in particular solution (normal form)

C = amplitude = $\sqrt{A^2 + B^2}$ (always positive)

ω = circular frequency

* v = linear freq = $\frac{\omega}{2\pi} = \frac{1}{T}$

T = period = $\frac{2\pi}{\omega}$

α : $A > 0, B > 0 = \tan^{-1}\left(\frac{B}{A}\right)$

$A < 0, B > 0 = \pi + \tan^{-1}\left(\frac{B}{A}\right)$

$A < 0, B < 0 = 2\pi + \tan^{-1}\left(\frac{B}{A}\right)$

phase angle

max position @ $x'(t) = 0$

* Normal solution form: $x(t) = A\cos(\omega t) + B\sin(\omega t)$

- type of solution dep. on discriminant of characteristic eq

$c^2 - 4mk$ where $m, c, k \neq 0$ (know graphs!)

- discriminant $> 0 \rightarrow$ roots $= \mathbb{R}$, distinct : OVERDAMPED – strong damper
- discriminant $= 0 \rightarrow$ roots $= \mathbb{R}$, rep : CRITICALLY DAMPED – "just right"
- discriminant $< 0 \rightarrow$ roots $= \mathbb{C}$: UNDERDAMPED – weak damper

only case w/ oscillations ; * equilibrium crossed when $y(t) = 0$ *

NONHOMOGENEOUS EQUATIONS – *Undetermined Coefficients*

- use when $f(x)$ is polynomial, exponential, sin/sinh, cos/cosh
- BASIC IDEA: "guess" y_p based on form of x

STEPS TO FIND y_p (particular solution):

1) set y_p = general form based on $f(x)$
2) Find derivatives of y_p, sub in for y – SIMPLIFY
3) Coeff. left = coeff right – solve for unknowns

* if y_p form is a duplicate of y_c, multiply by x until repeat is gone

- General Solution: $y = y_c + y_p$ (y_c: complementary solution (solve as if $f(x) = 0$); y_p: particular solution ("guess"))
  - always solve for y_c before y_p to check dupes
- if $f(x)$ = constant
  $y_p = A$ is sufficient

2 ways to solve w/ hyperbolics –
- rewrite as exponentials
- retain hyperbolic form

COMMON FORMS:

- exponential ($f(x) = e^x$)
  $y = Ae^x$ or $(Ax + B)e^x$
- 1st polynomial
  $y = Ax + b$
- 2nd polynomial
  $y = Ax^2 + bx + c$
- cos or sin
  $y = A\cos(x) + B\sin(x)$
- cosh as exponential
  $y = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$
- sinh as exponential
  $y = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

guess: $Ae^x + Be^{-x}$

also important:

$e^x = \cosh(x) + \sinh(x)$

$e^{-x} = \cosh(x) - \sinh(x)$

* $y_p = A\cosh(x) + B\sinh(x)$

$\cosh' = \sinh$ $\sinh' = \cosh$

no sign change * if x terms exist, should cancel

NONHOMOGENEOUS EQUATIONS: *Variations of Parameters*

- General Solution: $y = u_1(x)y_1 + u_2(x)y_2$ ("parameters") * Don't have to worry abt dupes w/ y_c!

- Find u_1, u_2 by solving system:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

OR (SIMPLIFIED)

$$y_p = -y_1\int \frac{y_2 f(x)}{W}dx + y_2\int \frac{y_1 f(x)}{W}dx$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

TRIG TO KNOW for this:

1) $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$
2) $\cos(a)\sin(b) = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$
3) $\cos(a)\sin(b) = \frac{1}{2}[\sin(a+b) - \sin(a-b)]$

* $\sin(-x) = -\sin(x)$
$\cos(-x) = \cos(x)$

EIGENVALUES AND EIGENVECTORS – for scalar equations

- eigenvalues (λ) – factors by which eigenvectors ($\vec{v}$) change ($\vec{v}$: define direction of matrix)
- Solve $\det(A - \lambda I) = 0$ for λ's
- Use resulting λ's to find $\vec{v}$'s w/ $(A - \lambda I)\vec{v} = 0$ (should get at least 1 zero row)

* $(A - \lambda I) \rightarrow$ subtract λ from main diagonal values
  ↳ λ = main diagonal values for triangular / diagonal matrix A

* if λ's are complex, $\vec{v}$'s will be complex pairs (half the work)

FIRST-ORDER SYSTEMS OF DIFF EQS

form:

$x'(t) = f(x, y, t)$ – $x, y \rightarrow$ dep var.

$y'(t) = g(x, y, t)$ – $t \rightarrow$ indp. var.

* n^{th} order $\rightarrow$ n 1^{st} order and vice versa
↳ use to solve higher-order systems

STEPS:

- define new variables to rep. $x^{(n-1)} \rightarrow x$
- write the diff eq for the new variables

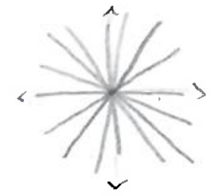
sub into OG equation, solve for variable rep. $x^{(n)}$ (should be a deriv. of one of the new variables)

* should be able to set up result as matrix equation: $\vec{x}' = p(t)\vec{x} + f(t)$ – nonhomogeneous term (not dep on other var.); $p(t)$: coeff. based on diff eq

↳ Basic properties –
- if $\vec{f}(t) = 0$, $\vec{x}' = \vec{p}(t)x$ is homogeneous system
  Solution: $\vec{x} = c_1x_1 + c_2x_2 + \cdots + c_nx_n \rightarrow$ solutions are linearly indp.
- if $\vec{p}(t)$ is $n \times n$ matrix $\rightarrow$ n solutions exist
- a linear combo of solutions is also a solution
- Wronskian of solutions : $W = |\vec{x}_1\ \vec{x}_2 \cdots \vec{x}_n| \rightarrow$ solutions as columns
- if $f(t) \neq 0 \rightarrow$ nonhomogeneous system – solution: $\vec{x} = \vec{x}_c + \vec{x}_p$

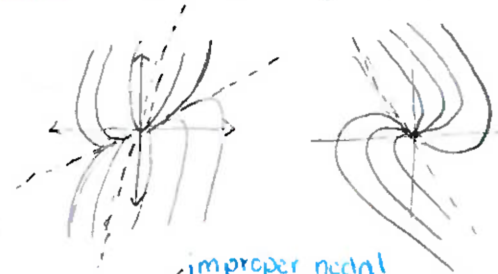
basically similar to a first-order scalar equation

CLASSIFYING SOLUTION CURVES → typical phase portraits



proper nodal - radial
(source or sink)

rep λ's w/ enough $\vec{v}$'s
↳ COMPLETED matrix



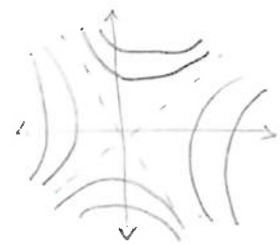
improper nodal
S-shape - (source or sink)

- 2 asymptotes
- rep λ's, not enough $\vec{v}$'s
↳ DEFECTIVE matrix

key: grow out of asymp.

- 1 straight line asymp.
- λ's are $\mathbb{R}$, share same sign distinct

key: all solutions grow from origin



saddle point - square-ish

- λ's are $\mathbb{R}$, opp. signs
- 2 asymptotes

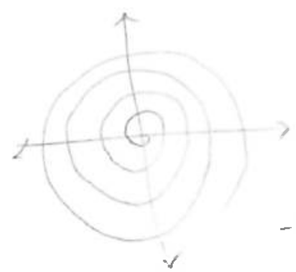
orientation given by real part of $\vec{v}$'s

* sin or cos means rotation

center
ellipses

λ's are purely imaginary (NO real part)
$\mathbb{C}$
discriminant is < 0

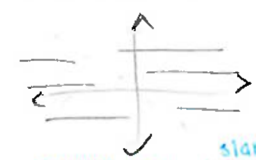
- direction given by choosing $\vec{x}$ and finding direction of tangent from $\vec{x}' = A\vec{x}$



spiral
(source: real part > 0 sink: real part < 0)

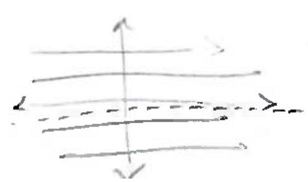
- has e^t (real part) and cos/sin (complex part)

- λ's are $\mathbb{C}$ and have real part
- direction det. by choosing $\vec{x}$ and finding tangent from $\vec{x}' = A\vec{x}$



parallel lines w/ slanted asymp.

one 0 λ and one $\mathbb{R}$ λ exists
(towards or away from dotted line)



parallel lines w/ parallel asymptote

- rep. $\lambda = 0$ w/ 2 lin. ind. $\vec{v}$'s
→ if x-comp of vector is 0, left to right
→ if y-comp of vector is 0, up and down

VARIATION OF PARAMETERS: more info

$u_1 = -\int \frac{y_2 f(x)}{W}$ OR $\Longrightarrow$ solve system of equations

$u_2 = \int \frac{y_1 f(x)}{W}$

$u_1' y_1 + u_2' y_2 = 0$

$u_1' y_1' + u_2' y_2' = f(t)$ — right hand side of standard form

2 options to solve $\longrightarrow$ $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$

1) row reduction

2) use matrix inverse $\vec{u}' = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$ } treat variables like normal numbers

* remember exponent prop. $a^m \cdot a^n = a^{m+n}$

MISCELLANEOUS: cumulative

- If given a diff eq. and asked if all solutions on $(-\infty, \infty)$ are vector spaces, solve diff eq, treat solutions as "vectors" and use them to test conditions of vector spaces
  - ↳ contains $\vec{0}$, closed under addition, closed under mult.
    - ↳ just one solution has to satisfy this one