MA 266 Lecture 1

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Sec 1.1 Differential Equations; Mathematical Models

Question: What is a differential equation?

A **differential equation** is

- unknown function $f(\cdot)$
- its derivatives $f', f'', f^{(3)}, \ldots, f^{(n)}, \ldots$

Example 1. ***(Types of equations)***

1. *Find x in* $x^2 + 6x + 1 = 0.$ Not a DE ; algebraic eq. (model static world)

   unknown var

2. *Find $f(t)$ in* $f(t)\cos(t) = e^t - \sin(t).$ Not a DE ; trascend. eq.

   unknown f'n No derivatives

3. *Find $y(t)$ in* $y'' + 10y' = e^t.$ Yes! a DE.

   unknown f'n

Question: Why do we study differential equations?

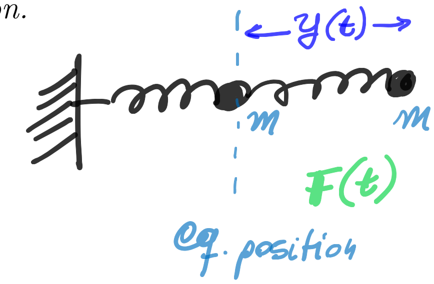
- Many natural phenomena; physical processes involve __change__.
- Because $\frac{dx}{dt} = f'(t)$ is the __rate__ at which $x = f(t)$ is __changing__.
- Differential equations to model __changing world__.

  phenomena that involves changing.

Q

Example 2. ***(An example of mathematical model — object-spring)***

Consider an object with a mass m attached to the end of a spring. The mass experiences a force $F(t)$. Formulate a differential equation to model its motion.



- **Notations**
  - $y(t)$: position @ time t
  - acceleration $a = \dfrac{d^2y}{dt^2}$
- **Physical Law: Newton's law**

$$F = m \cdot a = m \cdot \frac{d^2y}{dt^2}.$$

- **Forces that acted on the object**

$$F(t) = -k\, y(t)$$

where $k > 0$ is a constant.

Differential eq.:

$$-k\, y(t) = m \cdot \frac{d^2y}{dt^2} \qquad (1)$$

$$\frac{d^2y}{dt^2} + \omega^2 y(t) = 0 \quad ; \quad \omega = \sqrt{\frac{k}{m}} \qquad (2)$$

Remark The differential equation contains two constants: m, and k

- Formulate a physical process via mathematical model.
- Goal: i) Find the solution of (1)
  ii) Interpret the solution of (1)

Definitions

- The **order** of a differential equation is the **order** of the highest derivative involved in the ODE.

Example 3. ***(Find the order)***

$y(\cdot)$: unknown f'n.
x: independent var.

1. $4x^2y'' + y = 0$

order $= 2$.

2. $\underbrace{(y')^2}_{\geq 0} + \underbrace{y^2}_{\geq 0} = -1$

order $= 1$

$\boxed{\text{Note: } \nexists \text{ no } \mathbb{R}\text{-valued solution.}}$

3. $y^{(3)}x^2 + x^{10}y = \sin(x)$

order $= 3$

- The **general** form of an n-**th order** differential equation:

(3) $\quad F(\underbrace{x, y}_{2 \text{ vars}}, \underbrace{y', y'', \ldots, y^{(n)}}_{n \text{ derivative}}) = 0$

F is given $\mathbb{R}$-valued f'n input @ most $n+2$ vars

- We say $u(x)$ is a **solution** of the differential equation (3) if

i) $u', u'', \ldots, u^{(n)}$ exist $\quad$ ii) $F(x, u, u', u'', \ldots, u^{(n)}) = 0$.

- **Initial value problem** (IVP): DE together with an initial cond. (IC).

e.g.) $\quad \dfrac{dy}{dx} = f(x, y); \quad y(x_0) = y_0.$

- The solution to an ODE for a given IC is called **particular** solution.

"y_0 given"

- **General solution**
  - Without an IC, the ODE may have ∞-number solutions.
  - If we can write an expression for every solution $\equiv$ **general solution**.

family of solutions.

Example 4. ***(Population Dynamics)***
Consider the time rate of change of a population $P(t)$.

$\frac{dP(t)}{dt}$ ← population, ← time.

- **Notation**
  - constant birth rate $b > 0$
  - constant death rate $d > 0$

- **Differential equation**

$$\frac{dP(t)}{dt} = \underbrace{(b-d)}_{=:k} P(t) = k P(t) \qquad (4)$$

1. Check $P(t) = Ce^{kt}$ is a general solution of (4)

$$P'(t) = Ce^{kt} \cdot k = k \cdot \underbrace{(Ce^{kt})}_{=P(t)} = k \cdot P(t) \equiv (4) \checkmark$$

2. Suppose that the population at time $t = 0$ (hours, h) was 1000. Find the value of C

$$\text{IVP} \begin{cases} \frac{dP}{dt} = kP \\ P(0) = 1000 \end{cases} \qquad P(t=0) = Ce^{k0} = \boxed{C = 1000}$$

$$P(t) = 1000\, e^{kt}$$

3. Assume the population doubled after 1 hour, determine the value of k

$$P(t = 1h) = 2 \cdot P(0) = 2000. \Rightarrow (b > d)$$

$$P(t = 1h) = 1000\, e^{k \cdot 1} = 2000$$

$$\Rightarrow e^k = 2 \Rightarrow k = \ln(2)$$

4. Write the particular solution. Use it to predict the population after 1.5 hours

$$P(t) = 1000 \cdot e^{\ln(2) t} = 1000 \cdot 2^t$$

$$P(t = 1.5\, h) = 1000 \cdot 2^{(1.5)}$$

- **Ordinary** differential equations (ODE): the unknown f'n $y(\cdot)$ depends on a single indep. variable.
- **Partial** differential equations (PDE): If the unknown f'n $y(\cdot)$ is a function of ≥ 2 indep. variables. partial derivatives will be involved.

Example 5. ***(Thermal Diffusivity)***
Consider a one dimensional rod. The temperature $u(x,t)$ *satisfies the heat equation:*

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}.$$

where k (const) is the thermal diffusivity.

MA 266 Lecture 2

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Sec 1.2 Integrals as General/Particular Solutions

In this section, we discuss how to solve differential equations.

- Consider the **first order** equation:

order=1
$$\frac{dy}{dx} = f(x, y)$$
independent
unknown f' / dependent var

- Consider the simple case

(1)
$$\frac{dy}{dx} = \underbrace{f(x)}_{R.H.S.}$$
independent variable

Example 1. *Find the solution $y(x)$ of the simple case:*

$$\frac{dy}{dx} = f(x) \Rightarrow \int dy = \int f(x)\, dx.$$

(2) $$y(x) = \int f(x)\, dx + C.$$

general antiderivative

Remarks

- __(2)__ is the __general.__ solution. of (1)
- Involves an arbitrary constant __$C \in \mathbb{R}$.__.
- For every choice of __$C \in \mathbb{R}$__, __(2)__ is a particular solution of __(1)__.

- Consider the **Initial Value Problem** (IVP):

$$\underbrace{\frac{dy}{dx} = f(x)}_{\text{DE}}, \quad \text{and} \quad \underbrace{y(x_0) = y_0}_{\text{IC}}.$$

Example 2. *Find the* ***particular*** *solution of the IVP.*

- general solution.
- IC ⇒ particular solution

$$y(x) = \underbrace{\int f(x)\,dx}_{=:G(x)} + C \quad \text{particular antideriv}$$

$$y(x) = G(x) + C \qquad G'(x) = f(x)$$

$y(x_0) = y_0$. $\quad y_0 = y(x_0) = G(x_0) + C \Rightarrow C = y_0 - G(x_0)$

Example 3. *Find the particular solution $y(x)$ of the following IVP.*

Particular solution

$y(x) = G(x) - G(x_0) + y_0$.

$$\frac{dy}{dx} = \underbrace{\sin(x)}_{f(x)}, \qquad y(0) = 1.$$

- General solution

$$y(x) = \int \sin(x)\,dx + C.$$

$$= \underbrace{-\cos(x)}_{=:G(x)} + C.$$

$$G'(x) = \sin(x) = f(x)$$

- IC $y(0)=1$ → particular solution.

$$C = y_0 - G(x_0)$$

$$= 1 - (-\cos(0))$$

$$= 2.$$

particular solution:

$$\boxed{y(x) = -\cos(x) + 2}$$

Second Order Equations

simple.

- Consider the second-order differential equation of the special form:

independent variable

order = 2 $$\frac{d^2y}{dx^2} = g(x) \qquad (3)$$

Example 4. *Find the general solution of this second-order equation.*

i) $$\frac{dy}{dx} = \int \frac{d^2y}{dx^2}\, dx = \int g(x)\, dx = G(x) + \underline{\underline{c_1}}$$

$$\frac{dy}{dx} = \underbrace{G(x) + c_1}_{= v(x)}$$

$$y(x) = \int \frac{dy}{dx}\, dx = \int G(x) + c_1 \, dx + \underline{\underline{c_2}}$$

$$= \int G(x)\, dx + c_1 x + c_2.$$

Remark

- The above second-order differential equation can be solved by solving successively the

first-order diff. eq's

$$\frac{dy}{dx} = v(x)$$

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} = g(x)$$

Velocity and acceleration

Notation

$x\text{-axis.}$

- The motion of a particle along a straight line (the x-axis) is described by its position function:

$$x(t) = f(t)$$

- $\underline{\quad x(t) \quad}$ is the x-coordinate at time t. indep var.

F.O. DE $\quad \frac{dx}{dt} =: v(t) = f'(t)$

- The **velocity** $v(t)$ of the particle is:

- The **acceleration** $a(t)$ is:

$$\frac{dv}{dt} =: a(t) = \frac{d^2x}{dt^2}.$$

Example 5. *Find the general solution when the acceleration is constant $a(t) = a$.*

$a(t) = a \quad \forall t > 0$

"for all"

$$\frac{dv}{dt} = a.$$

general sol'n of v. $\quad v(t) = \int a\, dt = a \int dt + C.$

$$= at + C_1$$

$$\frac{dx}{dt} = v(t)$$

$$\frac{dx}{dt} = at + c_1$$

general sol'n of x. $\quad x(t) = \int [at + c_1]\, dt + C_2.$

$$x(t) = \frac{1}{2}at^2 + C_1 t + C_2.$$

Example 6. *Given an initial position $x(0) = x_0$ and initial velocity $v(0) = 0$, find the particular solution of the corresponding IVP.*

IVP: $\begin{cases} \frac{dv}{dt} = a \\ \frac{dx}{dt} = v(t) \\ v(0) = 0 \,;\; x(0) = x_0. \end{cases}$

$v(t) = at + C_1$

$x(t) = \frac{1}{2} a t^2 + C_1 t + C_2$

$v(0) = 0 \Rightarrow C_1 = 0$

$x(0) = x_0 \Rightarrow C_2 = x_0$

$x_0 = x(0) = \frac{1}{2} a (0)^2 + C_2 \Rightarrow C_2 = x_0.$

Example 7. *At 12:00 PM, a car starts from rest at point A and proceeds at constant acceleration along a straight road towards point B. The car reaches B at 12:50 PM with velocity of 60 miles/hour. Find the distance from A to B.*

12:00 PM, $t = 0$ → a 12:50 PM, $t = 5/6$ h

A B

$v_A = v(0) = 0.$ $v_B = v(5/6) = 60 \frac{\text{miles}}{h}.$

$\frac{dv}{dt} = a.$

$x_A = x(0) = 0.$

$x_B = x(5/6) = ?$

i) general solution

$v(t) = at + C_1.$

ii) particular sol'n:

$\boxed{C_1 = 0} \Rightarrow v(t) = at$

$60 = v(5/6) \Rightarrow a = \frac{6}{5} 60 = 72.$

$v(t) = 72t.$

iii) general solution of $x(t)$

$x(t) = \frac{1}{2} a t^2 + C_2$

iv) $\boxed{x(t) = \frac{1}{2} 72 t^2.}$

$x_B = x(5/6) = 25$ miles //

Vertical Motion and Gravitational Acceleration

Notation

- The weight W of a body is the force exerted on the body by gravity.

$$W = m \cdot g$$

- If we ignore the air resistance, then the acceleration $a = \frac{dv}{dt}$ is

$$a = -g$$
$$\frac{dv}{dt} = -g.$$

- The **velocity** equation is:

$$v(t) = -gt + C_1$$

- The **height** equation is:

$$y(t) = -\frac{1}{2} g t^2 + C_1 t + C_2$$

Example 8. *Suppose that a ball is thrown straight upward from the ground ($y_0 = 0$) with initial velocity $v_0 = 96$ ft/s (then $g = 32$ ft/s^2). Find the maximum height the ball attains.*

$v(t^*) = 0$ $\quad y(t^*)$

$y(0) = 0.$ $\quad v(0) = 96$ f/s. $(g = 32$ f/s$^2)$

$v(t^*) = 0.$

$\Rightarrow t^* = 3\underline{s}.$

i) part. sol'n of $v(t)$ $(C_1 = ?)$

$96 = v(0) = C_1$ $\qquad v(t) = -32t + 96.$

ii) part' sol'n of $y(t)$ $(C_2 = ?)$

$C_2 = 0$.

$$\boxed{y(t) = -\frac{1}{2}(32)t^2 + 96t.}$$

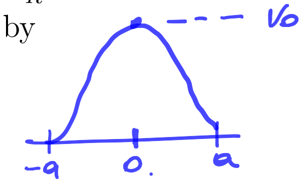
$y(t^*) = 144$ ft.

A Swimmer's Problem

Consider a northward-flowing river of width $w = 2a$. The lines $x = \pm a$ represent the banks of the river and the y-axis its center. Suppose that the velocity v_R at which the water flows increases as one approaches the center of the river. v_R is given by

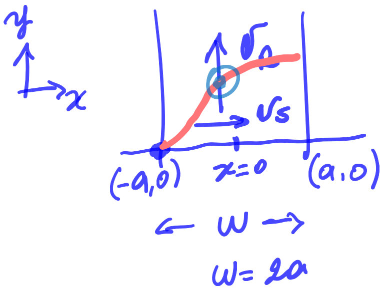
$x \in [-a, a]$

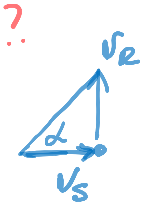
$$v_R = v_0\left(1 - \frac{x^2}{a^2}\right).$$



Example 9. *Suppose that a swimmer start at point $(-a, 0)$ on the west bank and swims due east (relative to the water) with constant speed v_S. His velocity (relative to the riverbed) has a horizontal component v_S and a vertical component v_R. Find the swimmer trajectory $y(x)$.*

$y(x) = ??$





$$\tan\alpha = \frac{v_R}{v_s} = \frac{\frac{dy}{\cancel{dt}}}{\frac{dx}{\cancel{dt}}} = \frac{dy}{dx}$$

$y(-a) = 0.$

IVP: $$\frac{dy}{dx} = \underbrace{\frac{v_0}{v_s}\left(1 - \frac{x^2}{a^2}\right)}_{RHS}; \quad y(-a) = 0.$$

• general solution:

$$y(x) = \int \underbrace{\qquad}_{RHS} dx + C.$$

$$= \frac{v_0}{v_s}\left(x - \frac{1}{3}\frac{x^3}{a^2}\right) + C.$$

$y(-a) = 0.$

$$0 = y(-a) = \frac{v_0}{v_s}\left(-a + \frac{1}{3}\frac{a^{\cancel{3}}}{\cancel{a^2}}\right) + c$$

$$= \frac{v_0}{v_s}\left(-\frac{2}{3}a\right) + C$$

$$C = \frac{v_0}{v_s} \cdot \frac{2}{3}a$$

$w = 1$ mile

$v_0 = 9$ miles/h.

$v_s = 3$ miles/h

$y(x) = ?$

$$\boxed{y(x) = 3x - 4x^3 + 1}$$

MA 266 Lecture 3

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Sec 1.3 Slope Fields and Solution Curves

Consider the first order differential equation:

independent
dependent

$$\frac{dy}{dx} = \underbrace{f(x, y)}_{R.H.S.}$$

Question: Can we find the solution $y(x)$ using

$$y(x) = \int f(x, y(x))dx + C \quad ? \qquad (1)$$

Answer: No.

- (1) involves the unknown f'n $y(x)$.

Def'n of slope

$f(x,y) = m = \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

Slope fields and Graphical solutions

- For each $(x,y) \in \mathbb{R}^2$, $\underbrace{f(x,y)}_{R.H.S}$ determines the slope of $y(x)$.

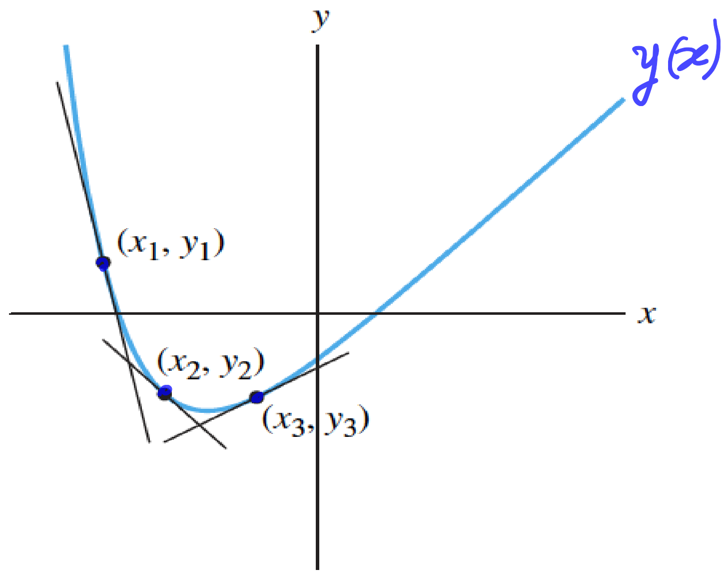
  $m = f(x,y)$

Definition. A solution $y(x)$ of the differential equation $\frac{dy}{dx} = f(x, y)$ is a curve in (x,y)-plane whose tangent lines at each (x,y) has:

slope $m = f(x,y)$

Example 1. *Consider **a** solution curve of*

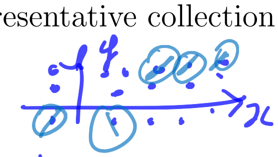
$$\frac{dy}{dx} = x - y$$



- point (x_1, y_1) $= (x_1, y(x_1))$ has slope $m = f(x_1, y_1) = x_1 - y_1$.
- point (x_2, y_2) $= (x_2, y(x_2))$ " " $m = f(x_2, y_2) = x_2 - y_2$
- point (x_3, y_3) $m = x_3 - y_3$.

Constructing slope fields.

- Consider a representative collection of points (x, y) in the plane. $\mathbb{R}^2$



- For each (x, y), we draw a "short" line segment having:

  proper slope $m = f(x, y)$

- The collection of line segments: slope field (directional field). vector field

of $\frac{dy}{dx} = f(x, y)$

Example 2. *Consider the population dynamics:*

$$\frac{dP(t)}{dt} = \underbrace{kP(t)}_{f(P)}.$$

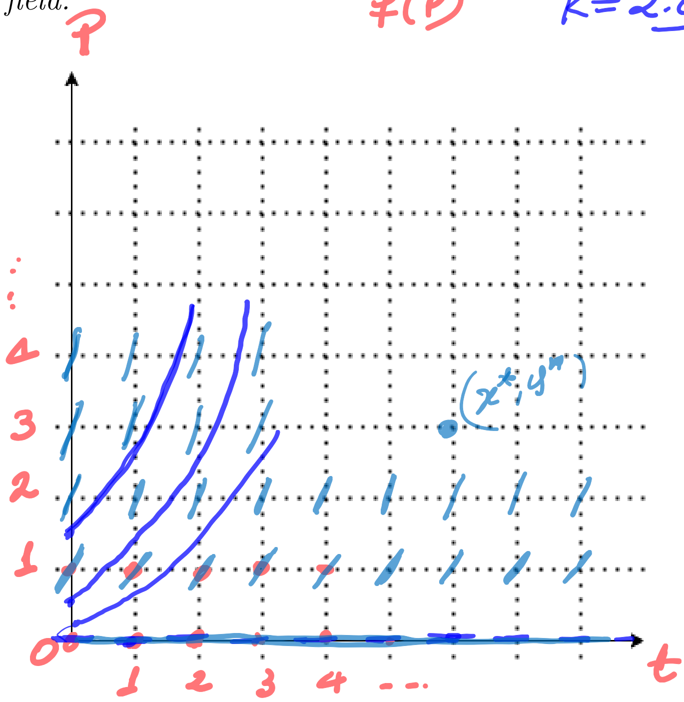
$t \geq 0$

$P(t) \geq 0$

$k = 2.0 = (b - d)$

Construct the slope field.

t	P	$m = kP(t)$
0	0	0
1	0	0
⋮		
1000	0	0
0	1	2
1	1	2
⋮		
1000	1	2
0	2	4

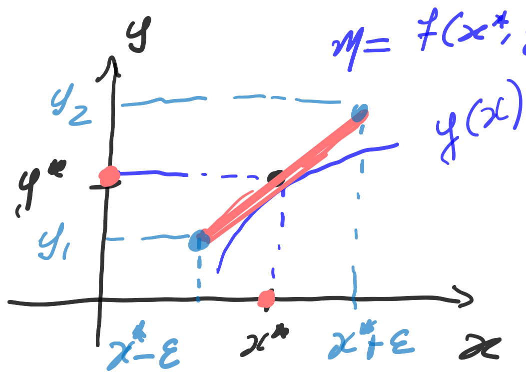


$P(t) = 0.$

- Recall the general solution:

$$P(t) = Ce^{\overset{2.0}{k}t}$$

- How do we draw the tangent line?



$$\frac{dy}{dx} = f(x, y)$$

$$m = \frac{y - y^*}{x - x^*}$$

$$y = m(x - x^*) + y^* \quad (\square)$$

Remark

- We use he slope field to study the properties / behavior of solutions of the ODE.

$x \to \infty$
$x \to 0$
$x \to -\infty$.

Example 3. *Consider the differential equation*

$$\frac{dy}{dx} = ky,$$

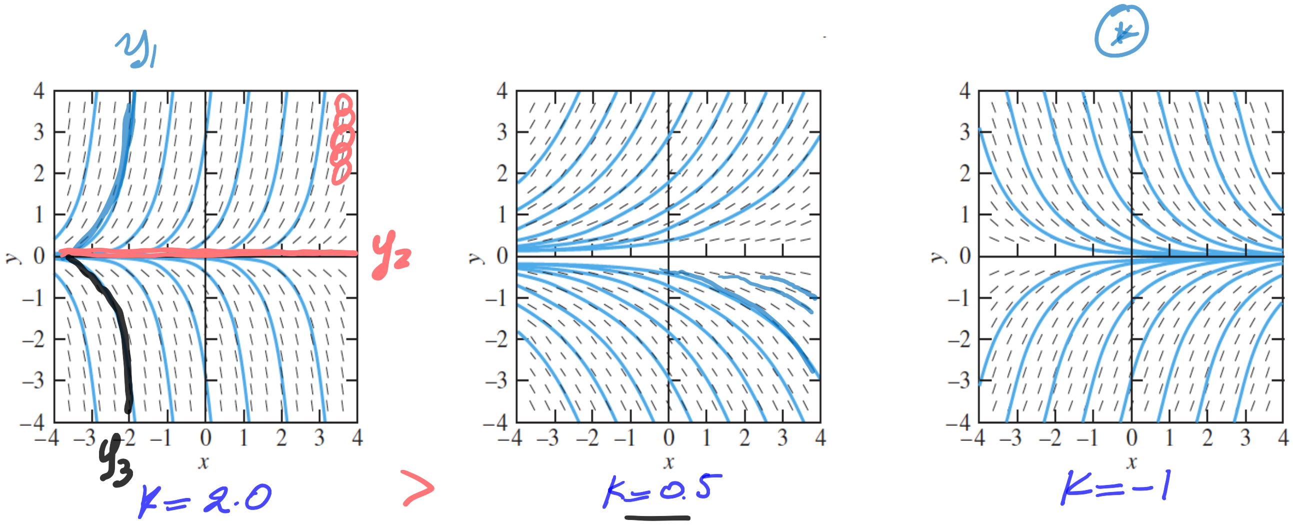
where $|k|$ *is the rate of change of* $y(x)$.

$x \in \mathbb{R}$ $(t \geq 0)$
$y \in \mathbb{R}$. $(P(t) \geq 0)$

- The general solution is:

$$y(x) = Ce^{kx}$$

- The solution curves and slope fields for $k = 2, 0.5, -1$:



Q: $\lim_{x \to \infty} y_1 = +\infty$

Q $\lim_{x \to \infty} y_2 = 0$

Q $\lim_{x \to \infty} y_3 = -\infty$

$\rightarrow$ same behavior

$\lim_{x \to \infty} y(x) = 0$

FAILS EXISTENCE

Existence of Solutions

Example 4. *Consider the IVP:*

$$\frac{dy}{dx} = \underbrace{\frac{1}{x}}_{f(x)}, \qquad y(0) = 0.$$

- General solution:

From lec-2

$$y(x) = \int \frac{1}{x}\,dx + C = \ln|x| + C$$

ln|x| is not defined @ $x = 0$

- Particular solution:

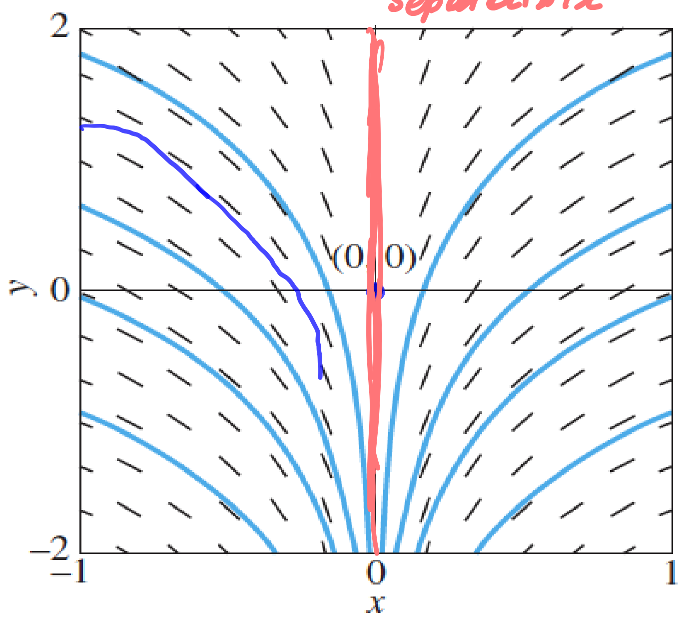
From lec-2

use IC: $0 = y(0) = \underline{\ln|0| + C}$

The slope field: ∴ The solution of the IVP does not exist

separatrix

$m = +\infty$



Remark

- The <u>slope field</u> forces all curves near <u>y-axis</u> ($y(x)$) to plunge downward so that none can pass through <u>(0,0)</u>.

Uniqueness of Solutions

Example 5. *Consider the IVP:*

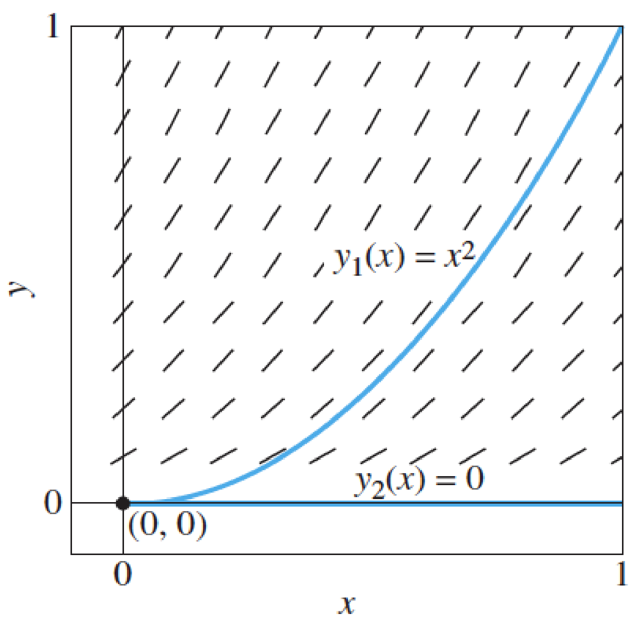
$$\frac{dy}{dx} = 2\sqrt{y}, \qquad y(0) = 0. \qquad (2)$$

- Check if $y_1(x) = x^2$ is a solution: of the IVP?
  - $\frac{dy_1}{dx} = 2x = 2\sqrt{y}$ ✓
  - $y_1(0) = 0^2 = 0$ ✓

FAILS UNIQUENESS

- Check if $y_2(x) = 0.$ is a solution:
  - $\frac{dy_2}{dx} = 0 = 2\sqrt{y_2}$ ✓
  - $y_2(0) = 0$ ✓

$\therefore$ the solution of (2) is not unique

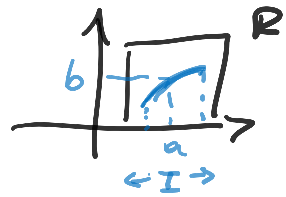


Local ∂f/∂y R.H.S

Theorem 1. *(Existence and Uniqueness of Solutions) Suppose that both the function $f(x, y)$ and its partial derivative $D_y f(x, y)$ are continuous on some rectangle R in the xy-plane that contains the point (a, b) in its interior. Then, for some open interval I containing the point a, the initial value problem*

$$\frac{dy}{dx} = f(x, y), \qquad y(a) = b \qquad \text{(IVP)}$$

has ***one and only one solution*** *that is defined on the interval I.*



Sufficient condition

$P \Rightarrow Q$

$\left\{\begin{array}{l} \text{statement}_1 \\ \vdots \\ \text{statement}_n \end{array}\right.$ ARE TRUE $\Rightarrow$ Q is TRUE

$P = \left\{\begin{array}{l} \bullet\ f(x,y) \text{ is continuous on } R \text{ containt } (a,b) \\ \text{and} \\ \bullet\ \frac{\partial f}{\partial y} \text{ is cont } \quad " \quad " \quad " \quad " \end{array}\right.$

$Q = \{ \circ$ (IVP) has one and only one solution on interval I

Example 6. *Consider*

$$\frac{dy}{dx} = 2\sqrt{y}$$

FAILS uniqueness

$P: \left\{\begin{array}{l} \bullet\ f(x,y) = 2\sqrt{y} \text{ is } C^0\text{? Yes } y \geq 0. \\ \bullet\ \frac{\partial f}{\partial y} = \frac{1}{\sqrt{y}} \text{ is } C^0\text{? is discontinuous on } y=0. \end{array}\right.$

Example 7. *Consider:*

$$x\frac{dy}{dx} = 2y$$

a) Check the existence and uniqueness of the IVP:

$$x\frac{dy}{dx} = 2y, \quad y(0) = b.$$

$(0, b)$

- $f(x,y) = \frac{2y}{x}$
- $\frac{\partial f}{\partial y} = \frac{2}{x}$

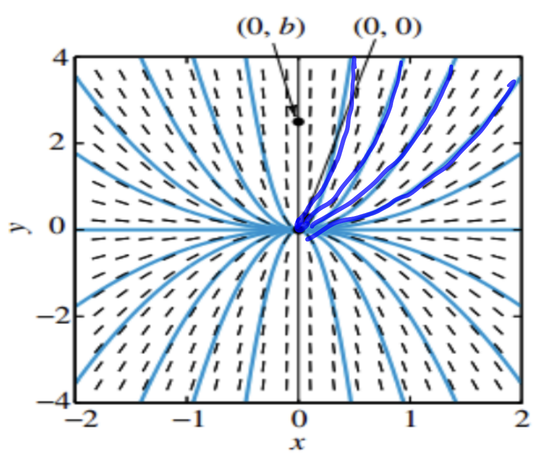
discont. @ $x=0$.

- Case $b = 0$: $y(0)=0.$

Consider $y(x) = Cx^2.$

- $\frac{dy}{dx} = 2Cx \frac{x}{x} = \frac{2Cx^2}{x} = \frac{2y}{x}$ ✓
- $y(0) = C(0)^2 = 0$ ✓

- Case $b \neq 0$:



b) Check the existence and uniqueness of the IVP:

$$x\frac{dy}{dx} = 2y, \quad y(a) = b, \quad a, b \neq 0. \qquad (\Delta)$$

$a = -1$

$b = 1.$

$y(-1) = 1$

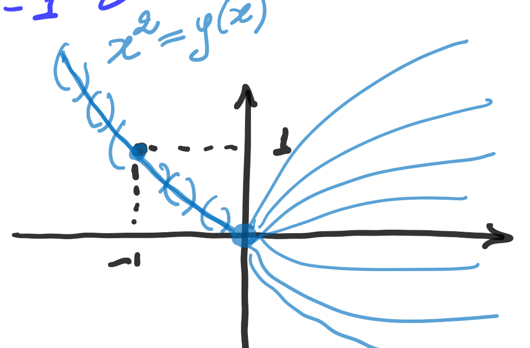
Apply Theorem:

- $f(x,y) = \frac{2y}{x}$ cont. @ (a,b) ? ✓
- $\frac{\partial f}{\partial y} = \frac{2}{x}$ " " " ? ✓

$\Rightarrow$ Solution exists and is unique.

Consider $y(x) = x^2$. Is this a solution of (Δ)?

- $\frac{dy}{dx} = 2x = 2x \cdot \frac{x}{x} = \frac{2y}{x}$,
- $y(-1) = 1$ ✓



$$y(x) = \begin{cases} x^2 & x \leq 0 \\ Cx^2 & x > 0 \end{cases}$$

MA 266 Lecture 4

Christian Moya, Ph.D.

Sec 1.4 Separable Equations and Applications

Defintion 1. *We call a first-order differential equation* separable *if it can be written as:*

$$\frac{dy}{dx} = f(x,y) = g(x) \cdot k(y) = \frac{g(x)}{h(y)} \quad (1)$$

where $h(y) = 1/k(y)$.

$$h(y) \cdot \frac{dy}{dx} = g(x) \quad (2)$$

Example 1. *Determine* $g(x)$ *and* $h(y)$

$$h(y) \cdot \frac{dy}{dx} = g(x)$$

1. *Consider*

$$x^3 \frac{dy}{dx} = e^{-y}$$

$$\underbrace{\frac{1}{e^{-y}}}_{h(y)} \frac{dy}{dx} = \underbrace{\frac{1}{x^3}}_{g(x)}$$

$g(x) =$ $1/x^3 = x^{-3}$ *and* $h(y) =$ $1/e^{-y} = e^{y}$.

2. *Consider*

$$\frac{dy}{dx} = \frac{x^{1000}}{y}$$

$g(x) =$ x^{1000} *and* $h(y) =$ y.

3. *Consider*

$$\frac{dy}{dx} = 100 \cdot (xy)^{3/5}$$

$g(x) =$ $100x^{3/5}$ *and* $h(y) =$ $1/y^{3/5}$.

Solving Separable Equations

1. The separable equation:

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

$$\iff h(y)\frac{dy}{dx} = g(x) \quad (3)$$

2. Write in the form:

$$h(y)\,dy = g(x)\,dx$$

3. Integrate both sides:

general solution of DE (implicitly) $\rightarrow$ $$\int h(y)dy = \int g(x)\,dx + \underline{C} \quad (4)$$

4. We only need the antiderivatives:

- $H(y) = \int h(y)\,dy$.
- $G(x) = \int g(x)\,dx$

Remark

- Equations (3) and (4) are equivalent:

$$\frac{\partial}{\partial x}\left(H(y(x))\right) = H'(y)\,y' = h(y)\frac{dy}{dx} = g(x) = \frac{\partial}{\partial x}\left(G(x)\right)$$

general solution of DE in implicit form $$\boxed{H(y) = G(x) + C}$$

Example 2. *Find the general solution of*

$$\frac{dy}{dx} = y\sin(x) \quad (5)$$

$g(x) = \sin x$

$h(y) = \frac{1}{y}$

- For $y \neq 0$, separating variables gives:

$$\frac{1}{y}\,dy = \sin x\,dx$$

- Integrating both sides:

$$\int \frac{1}{y}\,dy = \int \sin x\,dx + C$$

gen sol'n of DE in implicit form. →

$$\boxed{\ln|y| = -\cos x + C}$$

$$y(x) = \pm e^{-\cos x + C}$$

$$= \underbrace{\pm e^{C}}_{=A} \cdot e^{-\cos x}$$

$A = \pm e^{C}$

$\boxed{A \neq 0}$

general sol'n explicit form →

$$\boxed{y(x) = A \cdot e^{-\cos x}.}$$

Remark

- For $y(x)=0$, (5) has a singular solution.

Defintion 2. Singular *solutions are exceptional solutions that cannot be obtained by selecting a value for* A.

Example 3. *Solve the differential equation:*

$$(6) \qquad \frac{dy}{dx} = \frac{4-2x}{3y^2-5} \qquad \begin{aligned} g(x) &= 4-2x \\ h(y) &= 3y^2-5 \end{aligned}$$

- Separating variables:

$$(3y^2-5)\,dy = (4-2x)\,dx$$

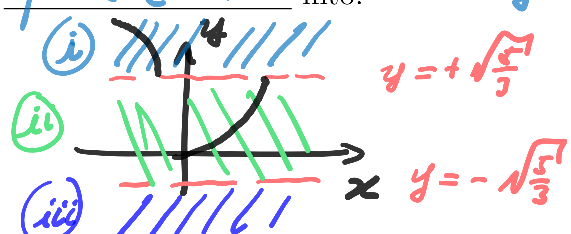
- Is (6) defined for all y? ________.

set $3y^2-5=0 \implies \frac{1}{h(y)} = k(y)$ or $\frac{dy}{dx} \nearrow +\infty$

- This implies that for (6), no solution curve can cross either of the horizontal lines:

$$3y^2-5=0 \begin{cases} \nearrow y = -\sqrt{\frac{5}{3}}. \\ \searrow y = +\sqrt{\frac{5}{3}} \end{cases}$$

- Divides the plane ($\mathbb{R}^2$) into: 3 regions.



- Integrating $\implies$ general solution:

$$\int (3y^2-5)\,dy = \int (4-2x)\,dx + C. \qquad \text{level curves / contour curves.}$$

gen. sol'n implicit.

$$\bullet\ y^3 - 5y = 4x - x^2 + C. \qquad \bullet\ F(x,y) = C.$$

$$F(x,y) =: y^3-5y-(4x-x^2) = C.$$

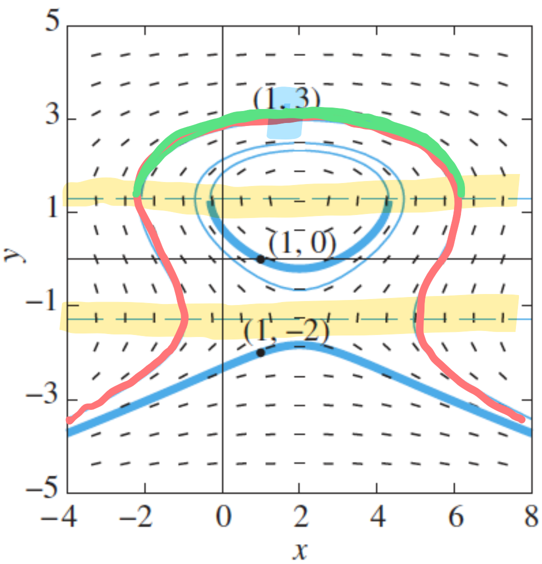
- **Question:** Can we solve the general solution for y? No !.

- solution curves of (6) are contained in level curves / contour curves of: $F(x,y)=C$

Example 4. *Solve the initial value problem:*

$$\frac{dy}{dx} = \frac{4-2x}{3y^2-5}; \qquad y(1) = 3.$$

contour / level curves $F(x,y) = C$



$y = +\sqrt{\frac{5}{3}}$

$y = -\sqrt{\frac{5}{3}}$.

- Compute C of the general solution: $y^3 - 5y = 4x - x^2 + C$

substitute $x=1$ and $y=3$ → $\boxed{C = 9}$

- our desired solution part. lies on the level curve:

$C=9$

$F(x,y) = y^3 - 5y - (4x - x^2) = 9$

Note: solution curve must pass through the point $(1,3)$

Example 5. *Find all solutions of the differential equation:*

$$\frac{dy}{dx} = 6x(y-1)^{2/3} \quad (8)$$

- Separation of variables gives:

Assuming $y \neq 1$.

$$\int \frac{1}{3(y-1)^{2/3}}\, dy = \int 2x\, dx$$

$$(y-1)^{1/3} = x^2 + C.$$

$$\boxed{y(x) = 1 + (x^2 + C)^3}$$

- Is $y(x) = 1.$ a **singular** solution?

  - It cannot be obtained for any C. ✓
  - sol'n of (8) ✓

$$\frac{dy}{dx} = 6x(y-1)^{2/3} \quad (8)$$

- Are a) $y(x) \equiv 1$ and b) $y(x) = 1 + (x^2 - 1)^3$ solutions of the IVP with IC: $y(1) = 1$?

- $y(x) = 1$
  - solution ✓
  - satisfies the IC

FAICS uniqueness

- $y(x) = 1 + (x^2 - 1)^3$
  - solution ✓
  - satisfies IC. ✓

Natural Growth and Decay

The differential equation:

$$\boxed{\frac{dx}{dt} = kx,} \qquad k \text{ a cosntant}$$

serves as a mathematical model for wide range of natural phenomena:

- Population dynamics
- Compound interest
- Radioactive decay

The **general** solution:

$$x(t) = ce^{kt}. \quad ✓$$

- Separating the variables and integrating:

$$\frac{dx}{dt} = kx.$$

$$h(x)\frac{dx}{dt} = g(t)$$

$$\int \frac{1}{x}\,dx = \int k\,dt + c.$$

$$\ln|x| = kt + c$$

- We solve for x:

$$\frac{dy}{dx} = ky.$$

$$x(t) = \underbrace{\pm e^{c}}_{=:C} e^{kt}$$

$$x(t) = C \cdot e^{kt} \quad ✓$$

- The particular solution for the IC $x(0) = x_o$ is:

$$\boxed{x(t) = x_0 e^{kt}}$$

Radioactive Decay

Consider a sample of material that contains $N(t)$ atoms of a certain radioactive isotope a time t. It has been observed that a constant fraction of those radioactive atoms will spontaneously decay during each unit of time. Consequently, the sample behaves exactly like the population dynamics with no births $b = 0$. The model for the $N(t)$ atoms is then

$$\boxed{\frac{dN}{dt} = -kN(t).}$$

Example 7. *An accident at a nuclear plant has left the surrounding area polluted with radioactive material that decays naturally. The initial amount of radioactive material is 15 (safe units), and 5 months later is still 10 su.*

- Write a formula given the amount $N(t)$ of radioactive material (in su) after t months.

$\cdot\, N(0) = 15$ su

$\cdot\, N(5) = 10$ su.

$\cdot$ general: $N(t) = \overset{=15}{c}\, e^{-kt}$.

$\cdot$ particular: $N(t) = 15 \cdot e^{-kt}$.

$\cdot$ Find k: $N(5) = 10$. s.u $\Rightarrow$ $10 = 15\, e^{-k5}$

$\frac{1}{e^{-k5}} = \frac{15}{10}$. $\Leftrightarrow$ $e^{k5} = \frac{15}{10}$.

$e^{k5} = \frac{\cancel{15}\,3}{\cancel{10}\,2}$.

$k = \frac{1}{5} \cdot \ln\left(\frac{3}{2}\right)$

$\cdot\, N(t) = 15 \cdot e^{\ln\left(\frac{3}{2}\right)^{-t/5}}$

$= 15 \cdot \left(\frac{2}{3}\right)^{t/5}$

- What amount of radioactive material will remain after 8 months?

$N(8) = ?$

$N(8) = 15 \cdot \left(\frac{2}{3}\right)^{8/5} \approx 7.84$ s.u.

- How long it will be until $N = 1$ su, so it is safe for people to return to the area?

$N(t) = 1$ solve for t.

$1 = 15 \cdot \left(\frac{2}{3}\right)^{t/5}$

$\frac{1}{15} = \left(\frac{2}{3}\right)^{t/5} \Leftrightarrow$ $\ln\left(\frac{1}{15}\right) = \frac{t}{5} \ln\left(\frac{2}{3}\right)$

$t = 5 \cdot \frac{\ln(1/15)}{\ln(2/3)}$

≈ 33.39 months.

MA 266 Lecture 4

Christian Moya, Ph.D.

Population dynamics

Example 1. *According to a census, the world's total population reached 6 billion persons in mid-1999, and was then increasing at a* ***rate*** *of about 212 thousand persons each day. Assuming that natural population growth at this rate continues:*

a) What is the annual growth rate k ?

Let $P(t)$ denote the population in billions and let t be the time in years. Then, the differential equation describing the population dynamics is:

$$\frac{dP}{dt} = \underbrace{(b-d)}_{=:k} P(t). \tag{1}$$

Set $t = 0$ for mid-1999. Then, the initial population is $P(0) = 6$ billion. Observe that the instantaneaous increase of population is

$$212 \text{ [thousand persons / day]} \equiv 0.000212 \text{ [billion persons / day]}$$

Hence, the efective increase at time $t = 0$ is:

$$P'(0) = 0.000212 \text{ [billion / day]} \cdot 365.25 \text{ [days / year]} \approx 0.07743 \text{ [billion / year]}.$$

Using the above result, we can compute k as follows:

$$k = \frac{P'(0)}{P(0)} \approx \frac{0.07743}{6} = 0.0129.$$

The *particular solution* is then

$$P(t) = 6 \cdot e^{0.0129 \cdot t} \tag{2}$$

b) What would be the population at the middle of the 21st century?

We use the particular solution (2) to make preditions:

$$P(t = 51 \text{ years}) = 6 \cdot e^{0.0129 \cdot 51} \approx 11.58 \text{ [billions]}$$

c) How long will it take the world to increase tenfold –thereby reaching 60 billion that some demographers believe to be the maximum for which the planet can provide food supplies?

Solve for t, the folowing equation:

$$60 = P(t) = 6 \cdot e^{0.0129 \cdot t}$$

The obtained time t is:

$$t = \frac{\ln(10)}{0.0129} \approx 178 \text{ [years]},$$

i.e., the population will reach 60 billion in the year 2177.

MA 266 Lecture 5

Christian Moya, Ph.D.

Sec 1.5 Linear First-Order Equations

Defintion 1. *A* <u>linear first-order diff. eq'n.</u> *is a differential equation of the form:*

$$\underbrace{\frac{dy}{dx}}_{\text{isolated}} + P(x)y = Q(x).$$

coefficients.

The coefficients $P(x)$ and $Q(x)$ are assumed to be ***continuous*** *in some interval on the x-axis.*

Example 1. *Determine if the following equations are linear:*

1. $\frac{dy}{dx} = -e^x sin(x)y + x^{2000}$.? yes!

   $P(x) = e^x sin(x)$
   $Q(x) = x^{2000}$

   $\frac{dy}{dx} + e^x sin(x) \cdot y = x^{2000}$

2. $\frac{dy}{dx} = x \cdot cos(y) + 2x$.? No!

   because $cos(y)$ is a nonlinear function.

$\frac{dy}{dx} + y = 0.$

$Q(x) = 0.$

$P(x) = 1$

start from.
$$\frac{dy}{dx} + P(x)\,y = Q(x) \quad (1)$$

METHOD: Solution of linear first-order equations

Step 1) Calculate the integrating factor:

$$\rho(x) = e^{\int P(x)\,dx}.$$

Step 2) Multiply both sides of the diff. eq. by $\rho(x)$.

$$(\Delta) \quad \boxed{e^{\int P(x)\,dx} \cdot \frac{dy}{dx} + e^{\int P(x)\,dx} \cdot P(x) \cdot y(x)} = e^{\int P(x)\,dx} \cdot Q(x).$$

LHS.

Step 3) Left hand side $\iff$ derivative of the product:

$$(\Delta) \equiv \underbrace{D_x}_{\frac{\partial}{\partial x}} [\rho(x) \cdot \underbrace{y(x)}]. = D_x\left[e^{\int P(x)\,dx} \cdot y(x)\right]$$
$$= e^{\int P(x)\,dx} \cdot \frac{dy}{dx} + e^{\int P(x)\,dx} \cdot P(x) \cdot y(x)$$

Step 4) Integrating both sides gives:

$$\int D_x[\rho(x) \cdot y(x)] = \int \rho(x) \cdot Q(x)\,dx + \underline{\underline{C}}$$
$$e^{\int P(x)\,dx}\,y(x) = \int e^{\int P(x)\,dx} \cdot Q(x)\,dx + C$$

Step 5) Solving y, we obtain the general solution:

explicit form.

$$y(x) = e^{-\int P(x)\,dx} \cdot \left(\int e^{\int P(x)\,dx} \cdot Q(x)\,dx + C.\right)$$

isolated

Example 2. *Find the general solution:*

$$(x^2+1)\frac{dy}{dx} + 3xy = 6x \iff \frac{dy}{dx} + \frac{3x}{(x^2+1)} y = \frac{6x}{(x^2+1)}$$

- Is this a *linear* equation? yes!.
- $P(x) = \frac{3x}{(x^2+1)}$ and $Q(x) = \frac{6x}{(x^2+1)}$.
- Integrating factor:

$$\rho(x) = e^{\int P(x)\,dx} = e^{\int \frac{3x}{(x^2+1)}dx} = e^{\frac{3}{2}\ln(x^2+1)} = (x^2+1)^{3/2}$$

- L.H.S. is the derivative of the product:

$$(x^2+1)^{3/2}\cdot\frac{dy}{dx} + (x^2+1)^{3/2}\cdot\frac{3x}{(x^2+1)}\, y = (x^2+1)^{3/2}\cdot\frac{6x}{(x^2+1)}$$

- Integrating both sides: $\int D_x\left[(x^2+1)^{3/2}\cdot y(x)\right] = \int (x^2+1)^{1/2}\cdot 6x \,.+ C$

$$(x^2+1)^{3/2}\cdot y(x) = 2(x^2+1)^{3/2} + C.$$

- General solution:

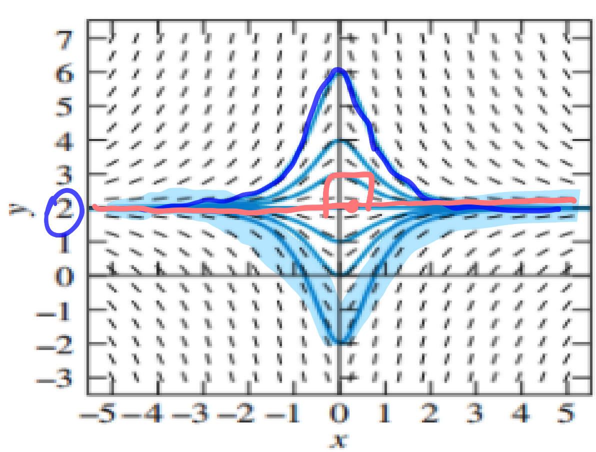
$$y(x) = 2 + C(x^2+1)^{-3/2}.$$

Q: $x \to \infty$.

$y(x) \to 2$.

$y(x) = 2$

$C = 0$



$y = 2$.

slope field

$y(0) = 2$.

Example 3. *Solve the initial value problem:*

$$\frac{dy}{dx} + y = 2, \quad y(0) = 0$$

- $P(x) =$ $\underline{\quad 1 \quad}$ and $Q(x) =$ $\underline{\quad 2 \quad}$.
- Integrating factor:

$$\rho(x) = e^{\int P(x)\,dx} = e^{x}.$$

- We know that:

$$D_x[\rho(x)\,y(x)] = \rho(x)\,Q(x)$$

- Integrating both sides: $\int D_x[e^x \cdot y(x)] = \int e^x \cdot 2\,dx + C.$

$$e^x \cdot y(x) = 2e^x + C$$

- General solution:

$$y(x) = 2 + \frac{C}{e^x} = 2 + Ce^{-x}.$$

- Particular solution:

Use IC: $y(0) = 0 = 2 + Ce^{-0}$

$\Rightarrow C = -2$

$$y(x) = 2 - 2e^{-x} = 2(1 - e^{-x})$$

$\blacksquare$

Example 4. *Solve the initial value problem:*

$$x\frac{dy}{dx} + y = 3xy, \quad y(1) = 0$$

- Linear first order form:

$$\frac{dy}{dx} + \frac{y}{x} = 3y$$

$$\Leftrightarrow \frac{dy}{dx} + y\left(\frac{1}{x} - 3\right) = 0.$$

- $P(x) =$ $\frac{1}{x} - 3$ and $Q(x) =$ 0.

- Integrating factor:

$$\rho(x) = e^{\int \frac{1}{x} - 3\, dx} = e^{\ln(x) - 3x} = xe^{-3x}.$$

- We know that:

$$D_x[\rho(x) \cdot y(x)] = \rho(x) \cdot Q(x)$$

- Integrating both sides:

$$\int D_x[xe^{-3x} \cdot y(x)] = \int 0.$$

$$xe^{-3x} \cdot y(x) = +C$$

- General solution:

$$y(x) = Cx^{-1}e^{3x}.$$

- Particular solution:

use IC $y(1) = 0$ $\Rightarrow$ $C = 0$

our particular solution is zero function $\boxed{y(x) = 0}$.

Example 5. *Find the general solution:*

$$\frac{dy}{dx} + y\cot x = \cos x$$

- $P(x) = $ $\underline{\cot x}$ and $Q(x) = $ $\underline{\cos x}$.
- Integrating factor:

$$\rho(x) = e^{\int \cot x\, dx} = e^{\ln(\sin x)}$$

$$\rho(x) = \sin x.$$

- We know that: $D_x[\sin x \cdot y(x)] = \sin x \cdot \cos x.$

- Integrating both sides:

$$\int D_x[\sin x \cdot y(x)] = \int \sin x \cdot \cos x \, dx + C$$

$$\sin x \cdot y(x) = \frac{1}{2}\sin^2 x + C.$$

- General solution:

$$\boxed{y(x) = \frac{1}{2}\sin x + \frac{C}{\sin x}}$$

Example 6. *Express the general solution of*

$$\frac{dy}{dx} = 1 + 2xy$$

*in terms of the **error function***

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

- Linear first order form: $\frac{dy}{dx} - 2xy = 1.$

- $P(x) =$ $-2x$ and $Q(x) =$ 1.

- Integrating factor:

$$\rho(x) = e^{\int -2x\, dx} = e^{-x^2}.$$

- We know that: $D_x[\rho(x) \cdot y(x)] = \rho(x) \cdot Q(x)$

$$\int D_x[e^{-x^2} \cdot y(x)] = \int e^{-x^2} \cdot 1 . dx + C$$

- Integrating both sides:

$$e^{-x^2} \cdot y(x) = \boxed{\int e^{-x^2} dx + C}$$

for some C'

$$\int e^{-x^2} dx + \underset{\text{constant}}{C'} = \int_0^x e^{-t^2} dt.$$

- General solution:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}$$

$$e^{-x^2} \cdot y(x) = \int_0^x e^{-t^2} dt \underbrace{-C' + C}_{+C.}$$

$$\int_0^x e^{-t^2} dt = \frac{\sqrt{\pi}}{2} erf(x)$$

$$y(x) = e^{x^2}\left[\frac{\sqrt{\pi}}{2} erf(x) + C\right]$$

MA 266 Lecture 6

Christian Moya, Ph.D.

Sec 1.5-b Linear First-Order Equations - part 2

- To solve the linear first order equation:

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{1}$$

- We use the integrating factor $\rho(x)$.

- Obtain the *explicit* general solution:

(2) $$y(x) = \frac{1}{\rho(x)} \cdot \left[\int Q(x) \cdot \rho(x)\, dx + \underline{\underline{C}} \right]$$

Theorem 1. *If the functions $P(x)$ and $Q(x)$ are continuous on the open interval I containing the point x_0, then the initial value problem*

$$y' + P(x)y = Q(x), \qquad y(x_0) = y_0$$

has a unique solution $y(x)$ on I, given by the formula in (2) *with an appropriate value of C.*

e.g) $\frac{dy}{dx} + xy = x^2$.

$y(a) = b$.

$P(x) = x$, $Q(x) = x^2$

x-axis.

- To find the particular solution, i.e., to find C,

IVP: $\rho(x) = e^{\int_{x_0}^{x} P(t)\, dt + C} = e^{C} \cdot e^{\int_{x_0}^{x} P(t)\, dt}$

$e^C > 0 \quad C = 0$

$\int_{x_0}^{x} P(t)\, dt + C = \int P(x)\, dx$

$\rho(x_0) = 1 \qquad y(x_0) = y_0.$

$$y(x) = \frac{1}{\rho(x)} \cdot \left[\int_{x_0}^{x} Q(t)\, \rho(t)\, dt_1 + C \right]$$

$$y_0 = y(x_0) = \frac{1}{\cancel{\rho(x_0)}} 1 \cdot \left[\int_{x_0}^{x_0} Q(t) \rho(t)\, dt + C \right]$$

0

$\Longrightarrow C = y_0.$

$$\boxed{y(x) = \frac{1}{\rho(x)} \cdot \left[\int_{x_0}^{x} Q(t)\, \rho(t)\, dt + y_0 \right]}$$

Example 1. *Solve the initial value problem*

$$x^2 \frac{dy}{dx} + xy = \sin(x), \qquad y(1) = y_0$$

- Linear first order form:

$I := \{x : x > 0\}$

- $P(x) =$ $1/x$ and $Q(x) =$ $\sin x / x^2$.

- With $x_0 = 1$, the integrating factor:

$$\rho(x) = e^{\int_1^x \frac{1}{t}\, dt} = e^{\ln(x)} = x.$$

- The desired particular solution is:

$$y(x) = \frac{1}{x} \cdot \left[\int_1^x \frac{\sin t}{t^{\cancel{2}}} \cdot \cancel{t}\, dt + y_0 \right]$$

$$\boxed{y(x) = \frac{1}{x} \cdot \left[\int_1^x \frac{\sin t}{t}\, dt + y_0 \right]}$$

Example 2.

a) Show that

$$y_c(x) = Ce^{-\int P(x)dx} \qquad y_c'(x)$$

is a *general* solution of $\frac{dy}{dx} + P(x)y = 0$.

The fund. th'm of calculus, for every C

$$y_c'(x) = \underbrace{Ce^{-\int P(x)dx}}_{=\,y_c(x)} \cdot [-P(x)].$$

$$y_c'(x) = -P(x)\, y_c(x) \iff \frac{dy_c}{dx} + P(x)\, y_c(x) = 0 \quad \blacksquare$$

b) Show that

$$y_p(x) = e^{-\int P(x)dx}\left[\int \left(Q(x)e^{\int P(x)dx}\right) dx\right]$$

$$y_p'(x)$$

is a particular solution of $\frac{dy}{dx} + P(x)y = Q(x)$.

Product rule + fund. th'm of calculus:

$$y_p'(x) = \underbrace{\cancel{e^{-\int P(x)dx}} \cdot Q(x) \cdot \cancel{e^{\int P(x)dx}}} + e^{-\int P(x)dx} \cdot [-P(x)] \cdot \int Q(x)e^{\int P(x)dx}dx.$$

$$= Q(x) - P(x) \cdot \underbrace{e^{-\int P(x)dx} \cdot \left[\int Q(x)e^{\int P(x)dx}dx\right]}_{y_p(x)}$$

c) Show that $y(x) = y_c(x) + y_p(x)$ is a general solution of $\frac{dy}{dx} + P(x)y = Q(x)$.

$$\frac{dy_p}{dx} + P(x)\, y_p(x) = Q(x) \quad \blacksquare$$

LHS: $$\frac{dy}{dx} + P(x)\, y = y'(x) + P(x)\, y(x)$$

$$= (y_c'(x) + y_p'(x)) + P(x) \cdot (y_c(x) + y_p(x))$$

$$= \underbrace{(y_c'(x) + P(x)\, y_c(x))}_{=0} + \underbrace{(y_p'(x) + P(x)\, y_p(x))}_{=Q(x)}$$

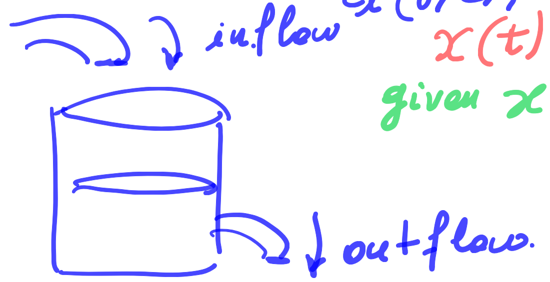
$$= Q(x) \quad \blacksquare \quad \text{R.H.S.}$$

"principle of superposition"

Mixture problems

Consider a mixture of a solute and a solvent (e.g., salt dissolved in water). There is both inflow and outflow. Our goal is to compute the amount $x(t)$ of the solute in the tank at time t, given the amount $x(0)$ at time $t = 0$. We assume that a solution with a concentration of c_i (g/L) of solute flows into the tank at constant rate r_i (L/s), and that the solution in the tank flows out at constant rate r_o (L/s).

c_i (g/L); r_i (L/s)



$x(t)$ of solute @ time t.

given $x(0)$ @ time $t=0$.

r_o (L/s)

c_o (g/L)

$$c_o = \frac{x(t)}{V(t)}$$

- The amount of solute $x(t)$ in the tank satisfies the differential equation:

if $r_i = r_o \Rightarrow V(t) =$ constant

IVP. $$\frac{dx}{dt} = r_i c_i - \frac{r_o}{V} x.$$

$x(0) = x_0$.

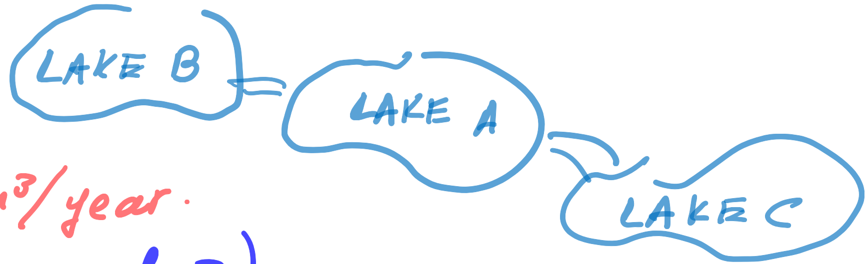
$\underline{V > 0}$

$P(t) = \frac{r_o}{V(t)}$

$Q(t) = r_i \cdot c_i$

Q:

Example 3. *Assume that a lake A has a volume of 480 km*^{3} *and that its rate of inflow from Lake B and outflow to Lake C are both 350 km*^{3} *per year. Suppose that* $t = 0$ *(years), the pollutant concentration is five times that of Lake B. If the outflow henceforth is perfectly mixed lake water, how long it will take to reduce the pollution concentration in Lake A to twice that of Lake B?*



- We have:

$V = 480 \text{ km}^3$

$r_i = r_o = r = 350 \text{ km}^3/\text{year}.$

$c_i = c$ (pollutant conc. of B)

$x(0) = 5 \cdot c \cdot V.$

$\boxed{c_A = 5c_B = 5c}$

$c = \frac{x}{V}$

$x_A(0) = c_A \cdot V$
$= 5cV.$

- The differential equation:

$$\frac{dx}{dt} = r_i \cdot c_i - \frac{r_o}{V} x(t)$$
$$= r \cdot c - \frac{r}{V} x(t)$$

- The particular solution:

$x(0) = 5cV.$

Integrating factor.

$$\rho(t) = e^{\int_0^t \frac{r}{V} ds} = e^{\frac{r}{V} \cdot t}.$$

$$x(t) = \frac{1}{\rho(t)} \cdot \left[x_0 + \int_0^t \rho(t)\, Q(t)\, dt\right]$$

$$x(t) = e^{-\frac{r}{V} \cdot t} \cdot \left[x_0 + \int_0^t e^{\frac{r}{V} \cdot t} r \cdot c\, dt\right]$$
$$= e^{-\frac{r}{V} \cdot t} \cdot \left[\underset{5 \cdot cV}{x_0} + \frac{\cancel{r} \cdot c \cdot V}{\cancel{r}} \cdot \left(e^{\frac{r}{V} \cdot t} - 1\right)\right]$$
$$= e^{-\frac{r}{V} \cdot t} \cdot \left[5cV + cV\left(e^{\frac{r}{V} t} - 1\right)\right]$$

$$\boxed{x(t) = cV + 4cVe^{-\frac{r}{V} \cdot t}}$$

- To find when $x(t) = 2 \cdot cV$, we solve:

$2cV = cV + 4cVe^{-\frac{r}{V} \cdot t}$

$\cancel{cV} = 4\cancel{cV} e^{-\frac{r}{V} \cdot t}$

$e^{\frac{r}{V} \cdot t} = 4$

$\frac{r}{V} \cdot t = \ln(4)$

$\Rightarrow t = \frac{V}{r} \cdot \ln(4)$

Example 4. *Rework the previous example for the case of Lake C, which empties to a river X and receives inflow from Lake A. The volume of Lake C is 1640 km^3 and an inflow-outflow rate of 410 km^3 per year.*

- We have:

$V = 1640.$

$r_i = r_o = r = 410 \ \frac{km^3}{year}.$

$x(0) = 5 \cdot c \cdot V$

$x(t) = 3cV\,?$

$C_C = 5\,C_B.$

$c \equiv$ concent. of pollutant in B

- The differential equation:

IVP:

$$\frac{dx}{dt} = r \cdot c - \frac{r}{V} \cdot x(t)$$

$$x(0) = 5 \cdot c \cdot V.$$

- The particular solution:

$$\boxed{x(t) = c \cdot V + 4cV e^{-\frac{r}{V} \cdot t}}$$

- To find when $x(t) = 3 \cdot cV$, we solve:

Solve for t:

$$3 \cdot cV = cV + 4\,cV e^{-\frac{r}{V} \cdot t}$$

$$\Longleftrightarrow$$

$$\cancel{2}\,\cancel{c}V = \overset{2}{\cancel{4}}\ cV e^{-\frac{r}{V} \cdot t}$$

$$\frac{1}{e^{-\frac{r}{V}t}} = 2.$$

$$e^{\frac{r}{V} \cdot t} = 2.$$

$$\boxed{t = \frac{V}{R} \cdot \ln(2)}$$

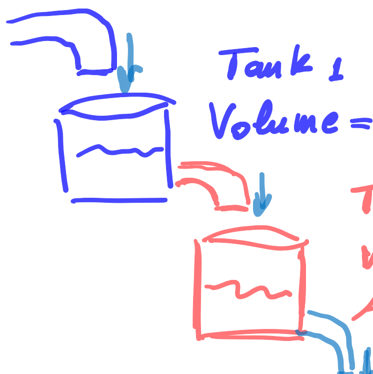
$$t = \frac{V}{R} \cdot \ln(2)$$

MA 266 Lecture 6

Christian Moya, Ph.D.

Cascade of Tanks

Example 1. *Suppose we have a cascade of tanks. Tank 1 initially contains 100 gal of pure ethanol and tank 2 initially contains 100 gal of water. Pure water flows into tank 1 at 10 gal/min and the two other flow rates are 10 gal/min. a) Find the amount $x(t)$ and $y(t)$ of ethanol in the two tanks at time $t \geq 0$.*



Tank 1
Volume $= V$

Amount of ethanol in tank 1
$x(t)$; $x(0) = 100$

Tank
Volume $= V$.
Amount of ethanol in tank 2.
$y(t)$; $y(0) = 0$.

Let $x(t)$ denote the amount of ethanol at time t for tank 1 and let $y(t)$ denote the amount of ethanol at time t for tank 2. Let's find the differential equation for $x(t)$. The data for tank 1 is:

- inflow rate: $r_i^1 = 10$ gal/min
- inflow concentration: $c_i^1 = 0$ (water)
- outflow rate: $r_o^1 = 10$ gal/min.
- volume: $V^1 = V = 100$ gals (remains constant).
- initial amount of ethanol: $x(0) = 100$ gal

The differential equation for $x(t)$ is then:

$$\frac{dx}{dt} = -\frac{r_o^1}{V}x(t) = -\frac{1}{10}x(t).$$

The above is a linear equation with $P(t) = 1/10$ and $Q(t) = 0$. So, the integrating factor is:

$$\rho(t) = e^{\int .1dt} = e^{t/10}.$$

The *general* solution is then

$$x(t) = e^{-t/10}\left(\int 0 \cdot \rho(t)dt + C\right) = Ce^{-t/10}.$$

Using the initial condition $x(0) = 100$, we find that **the amount of ethanol in tank 1** for $t \geq 0$ is:

$$x(t) = 100e^{-t/10}.$$

The data for tank 2 is:

- inflow rate: $r_i^2 = r_o^1 = 10$ gal/min
- inflow concentration: $c_i^2 = \frac{x(t)}{V}$
- outflow rate: $r_o^2 = 10$ gal/min.
- volume: $V = 100$ gals (remains constant).
- initial amount of ethanol: $y(0) = 0$ gal (only water)

The differential equation for $y(t)$ is

$$\begin{aligned}\frac{dy}{dt} &= r_i^2 c_i^2 - \frac{r_o^2}{V} y(t) \\ &= 10 \cdot \frac{x(t)}{V} - \frac{1}{10} y(t) \\ &= \frac{10}{100} \cdot 100e^{-t/10} - \frac{1}{10} y(t) \\ &= 10e^{-t/10} - \frac{1}{10} y(t).\end{aligned}$$

The above is a linear equation with $P(t) = \frac{1}{10}$ and $Q(t) = 10e^{-t/10}$. To solve the linear equation, we use the integrating factor:

$$\rho(t) = e^{t/10}$$

Thus, the general solution is:

$$\begin{aligned}y(t) &= e^{-t/10}\left(\int 10e^{-t/10}e^{t/10}dt + C\right) \\ &= e^{-t/10} \cdot (10t + C).\end{aligned}$$

Using the initial condition $y(0) = 0$, we find that $C = 0$. Thus, **the amount of ethanol in tank 2** for $t \geq 0$ is:

$$y(t) = 10te^{-t/10}.$$

Example 2. *Find the maximum amount of ethanol ever in tank 2.*

We know that for $t \geq 0$, the amount of ethanol in tank 2 is given by:

$$y(t) = 10te^{-t/10}.$$

To find the maximum value, we first need to solve

$$y'(t^*) = 0$$

for t^*. Using the product rule, we have

$$\begin{aligned} y'(t) &= 10(e^{-t/10} - \frac{t}{10}e^{-t/10}) \\ &= e^{-t/10}(10 - t). \end{aligned}$$

The above equation is zero when $t^* = 10$. We now check if $y(10)$ is a maximum.

- for $t \in [0, 10)$, $y'(t) = e^{-t/10}(10 - t) > 0$. So, $y(t)$ increases for $t \in [0, 10)$.
- for $t > 10$, $y'(t) = e^{-t/10}(10 - t) < 0$. So, $y(t)$ decreases for $t > 0$.

The above implies that $y(t)$ reaches its maximum at $t = 10$ min. Thus, **the maximum amount of ethanol in tank 2** is:

$$y(10) = 100e^{-1} \approx 36.79 \text{ gal}.$$

MA 266 Lecture 7

Christian Moya, Ph.D.

Sec 1.6-a Substitution Methods

- Consider the first order differential equation:

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

ind. var

dep. var

- Suppose there exists a function:

$$(2) \qquad v = \alpha(x, y)$$

- Suppose we can solve (2) for y:

$$y = \gamma(x, v)$$

- Then, by applying the chain rule:

$$(3) \qquad \frac{dy}{dx} = \frac{\partial \gamma}{\partial x} + \frac{\partial \gamma}{\partial v} \cdot \frac{dv}{dx} = \underbrace{\gamma_x + \gamma_v \cdot \frac{dv}{dx}}_{R.H.S.}$$

- Replacing R.H.S (3) for $\frac{dy}{dx}$ in 1, and solving for dv/dx:

$$(4) \qquad \boxed{\frac{dv}{dx} = g(x, v)}$$

- If this eq'n is *linear or separable*, then we can apply the methods from Sec. 1.4 or 1.5.

if $v = v(x)$ is the sol'n of (4), then $y = \gamma(x, v(x))$ is the solution for (1)

Example 1. *Solve the differential equation:*

$$x\frac{dy}{dx} = y + 2\sqrt{xy}.$$

- For $x, y : x \cdot y > 0$, we rewrite the differential equation as:

$$\frac{dy}{dx} = \frac{y}{x} + 2\sqrt{\frac{y}{x}} = F\left(\frac{y}{x}\right)$$

- Let's try the substitution:

$$v = \alpha(x,y) = \frac{y}{x}. \Rightarrow y = x \cdot v = \gamma(x, v)$$

- Then by the chain rule

$$\frac{dy}{dx} = \gamma_x + \gamma_v \cdot \frac{dv}{dx} = v + x \cdot \frac{dv}{dx}$$

Since $\frac{dy}{dx} = \frac{y}{x} + 2\sqrt{\frac{y}{x}} = v + 2\sqrt{v}$

- So, the transformed equation is $\cancel{v} + x \cdot \frac{dv}{dx} = \cancel{v} + 2\sqrt{v}$

$$x \cdot \frac{dv}{dx} = 2\sqrt{v}$$

- Separating variables:

$$\int \frac{dv}{\sqrt{v}} = \int \frac{2}{x}\, dx. + C.$$

$$2 \cdot \sqrt{v} = 2 \cdot \ln(x) + C$$

- The general solution is:

$$\sqrt{v} = \ln(x) + C.$$

$$v = (\ln(x) + C)^2.$$

Since $y = x \cdot v \Rightarrow \boxed{y(x) = x \cdot (\ln(x) + C)^2}$

Homogeneous Equations

Defintion 1. *A* homogeneous. *first-order differential equation is one that can be written in the form:*

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \qquad (5)$$

- If we make the substitution:

$$v = \frac{y}{x} \Rightarrow y = x \cdot v.$$

$$\frac{dy}{dx} = y_x + y_v \cdot \frac{dv}{dx} = v + x \cdot \frac{dv}{dx}. \qquad (6)$$

- Then (5) is transformed into the separable eq'n.:

$$v + x \cdot \frac{dv}{dx} = \frac{dy}{dx} = F\Big(\underbrace{\frac{y}{x}}_{=:v}\Big) = F(v)$$

$$\Rightarrow \quad v + x \cdot \frac{dv}{dx} = F(v) \Leftrightarrow \boxed{x \cdot \frac{dv}{dx} = F(v) - v}$$

- Thus every homogeneous first-order differential equation can be reduced to an integration problem by means of the substitutions in (6).

$$v = \frac{y}{x}.$$

Example 2. *Find general solutions of the differential equation:*

$$xy^2\frac{dy}{dx} = x^3 + y^3$$

- For $x, y \neq 0.$, we rewrite the differential equation as:

$$\frac{dy}{dx} = \frac{x^{\cancel{3}2}}{\cancel{x}y^2} + \frac{y^{\cancel{3}}}{xy^{\cancel{2}}} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)$$
$$= F\left(\frac{y}{x}\right)$$

- Substituting

$$v = \frac{y}{x} \Rightarrow y = x \cdot v.$$

$$x \cdot \frac{dv}{dx} = F(v) - v$$
$$= \left(\frac{1}{v^2}\right) + \cancel{v} - \cancel{v}$$

- Separating variables:

$$x \cdot \frac{dv}{dx} = \frac{1}{v^2}.$$

$$\int v^2 \, dv = \int \frac{1}{x} \, dx + C.$$

$$\frac{v^3}{3} = \ln(x) + C$$

- The general solution is:

$$v^3 = 3\ln(x) + C.$$

$$\boxed{y^3 = x^3 \cdot (3\ln(x) + C)}$$

* $v = y^3 \Rightarrow$ linear eq.

$$\frac{dv}{dx} - \frac{3}{x} v = 3x^3$$

if $n=1$ $\quad \frac{dy}{dx} + (P(x) - Q(x))y = 0.$

Bernoulli Equation

Consider:

$$(7) \qquad \frac{dy}{dx} + P(x)y = Q(x)y^n$$

The above equation is called a Bernoulli eq'n. If either $n=0$ or $n=1$, linear eq.

Otherwise

The substitution

$$v = y^{1-n}.$$

Homework.

Transforms (7) into linear eq'n:

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

Example 3. *Consider the homogeneous equation:*

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

- This is a *Bernoulli* equation:

$$\frac{dy}{dx} = \frac{4x^2}{2xy} + \frac{3y^2}{2xy} = \frac{2x}{y} + \frac{3y}{2x}$$

$$\boxed{\frac{dy}{dx} - \frac{3}{2x}\cdot y = 2x \cdot y^{-1}}$$

$P(x) = -\frac{3}{2x}$

$Q(x) = 2x$

$n = -1 \Rightarrow 1-n = +2$

- We use the substitution:

$v = y^{1-n} = y^2 \Rightarrow y = v^{1/2} = \underset{\sim}{g(x,v)}$.

- This gives:

i) chain rule

$$\frac{dy}{dx} = \cancel{g_x}^{\,0} + g_v \cdot \frac{dv}{dx}.$$

$$= \frac{1}{2} v^{-1/2} \cdot \frac{dv}{dx}$$

$$\frac{1}{2} v^{-1/2} \cdot \frac{dv}{dx} - \frac{3}{2x} v^{1/2} = 2x \cdot \frac{1}{v^{1/2}}.$$

$(\cdot 2 \cdot v^{1/2})$

$$\frac{dv}{dx} - \underbrace{\frac{3}{x}}_{P'(x)} \cdot v = \underbrace{4x}_{Q'(x)}.$$

Integrating factor $\rho(x) = e^{\int -\frac{3}{x}dx} = e^{\ln(x)^{-3}} = x^{-3}.$

$$\int D_x(x^{-3}\cdot v) = \int 4x \cdot x^{-3}. + C$$

$$x^{-3}\cdot v = -\frac{4}{x} + C.$$

We know $v = y^2 \Rightarrow x^{-3}\cdot y^2 = -\frac{4}{x} + C.$

$$\boxed{y^2 = -4x^2 + Cx^3}$$

Example 4. *Find the general solution of the differential equation:*

$$\frac{dy}{dx} = y + y^3$$

- Rewrite the differential equation as:

$$\frac{dy}{dx} - y = y^3.$$

$P(x) = -1$
$Q(x) = 1$
$n = 3$

"Bernoulli eq'n"

- We use the substitution:

$$v = y^{1-n} = y^{-2}. \Rightarrow y = v^{-1/2}.$$

- The substitution gives:

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)\cdot Q(x)$$

$$\frac{dv}{dx} + (-2)(-1)\cdot v = (-2)\cdot 1.$$

$$\boxed{\frac{dv}{dx} + 2v = -2}$$

Integrating factor:

$$\rho(x) = e^{\int 2\,dx} = e^{2x}.$$

$$\int D_x\left(e^{2x}\cdot v\right) = \int -2\, e^{2x}\,dx + C.$$

$$e^{2x}\cdot v = -e^{2x} + C.$$

$$v = -1 + Ce^{-2x}$$

We know $v = y^{-2} \Rightarrow$ $y^{-2} = -1 + Ce^{-2x}$

$$\Rightarrow \boxed{y^2 = \frac{1}{Ce^{-2x} - 1}.}$$

Example 5. *The equation*

$$\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x)$$

is called a ***Riccati Equation****. Suppose that one particular solution $y_1(x)$ of this equation is known. Show that the substitution:*

$$\boxed{\frac{dy_1}{dx} = A(x)y_1^2 + B(x)y_1 + C(x)}$$

$$y = y_1 + \frac{1}{v}$$

transforms the Riccati equation into the linear equation:

$$\frac{dv}{dx} + (B + 2Ay_1)v = -A.$$

use $y = y_1(x) + \frac{1}{v}$.

$$\frac{dy}{dx} = \frac{dy_1}{dx} - \frac{1}{v^2}\cdot\frac{dv}{dx}.$$

$$\cancel{\frac{dy_1}{dx}} - \frac{1}{v^2}\frac{dv}{dx} = A(x)y^2 + B(x)y + C(x)$$

$$= A(x)\left(y_1^2 + 2\frac{y_1}{v} + \frac{1}{v^2}\right) + B(x)\cdot\left(y_1 + \frac{1}{v}\right) + C(x)$$

$$= \underbrace{\cancel{A(x)\cdot y_1^2 + B(x)\cdot y_1 + C(x)}}_{\frac{dy_1}{dx}} + A(x)\cdot\left(\frac{2y_1}{v} + \frac{1}{v^2}\right) + \frac{B(x)}{v}.$$

$$-\frac{1}{v^2}\frac{dv}{dx} = A(x)\cdot\left(\frac{2y_1}{v} + \frac{1}{v^2}\right) + \frac{B(x)}{v}.$$

$(\cdot\, -v^2)$ $$\frac{dv}{dx} = -A(x)\cdot\left(2y_1 v + 1\right) - B(x)v.$$

$$= -(2y_1A + B)\cdot v - A(x)$$

$$\boxed{\frac{dv}{dx} + (B + 2Ay_1)\cdot v = -A}$$

$A \stackrel{\Delta}{=} A(x)$

$B \stackrel{\Delta}{=} B(x)$

MA 266 Lecture 8

Christian Moya, Ph.D.

Sec 1.6-b Exact Differential Equations

Recall that the *general* solution of

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

is often defined *implicitly* by:

$$F(x, y(x)) = C. \tag{2}$$

We can recover (1) from (2) as follows:

$$\frac{\partial}{\partial x} F(x, y(x)) = \frac{\partial}{\partial x} C^{\,0} \implies \underbrace{\frac{\partial F}{\partial x}}_{=: M(x,y)} + \underbrace{\frac{\partial F}{\partial y}}_{=: N(x,y)} \frac{dy}{dx} = 0.$$

differential form

$(\cdot\, dx)$:

$$\boxed{M(x,y)\,dx + N(x,y)\,dy = 0.} \quad (3)$$

- The general first-order differential eq'n $y' = f(x, y)$ can be written in this form with:

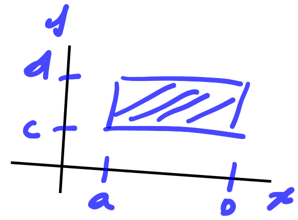
  $$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} \implies \text{if} \quad \begin{aligned} M(x,y) &= f(x,y) \\ N(x,y) &= -1 \end{aligned}$$

- As a result, if the exists a function $F(x, y)$ such that:

  $$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N.$$

- $F(x,y) = C$ defines a general solution of (3).

- In this case: (3) an exact differential eq.



Theorem 1. *Suppose that the functions $M(x, y)$ and $N(x, y)$ are continuous and have continuous first-order partial derivatives in the open rectangle $R : a < x < b, c < y < d$. Then the differential equation* differential form.

$$M(x, y)dx + N(x, y)dy = 0$$

is exact in R if and only if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \tag{3}$$

at each point of R. That is, there exists a function $F(x, y)$ defined on R with $\partial F/\partial x = M$ and $\partial F/\partial y = N$ if and only if (3) holds on R.

(*) if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is TRUE $\Rightarrow$ we can construct a $F(x,y)$ s.t. $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$

i) $\forall g(y)$:

$F(x,y) = \int M(x,y)\,dx + g(y)$

$\frac{\partial F}{\partial x} = M(x,y)$

ii) select $g(y)$ s.t $\frac{\partial F}{\partial y} = N$.

$N(x,y) = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y)$

$g'(y) = N - \frac{\partial}{\partial y}\int M(x,y)\,dx$.

$= \int \left(N - \frac{\partial}{\partial y}\int M(x,y)\,dx\right) dy$.

iii) $\frac{\partial}{\partial x}\left(N - \frac{\partial}{\partial y}\int M\,dx\right) = \frac{\partial N}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial x}\int M(x,y)\,dx$

$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$.

$$\boxed{F(x,y) = \int M(x,y)\,dx + \int\left(N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx\right)dy}$$

Example 1. *Verify that the following differential equation is exact; then solve it.*

$$\frac{dy}{dx} = -\frac{3x^2 + 2y^2}{4xy + 6y^2}$$

- Rewriting the equation in *differential* form gives:

$$\underbrace{(3x^2+2y^2)}_{=:M(x,y)}dx + \underbrace{(4xy+6y^2)}_{=:N(x,y)}dy = 0. \quad (4)$$

- We now check if ___(4)___ is *exact*:

$$\frac{\partial M}{\partial y} = 4y = \frac{\partial N}{\partial x} \checkmark \quad \text{Exact!}$$

Sol'n:

i) $\frac{\partial F}{\partial x} = M(x,y) \Rightarrow F(x,y) = \int M(x,y)\,dx + g(y)$

$$= \int (3x^2+2y^2)\,dx + g(y)$$

$$= x^3 + 2xy^2 + g(y)$$

ii) $\frac{\partial F}{\partial y} = N(x,y) \Leftrightarrow \cancel{4xy} + g'(y) = N(x,y)$

$$= \cancel{4xy} + 6y^2$$

$$g(y) = \int 6y^2\,dy = 2y^3.$$

$$\boxed{\begin{aligned} F(x,y) &= \int M(x,y)\,dx + g(y) \\ &= x^3 + 2xy^2 + 2y^3 = C. \end{aligned}}$$

Reducible Second-Order Equations

A *second-order differential equation* has the general form:

$$F(x, y, y', y'') = 0 \quad (5)$$

If *either* the independent var x or the dependent var y is missing from a second-order equation, then it can be easily reduced to a first-order equation.

Dependent variable y missing.

- If y is missing, (5) takes the form:

$$F(x, y', y'') = 0.$$

- Then the substitution:

$$p = y' = \frac{dy}{dx} \Rightarrow y'' = p'$$

- results in:

$F(x, \quad y', y'') = 0.$

$$\Rightarrow \boxed{F(x, p, p') = 0.}$$

- If we can solve this equation for a general solution $p(x, C_1)$,

$$y(x) = \int y'(x)\, dx = \int p(x, C_1)\, dx + C_2.$$

- Observe that the solution involves 2 constants C_1 and C_2.

Example 2. *Find a general solution of the differential equation:*

$$xy'' = y'$$

- Since the ______ y ______ is missing, we use the substitution:

$$p = y' \Rightarrow p' = y''$$

- This leads to: $x \cdot \frac{dp}{dx} = p.$

- Separating variables gives:

$$\int \frac{dp}{p} = \int \frac{dx}{x} + C.$$

$$\ln(p) = \ln(x) + C.$$

$$e^{\ln(p)} = \underbrace{e^{C}}_{\tilde{C}} e^{\ln(x)}$$

$$\boxed{p = Cx}$$

use $p = \frac{dy}{dx}$

$$y(x) = \int Cx\, dx + B.$$

$$\boxed{y(x) = \frac{1}{2} C x^2 + B}$$

Independent variable x missing.

- If x is missing, $F(x, y, y', y'')$ takes the form:

$$F(y, y', y'') = 0.$$

- Then the substitution:

$$p = y' \implies y'' = \frac{dp}{dx} = \frac{dp}{dy} \cdot \underbrace{\frac{dy}{dx}}_{p}$$

- results in: $y'' = p \cdot \frac{dp}{dy}.$

dep. var

$$F\left(y, p, p \cdot \frac{dp}{dy}\right) = 0$$

- If we can solve this equation for a general solution $p(y, C_1)$,

- Assuming $y' \neq 0.$:

$$x(y) = \int \frac{dx}{dy} \cdot dy. = \int \frac{1}{dy/dx}$$

$$= \underbrace{\int \frac{dy}{p(y, C_1)}}_{=: P(y, C_1)} + C_2.$$

Implicit form:

$$\boxed{x(y) = P(y, C_1) + C_2}$$

Example 3. *Find a general solution of the differential equation:*

$$yy'' + (y')^2 = yy'$$

- Since x is missing, we use the substitution:

$$p = y' = \frac{dy}{dx} \Longrightarrow y'' = \frac{dp}{dy} \cdot \frac{dy}{dx} = p \cdot \frac{dp}{dy}.$$

- This leads to:

$$\cancel{y}p \cdot \frac{dp}{dy} + (p)^2 = y \cdot \cancel{p}$$

$$\boxed{\frac{dp}{dy} + \frac{1}{y} \cdot p = 1.}$$

linear.

Use int. factor.

$$\rho(y) = e^{\int \frac{1}{y} dy} = y.$$

$$\int D_y (y \cdot p) = \int y + C.$$

$$y \cdot p = \frac{y^2}{2} + C.$$

$$\boxed{p = \frac{y^2 + C}{2y}.}$$

use $p = \frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{y^2 + C}{2y}.$$

separating var's

$$\int \frac{2y}{y^2 + C} dy = \int dx + B.$$

$$\boxed{\ln(y^2 + C) = x + B.}$$

Solve for y!

$$y(x) = \pm \sqrt{A + Be^x}.$$

Example 4. *Find a general solution of the differential equation:*

$$y'' = 2y(y')^3$$

- Since x is missing, we use the substitution:

$$y' = p \quad \text{and} \quad y'' = p \cdot \frac{dp}{dy}$$

- This leads to:

$$p \cdot \frac{dp}{dy} = 2y p^3.$$

$$\frac{dp}{dy} = 2y p^2.$$

Separating var's:

$$\int \frac{dp}{p^2} = \int 2y \, dy + C.$$

$$-\frac{1}{p} = y^2 + C$$

$$\boxed{p = -\frac{1}{y^2+C}}$$

use $p = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{1}{y^2+C}.$

Separating var's.

$$\int y^2 + C \, dy. = -\int dx + B.$$

$$\frac{1}{3}y^3 + Cy = -x + B.$$

$$\boxed{y^3 + 3x + Ay + B = 0}$$