Line integrals :

↳ Accumulation of $f(x,y,z)$ along a parametrized curve, c.

$$\int_C f(x,y,z)\, ds$$

Surface integrals :

↳ Accumulation of $f(x,y,z)$ all over a surface S.

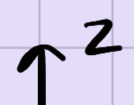
$$\iint_S f(x,y,z)\, dS$$

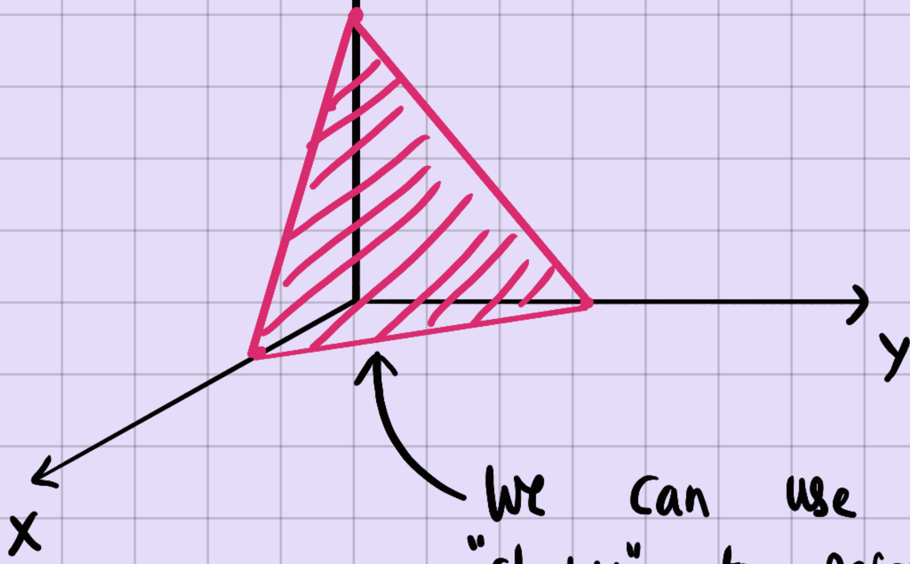
Parameterizing the surface S :

↳ A dimension higher than the line, so $\vec{r}(u,v)$.

$\vec{r}(u,v)$, for $u \in [\ \ ]$ and $v \in [\ \ ]$

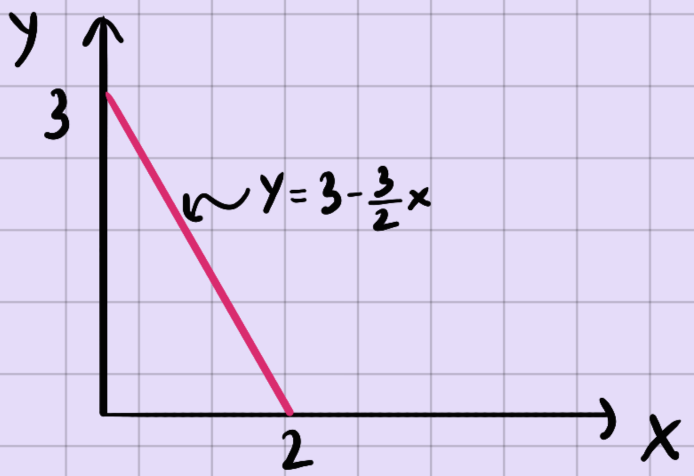
1) Parameterize the part of $3x + 2y + z = 6$ in the first octant.





We can use the xy "shadow" to parameterize the surface :

$\Rightarrow \vec{r}(x,y) = \langle x, y, \underset{}{z} \rangle$ ← $6-2y-3x$



$0 \leq x \leq 2$

$0 \leq y \leq 3 - \frac{3}{2}x$

So, $\vec{r}(u,v) = \vec{r}(x,y) = \langle x, y, 6-2y-3x \rangle$

$x \in [0,2]$

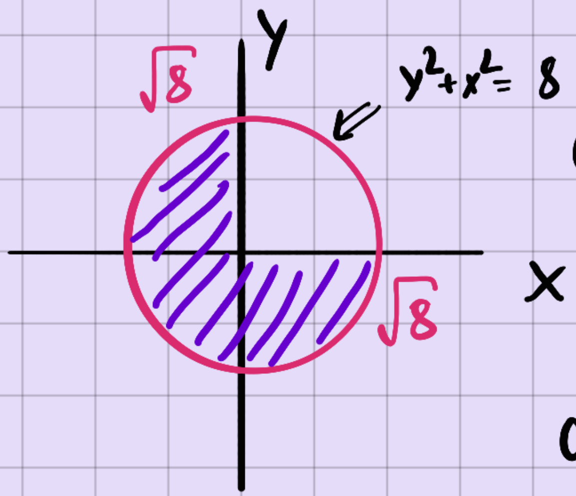
$y \in [0, 3-1.5x]$

2) $x^2+y^2+z^2 = 9$ above $z=1$ in the <u>first octant</u>.

$\Rightarrow z = \sqrt{9-x^2-y^2}$

$$1 \le z \le \sqrt{9-x^2-y^2}$$

At $z=1$:



$$0 \le x \le \sqrt{8}$$

$$0 \le y \le \sqrt{8-x^2}$$

$$\vec{r}(x,y) = \left\langle x, y, \sqrt{9-x^2-y^2} \right\rangle$$

$$0 \le x \le \sqrt{8}, \quad 0 \le y \le \sqrt{8-x^2}$$

3) $x^2+y^2+z^2=9$ above $z=1$ in the first octant, in cylindrical coordinates.

$$1 \le z \le \sqrt{9-x^2-y^2}$$

$$0 \le r \le \sqrt{8}$$

$$0 \le \theta \le \pi/2$$

Say
$u=r$, $v=\theta$

$x = r\cos\theta = u\cos v$

$x = r\cos\theta = u\cos v$

$y = r\sin\theta = u\sin v$

$$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ \sqrt{9-r^2}\ \rangle$$

$$0 \le u \le \sqrt{8}, \qquad 0 \le v \le \pi/2$$

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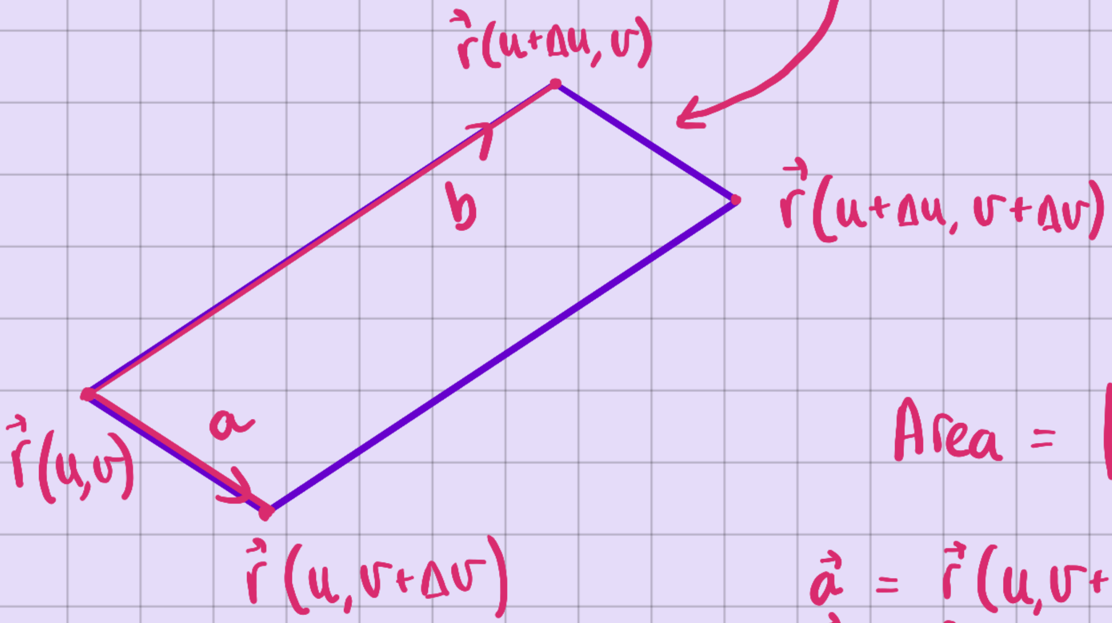
dS ?

↳ Area of small patch of surface.



$$\text{Area} = |\vec{a} \times \vec{b}|$$

$$\vec{a} = \vec{r}(u, v+\Delta v) - \vec{r}(u,v)$$
$$\vec{b} = \vec{r}(u+\Delta u, v) - \vec{r}(u,v)$$

Since :

$$f_x = \lim_{h\to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$\Rightarrow f_x \approx \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$\Rightarrow f(x+\Delta x, y) - f(x,y) = f_x \, \Delta x$$

$$\vec{a} = \vec{r}(u, v+\Delta v) - \vec{r}(u,v) = r_v \, \Delta v$$

$$\vec{b} = \vec{r}(u+\Delta u, v) - \vec{r}(u,v) = r_u \, \Delta u$$

$$\Rightarrow dS = |\vec{a} \times \vec{b}| = |\vec{r}_v \times \vec{r}_u| \, \Delta u \, \Delta v$$

$$dS = |\vec{r}_v \times \vec{r}_u| \, du \, dv$$

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1) $\displaystyle\iint_S (x+y) \, dS$ $\quad S$: $3x+2y+z=6$ in first octant above $(x,y) \in [0,1), [0, 3/2]$

let $x = u$, $y = v$.

$\vec{r}(u,v) = \langle u, v, 6-3u-2v \rangle$, $0 \le u \le 1$, $0 \le v \le 3/2$

$\vec{r}_u = \langle 1, 0, -3 \rangle$, $\vec{r}_v = \langle 0, 1, -2 \rangle$

$$\vec{r}_u \times \vec{r}_v = \langle 1, 0, -3 \rangle \times \langle 0, 1, -2 \rangle$$

$$= \langle 3, 2, 1 \rangle$$

$$dS = |\langle 3, 2, 1 \rangle| \, du \, dv = \sqrt{14} \, du \, dv$$

$$\sqrt{14}\int_0^1\int_0^{3/2} u+v \; dv\, du$$

$$= \sqrt{14}\int_0^1 \left[uv + \frac{v^2}{2}\right]_0^{3/2} du$$

$$= \sqrt{14}\int_0^1 \frac{3}{2}u + \frac{9}{8} \; du$$

$$= \sqrt{14}\left[\frac{3}{4}u^2 + \frac{9}{8}u\right]_0^1 = \sqrt{14}\left[\frac{6}{8} + \frac{9}{8}\right]$$

$$= \frac{15\sqrt{14}}{8}$$

Shortcuts for the coordinate systems:

a) Cartesian:

$$\vec{r}(u,v) = \langle \underset{x}{u}, \underset{y}{v}, \underset{z}{f(u,v)} \rangle$$

$$\vec{r}_u = \langle 1, 0, f_u \rangle, \quad \vec{r}_v = \langle 0, 1, f_v \rangle$$

$$dS = \left| \vec{r}_u \times \vec{r}_v \right| \, du\, dv$$

$$dS = \sqrt{1 + f_u^2 + f_v^2} \quad du\, dv$$

$$\iint_S g(x,y,z)\, \underline{dS}$$

$$\sqrt{1 + f_x^2 + f_y^2} \quad dA$$

$dx\,dy$

$dy\,dx$

So if $z = f(x,y)$, we can rewrite

$$dS = \sqrt{1 + f_x^2 + f_y^2}$$

b) Cylindrical :

For $z = f(r,\theta)$,

$$dS = \sqrt{f_r^2 + \frac{1}{r^2} f_\theta^2 + 1} \cdot r \; dr\, d\theta$$

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1) Surface area of $z = 11 - x^2 - y^2$ below $z = 2$.

$x = u$, $y = v$

$\Rightarrow \vec{r}(u,v) = \langle u, v, 11 - u^2 - v^2 \rangle$, $(u,v) \in [0,3]$

$z = f(x,y) = 11 - x^2 - y^2$

$\Rightarrow dS = \sqrt{1+(-2u)^2+(-2v)^2}\ du\, dv$

$= \sqrt{1+4u^2+4v^2}\ du\, dv$

Use polar :

$dS = r\sqrt{1+4r^2}\ dr\, d\theta$

SA : $\int_0^{2\pi}\int_0^3 r\sqrt{1+4r^2}\ dr\, d\theta$

$u = 1+4r^2 \Rightarrow du = 8r$

$\frac{1}{8}\int_0^{2\pi}\int_1^{37} u^{1/2}\ du\, d\theta = \frac{1}{8}\int_0^{2\pi} \frac{2}{3}u^{3/2}\Big|_1^{37}\ d\theta$

$= \frac{\pi}{6}\left[37^{3/2} - 1\right]$

2) Surface area of $z = \frac{x^2}{2} + \frac{y^2}{2}$ between $x^2+y^2=8$

and $x^2+y^2=24.$

$$\vec{r}(u,v) = \langle u,\ v,\ \tfrac{u^2}{2}+\tfrac{v^2}{2} \rangle$$

$$\Rightarrow f(x,y) = f(u,v) = \frac{u^2}{2} + \frac{v^2}{2}$$

$$f_u = u \ , \ f_v = v$$

$$dS = \sqrt{1+u^2+v^2}\ du\, dv$$

Using Polar :

$$dS = r\sqrt{1+r^2}\ dr\, d\theta \ , \quad \sqrt{8} \le r \le \sqrt{24}$$

$$0 \le \theta \le 2\pi$$

$$SA = \int_0^{2\pi} \int_{\sqrt{8}}^{\sqrt{24}} r\sqrt{1+r^2}\ dr\ d\theta$$

$r^2+1=u$

$du = 2r\,dr$

$$= \pi \int_9^{25} u^{1/2}\ du = \pi \left[\frac{2}{3} u^{3/2}\right]_9^{25}$$

$$= \frac{2\pi}{3}\left[125-27\right]$$

$$= \frac{2\pi}{3} \times 98$$

$$= \frac{196\pi}{3}$$

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1) Write down in the form $\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$, in cartesian.

2) $dS = \sqrt{1 + f_x^2 + f_y^2}\ dx\, dy$

3) Choose the appropriate coordinate system, and parametrize accordingly.

↳ $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$.

↳ $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$.

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3) Surface area of a cone $z^2 = 4(x^2+y^2)$ for $0 \le z \le 20$.

cylindrical

$$f(u,v) = \langle u, v, \sqrt{4(u^2+v^2)} \rangle, \quad x = u,\ y = v.$$

Since $z_{max} = 20 \Rightarrow x^2+y^2 = 100$

$z_{min} = 0 \Rightarrow x^2+y^2 = 0$

$$f(x,y) = z = \sqrt{4(x^2+y^2)} = 2(x^2+y^2)^{1/2}$$

$$f_x = 2x\,(x^2+y^2)^{-1/2}, \quad f_y = 2y\,(x^2+y^2)^{-1/2}$$

$$\Rightarrow dS = \sqrt{1+f_x^2+f_y^2}\; dy\,dx$$

$$= \sqrt{1+\frac{4x^2}{x^2+y^2}+\frac{4y^2}{x^2+y^2}}\; dA = \sqrt{\frac{5\cancel{x^2}+5\cancel{y^2}}{\cancel{x^2+y^2}}}\; dA$$

$$= \sqrt{5}\, dA$$

$$SA: \iint_{dA} \sqrt{5}\, dx\, dy \qquad 0 \le x^2+y^2 \le 100$$

$$0 \le r \le 10$$

$$\Rightarrow \int_0^{2\pi}\int_0^{10} r\sqrt{5}\, dr\, d\theta \qquad 0 \le \theta \le 2\pi$$

$$\Rightarrow 2\pi\sqrt{5}\left[\frac{r^2}{2}\right]_0^{10} = 2\pi\sqrt{5}\,(50)$$

$$= 100\sqrt{5}\,\pi$$

4) Surface area of $z = 2x^2$ for $-2 \le x \le 2$, $0 \le y \le 4$.

$$z = f(x,y) = 2x^2$$

$$f_x = 4x, \quad f_y = 0 \quad \Rightarrow \quad dS = \sqrt{1+16x^2}\ dx\, dy$$

$$SA = \int_0^4 \int_{-2}^{2} \sqrt{1+16x^2}\ dx\, dy$$

5) Surface area of $z = 5(x^2+y^2)$ for $z \in [0, 125]$

$$f(x,y) = 5(x^2+y^2)$$

$$\Rightarrow f_x = 10x, \quad f_y = 10y \quad \Rightarrow dS = \sqrt{1+100x^2+100y^2}\ dxdy$$

$$dS = \sqrt{1+100(x^2+y^2)}\ dx\, dy$$

At $z_{max} = 125$, $x^2+y^2 = 25 \Rightarrow r = 5$

At $z_{min} = 0$, $x^2+y^2 = 0 \Rightarrow r = 0$

$$SA = \int_0^5 \int_0^{2\pi} r\sqrt{1+100r^2}\ d\theta\, dr$$

6) $S: \; z = xy$ within $x^2+y^2=3$. Find:

$$\iint_S (z+1)\, dS$$

$\Rightarrow \; r(x,y) = \langle x, y, xy \rangle \leftarrow f(x,y) = xy$

$\Rightarrow \; dS = \sqrt{1+f_x^2+f_y^2}\; dx\,dy$

$$dS = \sqrt{1+x^2+y^2}\; dx\,dy$$

$$\iint_S (xy+1)\sqrt{1+x^2+y^2}\; dx\,dy$$

within $x^2+y^2=3$

$\hookrightarrow \; r \in [0, \sqrt{3}]$

$\theta \in [0, 2\pi]$

$$\int_0^{2\pi}\int_0^{\sqrt{3}} r\,(r\cos\theta\, r\sin\theta + 1)\sqrt{1+r^2}\; dr\, d\theta$$

$$= \int_0^{2\pi}\left[\int_0^{\sqrt{3}} r^3\cos\theta\sin\theta\sqrt{1+r^2} + r\sqrt{1+r^2}\; dr\right]$$

$$= \int_0^{\sqrt{3}} r^3\sqrt{1+r^2}\int_0^{2\pi} \frac{1}{2}\sin 2\theta\; d\theta\, dr + \int_0^{2\pi}\frac{1}{2}\int_0^{\sqrt{3}} 2r\sqrt{1+r^2}\; dr\, d\theta$$

$$\int_0 r\sqrt{\ } \quad \int_0 \frac{\ }{2}\sin 2\theta\, d\theta\, dr \qquad \int_0 \frac{\ }{2}\int_0 2r\sqrt{\ }\, dr\, d\theta$$

$$= \cancel{\int_0^{\sqrt{3}} r^3\sqrt{1+r^2}\left[\frac{-1}{2}\cos 2\theta\right]_0^{2\pi}} \to 0$$

$$\frac{-1}{2}\left[1-1\right] = 0$$

$u = 1+r^2$

$du = 2r\,dr$

$$2\pi\cdot\frac{1}{2}\int_1^4 u^{1/2}\,du$$

$$= \frac{14\pi}{3}$$

7) S: part of the sphere $x^2+y^2+z^2=1$ above $z=\frac{1}{2}$.

Compute $\displaystyle\iint_S 12z^2\,dS$

Using spherical coordinates :

$\Rightarrow r(\theta,\phi) = \langle \rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi\rangle$

$\Rightarrow dS = |r_\theta \times r_\phi|\ d\phi\, d\theta$

$\hookrightarrow r_\theta = \langle -\rho\sin\phi\sin\theta,\ \rho\sin\phi\cos\theta,\ 0\rangle$

$$r_\phi = \langle \rho\cos\phi\cos\theta,\ \rho\cos\phi\sin\theta,\ -\rho\sin\phi \rangle$$

$$\vec{n} = \left\langle -\rho^2\sin^2\phi\cos\theta,\ -\rho^2\sin^2\phi\sin\theta,\ -\rho\sin^2\theta\sin\phi\cos\phi - \rho\cos^2\theta\cos\phi\sin\phi \right\rangle$$

$$= \langle -\rho^2\sin^2\phi\cos\theta, -\rho^2\sin^2\phi\sin\theta,\ -\ \rho\sin\phi\cos\phi \rangle$$

$$|\vec{n}| = \sqrt{\rho^4\sin^4\phi(\cos^2\theta + \sin^2\theta) + \ \rho^2\sin^2\phi\underbrace{\cos^2\phi}_{\hookrightarrow 1-\sin^2\phi}}$$

$$= \sqrt{\cancel{\rho^4\sin^4\phi} + \ \rho^4\sin^2\phi - \ \cancel{\rho^4\sin^4\phi}}$$

$$= \ \rho^2\sin\phi$$

$$dS = \ \rho^2\sin\phi \quad d\phi\ d\theta \qquad [\rho = 1]$$

$$\int_0^{2\pi}\int_0^{\pi/3} 12\cos^2\phi\sin\phi \quad d\phi\ d\theta$$

$\hookrightarrow 0 \le \theta \le 2\pi$

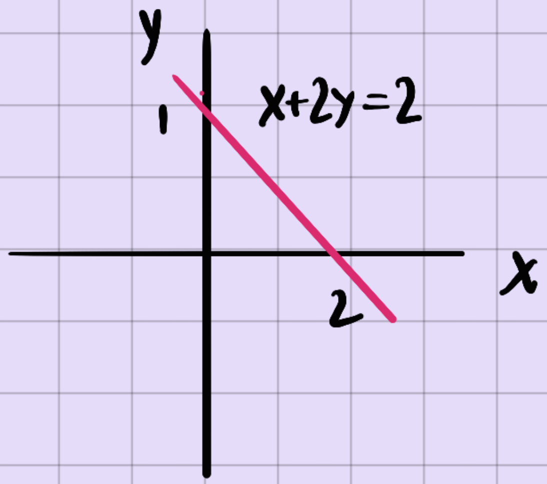
$z = 1/2$

$\Rightarrow \cos\phi = 1/2 \ \Rightarrow \ \phi = \pi/3$

$\hookrightarrow \ 0 \le \phi \le \pi/3$

8) Area of $x + 2y + 2z = 2$ in the first octant.

$$r(x,y) = \langle x, y, 1-y-\tfrac{x}{2} \rangle, \quad x \in [0,2],\ y \in [0,1]$$



$$f(x,y) = z = 1-y-\tfrac{x}{2}$$

$$f_x = -\tfrac{1}{2}, \quad f_y = -1$$

$$dS = \sqrt{1+1+\tfrac{1}{4}}\ dx\,dy$$

$$= \sqrt{9/4}\ dx\,dy$$

$$0 \le x \le 2$$

$$0 \le y \le 1-\tfrac{x}{2}$$

$$\int_0^2 \int_0^{1-\frac{x}{2}} \frac{3}{2}\ dy\,dx$$

$$\Rightarrow \frac{3}{2}\int_0^2 \left[1-\frac{x}{2}\right] dx = \frac{3}{2}\left[x - \frac{x^2}{4}\right]_0^2$$

$$= \frac{3}{2}\left[2-1\right] = \frac{3}{2}$$

9) $\vec{F} = \langle -y, x, z^3 \rangle$ is part of the sphere $\underline{x^2+y^2+z^2=4}$ above $z=1$, with $\underline{\text{upward orientation.}}$

$\hookrightarrow n_z > 0$

$\hookrightarrow \rho = 2$

$$\iint_S (\nabla \times F) \cdot dS$$

$$\vec{r}(\theta,\phi) = \langle 2\sin\phi\cos\theta,\ 2\sin\phi\sin\theta,\ 2\cos\phi\rangle$$

$$\vec{r}_\theta = \langle -2\sin\phi\sin\theta,\ 2\sin\phi\cos\theta,\ 0\rangle$$

$$\vec{r}_\phi = \langle 2\cos\theta\cos\phi,\ 2\cos\phi\sin\theta,\ -2\sin\phi\rangle$$

$$\vec{n} = \langle -4\sin^2\phi\cos\theta,\ -4\sin^2\phi\sin\theta,\ -4\sin\phi\cos\phi\rangle \quad X$$

$$\hookrightarrow \vec{n} = \langle 4\sin^2\phi\cos\theta,\ 4\sin^2\phi\sin\theta,\ 4\sin\phi\cos\phi\rangle$$

$$\vec{\nabla}\times\vec{F} = \langle \partial_x, \partial_y, \partial_z\rangle \times \langle -y, x, z^3\rangle$$

$$= \langle 0,\ 0,\ 1+1\rangle = \langle 0,0,2\rangle$$

$$\iint_S \langle 0,0,2\rangle\cdot\langle 4\sin^2\phi\cos\theta,\ 4\sin^2\phi\sin\theta,\ 4\sin\phi\cos\phi\rangle\, d\theta\, d\phi$$

$$\Rightarrow 8\iint_S \sin\phi\cos\phi\ d\phi\, d\theta$$

$$\Rightarrow 4\int_0^{2\pi}\int_0^{\pi/3} \sin 2\phi\ d\phi\, d\theta \qquad \theta\in[0,2\pi]$$

$$1 \le 2\cos\phi \le 2$$

$$\Rightarrow 8\pi\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/3} \qquad \phi = \pi/3,\ \phi = 0$$

$$\Rightarrow -4\pi\left[-\frac{1}{2}-1\right] = \cancel{4}\pi\cdot\frac{3}{\cancel{2}} = \underline{\underline{6\pi}}$$

Surface integrals are for 3 different purposes :

1) Surface Area and weighted integrals :

For simple surfaces without radial ease, write the parametrized surface as :

$$r(\underset{x}{u},\underset{y}{v}) = \langle u,\ v,\ \underbrace{f(u,v)}_{z=f(x,y)}\rangle$$

Note that sometimes it may be easier to write x as a function of y and z or y as a function of x and z.

Now, the area of the surface $= |r_u \times r_v|$

$$\iint_S F(x,y,z)\ \underline{ds}$$

$$|r_u \times r_v| = \sqrt{1+f_x^2+f_y^2} \quad \text{(in cartesian)}$$

$$\Rightarrow SA = \iint_S \sqrt{1+f_x^2+f_y^2}\; f(x,y,z)\; dx\, dy$$

Can convert to polar to simplify the computation.

For spheres, use spherical coordinate system:

$$\hookrightarrow \vec{r}(\phi,\theta) = \langle \underbrace{\rho\sin\phi\cos\theta}_{x}, \underbrace{\rho\sin\phi\sin\theta}_{y}, \underbrace{\rho\cos\phi}_{z} \rangle$$

$$dS = |r_\phi \times r_\theta| = \rho^2 \sin\phi \; d\theta\, d\phi$$

$$\iint_S \underbrace{f(x,y,z)}_{\text{modified using } \vec{r}(\theta,\phi)} \; \rho^2\sin\phi \; d\phi\, d\theta$$

$S \leftarrow (\phi,\theta)$

$\leftarrow 0 \le \theta \le 2\pi$ (mostly)

For cylindrical coordinates, use z and θ as the parameters:

$$\vec{r}(\theta, z) = \langle r\cos\theta, r\sin\theta, z \rangle$$

$$dS = |r_\theta \times r_z| \, d\theta \, dz.$$

More than likely, you will only need spherical and Cartesian systems for parametrization.

To find the bounds of integration, use the constraint given - the octant or the surface within which we are finding the surface Area, etc.

To find SA, $\iint_S 1 \, dS$.

2) Flux : $\iint_S \vec{F} \cdot \vec{n} \, dS$

Write the surface equation and find the normal vector $r_u \times r_v$.

$$\iint_S \underbrace{\vec{F}(x,y)}_{\text{rewrite in terms of } u \,\&\, v.} \cdot (r_u \times r_v) \, du \, dv$$

3) Circulation : Stokes' theorem

$$\int \vec{F} \cdot d\vec{r} = \iint (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

$$\oint \vec{F}\cdot d\vec{r} = \iint_S \underbrace{(\nabla\times \vec{F})}_{\text{curl}}\cdot \underbrace{dS}_{dS = r_u \times r_v}$$

Remember Green's theorem :

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_A \underbrace{g_x - f_y}_{\text{curl of F when F is two-dimensional}} \; dA \quad dx\,dy$$

$$\langle \partial_x , \partial_y , \partial_z \rangle$$
$$\times$$
$$\langle f , g , 0 \rangle$$
$$\Rightarrow \langle 0 , 0 , g_x - f_y \rangle$$

If $\vec{F}$ is 3 dimensional : $\vec{F} = \langle f, g, h\rangle$, then the concept of Green's theorem $\rightarrow$ Stokes' theorem.

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\; dS$$

Boundary curve of the 3-D surface

any surface