MA 266 Lecture 9

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Sec 2.1 Population Models

- Earlier we used the exponential differential equation

$$\frac{dP}{dt} = k P(t) \quad ; \quad P(0) = P_0.$$

  with solution

$$P(t) = P_0 e^{kt}.$$

  to model natural population growth.

- Assumed that the birth β and death δ rates were constant.
- Consider a more general pop. model that allows for *nonconstant* birth/death rates.

Variable birth and death rates

- $\beta(t)$ births per unit of population per unit of time at time t.
- $\delta(t)$ deaths per unit of population per unit of time at time t.
- Over the time interval $[t, t + \Delta t]$ there are then roughly

$$i)\ \underset{\text{"births"}}{\beta(t) \cdot P(t) \cdot \Delta t} \qquad ii)\ \underset{\text{"deaths"}}{\delta(t) \cdot P(t) \cdot \Delta t}$$

- Thus the change in population over this time interval is

$$\Delta P(t) = \text{births} - \text{deaths} = \big(\beta(t) \cdot P(t) - \delta(t) \cdot P(t)\big)\Delta t$$

$$\Longleftrightarrow \frac{\Delta P(t)}{\Delta t} = \big(\beta(t) - \delta(t)\big) \cdot P(t)$$

- Taking the limit as $\Delta t \to 0$ gives the general population eq'n:

$$\boxed{\frac{dP}{dt} = (\beta(t) - \delta(t)) \cdot P(t)}$$

- If β and δ are constant $\implies$ *natural growth equation* with $k = \beta - \delta$.

- But it also includes the possibility that β and δ vary with t.

Example 1. *Suppose that when a certain lake is stocked with fish, the birth β and death δ rates are both inversely proportional to $\sqrt{P(t)}$. Show that*

$$P(t) = \left(\frac{1}{2}kt + \sqrt{P_o}\right)^2,$$

where k is constant and $P(0) = P_o$.

$$\beta(t) = k_1/\sqrt{P} \qquad \delta(t) = k_2/\sqrt{P}.$$

- General population equation:

$$\frac{dP}{dt} = (\beta(t) - \delta(t)) \cdot P(t)$$

$$= \frac{k_1 - k_2}{\sqrt{P}} \cdot P = \underbrace{(k_1 - k_2)}_{=:k} \cdot \sqrt{P}. \quad ; P(0) = P_0.$$

$$\boxed{\frac{dP}{dt} = k \cdot \sqrt{P}}$$

- Separating variables:

$$\int \frac{dP}{\sqrt{P}} = \int k \cdot dt + C.$$

$$2 \cdot \sqrt{P} = k \cdot t + C.$$

$$\Rightarrow \boxed{P(t) = \left(\tfrac{1}{2}kt + C\right)^2}$$

using $P(0) = P_0 \Rightarrow C = \sqrt{P_0}$.

$$\boxed{P(t) = \left(\tfrac{1}{2}kt + \sqrt{P_0}\right)^2.}$$

The logistic equation

$\searrow$ $P(t)$ $\nearrow$

- Often, the $\beta(t)$ of a population *decreases* as the population itself grows.
- Model $\Longrightarrow$ the $\beta(t)$ is a *linear* decreasing function of the population P:

$$\beta(t) = \beta_0 - \beta_1 P(t) .$$

where $\beta_0, \beta_1 > 0.$ are positive constants.

- If $\delta = \delta_0.$, then our general population equation:

$$\frac{dP}{dt} = (\beta - \delta) \cdot P = (\beta_0 - \beta_1 P(t) - \delta_0) \cdot P.$$

- We can rewrite this as

$$\boxed{\frac{dP}{dt} = aP - bP^2} \quad (1)$$

where $a = \beta_0 - \delta_0.$ $\quad b = \beta_1$

- If $a, b > 0$, then this equation is called the (1) logistic equation..
- Useful to rewrite (1) in the form:

$$\boxed{\frac{dP}{dt} = k \cdot P(M - P)}$$

where $k = b$ $\quad M = \frac{a}{b}$ "constants"

Limiting populations and carrying capacity

- Consider the logistic initial value problem:

$$\frac{dP}{dt} = k \cdot P(M-P) \quad ; \quad P(0) = P_0.$$

- Solution:

$$\int \frac{dP}{P(M-P)} = \int k\,dt + C$$

partial fractions.

$$\frac{1}{M}\int\left(\frac{1}{P} + \frac{1}{M-P}\right)dP = \int k\,dt + C.$$

$$\ln(P) - \ln(M-P) = Mkt + C.$$

$$\frac{P(t)}{M-P(t)} = \underbrace{e^{C}}_{=:B} e^{Mkt} = B e^{Mkt}$$

general sol'n of logist. eq'n.

- Substituting $t=0$ and $P = P_0 \neq M$.

$$B = \frac{P_0}{M-P_0} \Rightarrow \frac{P(t)}{M-P(t)} = \frac{P_0}{(M-P_0)e^{-Mkt}}.$$

$$\boxed{P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-Mkt}}}$$

- If $P_0 = M$:

$P(t) = M. \quad \forall t > 0.$

"equilibrium solution"

- Otherwise:

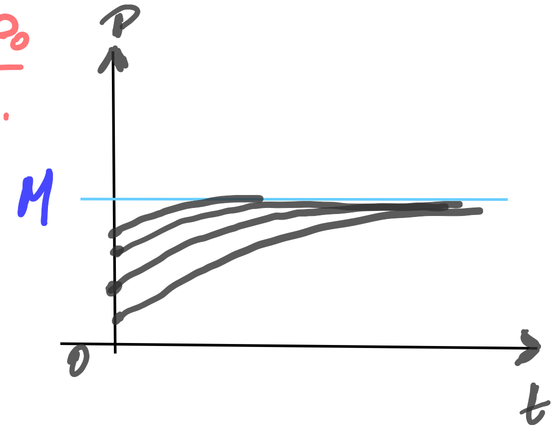
i) $0 < P_0 < M$ or ii) $P_0 > M$.

- If $0 < P_0 < M.$, then the logistic equation shows that:

i) $P(t) = \dfrac{MP_0}{P_0 + \underbrace{(M-P_0)}_{>0} e^{-Mkt}} < \dfrac{MP_0}{P_0}$.

$\Rightarrow P(t) < M.$

ii) $\dfrac{dP}{dt} = \underbrace{k}_{\geq 0.} \cdot P \underbrace{(M-P)}_{>0} > 0.$

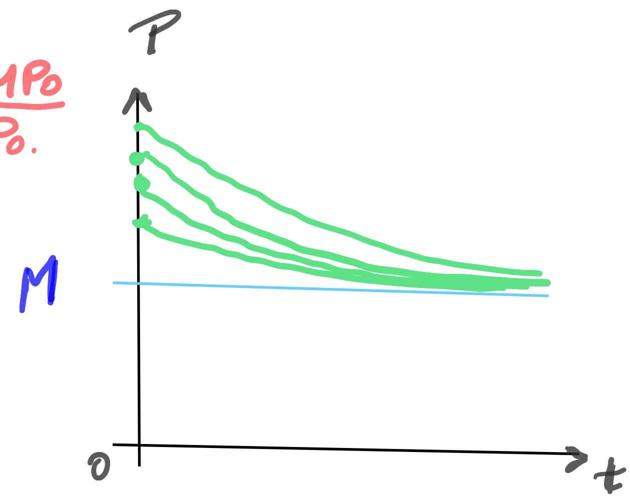


- However, if $P_0 > M$, then

i) $P(t) = \dfrac{MP_0}{P_0 + \underbrace{(M-P_0)}_{<0} e^{-Mkt}} > \dfrac{MP_0}{P_0}$.

$\Rightarrow P(t) > M.$

ii) $\dfrac{dP}{dt} = \underbrace{k}_{\geq 0} \cdot P \underbrace{(M-P)}_{<0} < 0$



- In either case we find that:

$$\lim_{t \to +\infty} \frac{MP_0}{P_0 + (M-P_0)e^{-Mkt}} = \underline{M}.$$

- It approaches the finite limiting population or carrying capacity. as $t \to +\infty$.

Example 2. *Consider a population $P(t)$ satisfying the logistic equation:*

$$\frac{dP}{dt} = \underbrace{aP}_{B(t)} - \underbrace{bP^2}_{D(t)},$$

where $B(t) = aP(t)$ is the time rate at which births occur and $D = bP(t)^2$ is the rate at which deaths occur. If the initial population is $P(0) = P_0$, and B_0 births per month and D_0 deaths per month are occurring at time $t = 0$, show that the limiting population is $M = B_0P_0/D_0$.

- Data:

$B(t) = a\,P(t)$ $\quad B_0$

$D(t) = b.P(t)^2$ $\quad D_0.$

$P(0) = P_0$

- Logistic equation form:

$$\frac{dP}{dt} = k.P(M-P) \quad \therefore \quad \frac{dP}{dt} = bP.\left(\frac{a}{b} - P\right)$$

- The *limiting population* is then:

$$M = \frac{a}{b}.$$

a, b?

$B_0 = aP(0) = aP_0$

$D_0 = bP(0)^2 = bP_0^2$

$$M = \frac{\frac{B_0}{P_0}}{\frac{D_0}{P_0^2}} = \frac{B_0 . P_0}{D_0} \quad //$$

Other applications of the logistic population model

1. *Limited environment situation.*

   - A certain environment can support at most $\underset{\sim}{M}$ individuals.
   - It is then reasonable to expect the growth rate $\beta - \delta$ to be proportional to $M - P$.
   - Then

$$\beta - \delta = k \cdot (M - P)$$

$$\frac{dP}{dt} = (\beta - \delta) P \equiv kP \cdot (M - P).$$

2. *Competition situation.*

   - If the birth rate β is constant but the death rate δ is proportional to P:

$$\delta = \underset{\sim}{\alpha} P(t)$$

$$\Rightarrow \frac{dP}{dt} = (\beta - \alpha P) \cdot P \equiv k \cdot P (M - P)$$

   - This could be a reasonable working hypothesis in a study of a cannibalistic population, in which all deaths result from chance encounters between individuals.

3. *Joint proportion situation.*

   - Let $P(t)$ denote the number of individuals in a constant-size population $\underset{\sim}{M}$ who are infected with a certain contagious and incurable disease, which is spread by chance encounters.
   - Then

$$\frac{dP}{dt} = k \cdot \underbrace{(\text{\# with disease})}_{=: P(t)} \cdot \underbrace{(\text{\# without disease})}_{=: M - P(t)}$$

$$\frac{dP}{dt} = kP(M - P)$$

   - The dynamics of the spread of a rumor in a population of M individuals is identical.

Example 3. *Suppose that a community contains 15000 people who are susceptible to a contagious disease. At time $t = 0$, the number of people who have developed the disease is 5000 and is increasing at 500 per day. Assume that $N'(t)$ is proportional to the product of the numbers of those who have caught the disease and of those who have not. How long will it take for another 5000 people to caught the disease?*

- Data

$M = 15$ (in thousands)

$N(t) = \#$ with some disease (in thousands).

$N(0) = 5$ $\qquad N'(0) = 0.5$ (in thousands).

- Population equation:

$N(t) = 10.\ ?$

$$\text{IVP} \quad \begin{cases} N'(t) = k \cdot N(t) \cdot (\underset{\text{"}M\text{"}}{15} - N(t)) \\ N(0) = 5. \end{cases}$$

$$k = \frac{N'(t)}{N(t) \cdot (15 - N(t))} = \frac{N'(0)}{N(0) \cdot (15 - N(0))} = 0.01.$$

Solution:

$$N(t) = \frac{M N(0)}{N(0) + (M - N(0)) e^{-\overbrace{Mk}^{15 \cdot 0.01} t}} = \frac{15}{1 + 2 e^{-0.15 t}}.$$

- Time for another 5000 people to caught the disease: $N(t) = 10\ ?$

$$10 = \frac{15}{1 + 2e^{-0.15 t}} \Rightarrow 2e^{-0.15t} = \frac{1}{2}.$$

$$e^{0.15t} = 4$$

$$\Rightarrow \boxed{t = \frac{\ln(4)}{0.15}}$$

Example 4. *Consider a breed of rabbits whose birth and death rates β and δ are each proportional to the rabbit population $P = P(t)$, with $\beta > \delta$.* ***(a)*** *Show that*

$$P(t) = \frac{P_0}{1 - kP_0 t},$$

where k is constant. What when $t \to \frac{1}{kP_0}$?

- Data

$$\beta = k_1 \cdot P(t) \qquad \delta = k_2 \cdot P(t)$$

- Population equation:

$$\frac{dP}{dt} = (\beta - \delta) \cdot P(t) = (\underbrace{k_1 - k_2}_{=:\,k}) P(t)^2.$$

$$\beta > \delta \Rightarrow k > 0.$$

- Separating variables:

$$\frac{dP}{dt} = k \cdot P(t)^2.$$

$$\int \frac{dP}{P^2} = \int k\, dt + C.$$

using $P(0) = P_0$.

$$-\frac{1}{P} = kt + C.$$

$$C = \frac{1}{P_0}.$$

$$\boxed{P(t) = \frac{1}{C - kt}}$$

$$\boxed{P(t) = \frac{P_0}{1 - \underbrace{kP_0 t}_{>0}}}$$

- As $t \to \frac{1}{kP_0}$:

as $t \to \frac{1}{kP_0}$ $\quad P(t) \to +\infty$: "population explosion"

Example 5. *Repeat the previous example with $\beta < \delta$ and $t \to \infty$.*

$$P(t) = \frac{P_0}{\underbrace{1 - \underbrace{kP_0 t}_{<0}}_{>0}}$$

as $t \to \infty$ $\quad P(t) \to 0.$

"extiction situation"

MA 266 Lecture 10

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Sec 2.2 Equilibrium Solutions and Stability

Qualitative analysis

- Often we use explicit solutions of differential eq's to answer specific numerical questions.
- However, when the DE impossible to solve explicitly, it is often possible to extract

  qualitative information about general properties of solutions.
- For example,

  • $x(t) \nearrow +\infty$ as $t \to \infty$.

  • $x(t) \to C < +\infty$ as $t \to \infty$

Example 1. *(Newton's Law of Cooling:) Let:*

- temperature of body: $x(t)$
- initial temperature: $x(0) = x_0$.
- body immersed in a medium with temperature = A.

Newton's law of cooling:

$$\frac{dx}{dt} = -k(x - A) \quad ; \quad (k > 0)$$

Separating variables:

$$\int \frac{dx}{(x-A)} = \int -k\,dt + c$$

Q: $\lim_{t\to\infty} x(t) =$? particular sol'n : $x(t) = A - (x_0 - A)e^{-kt}$.

$$x(t) \to A. < +\infty.$$

Is $x(t) = A.$ a solution? YES !

Autonomous equations

Defintion 1. *An* autonomous. *first-order differential equation takes the form:*

$$\boxed{\frac{dx}{dt} = f(x)} \quad \text{only on dep. var } x. \quad (1)$$

That is, the R.H.S. is function of x.

Critical points and equilibrium solutions

Defintion 2. *The critical points of* (1) *are the solutions of the algebraic equation:*

$$\boxed{f(x) = 0.}$$

$\Rightarrow f(c) = 0.$

If $x = c$ is a critical point of this equation. Then

$x(t) = c$ is a constant solution of (1)

- Such a solution is called an equilibrium. of the differential equation. $x(t) = c$ $\forall t > 0$.
- Qualitative information (behavior) can be described in terms of *critical points*.

Example 2. *Find the critical points of:*

a) $\frac{dx}{dt} = x^2 - 4$

b) $\frac{dx}{dt} = (2 - x)^3$

a) $f(x) = (x^2 - 4) = 0$

$\Rightarrow$ $c_1 = +2$, $c_2 = -2$.

b) $f(x) = (2 - x)^3 = 0$

$\Rightarrow$ $c = 2$.

Stability of critical points (Stable vs. Unstable)

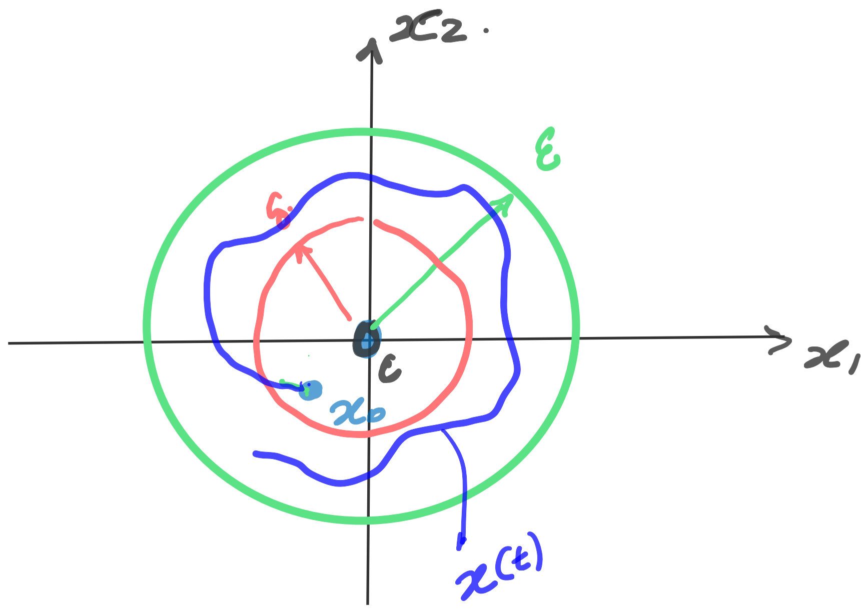
Defintion 3. *A critical point* $x = c$ *of* $\frac{dx}{dt} = f(x)$ *is* ***stable*** *provided that:*

$\Rightarrow f(c) = 0$

- If x_0 is sufficiently close to c, then $x(t)$ remains close to c for all $t > 0$.
- (**Formally**) the critical point is *stable* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x_0 - c| < \delta \implies |x(t) - c| < \varepsilon \quad \forall\, t > 0.$$

- Otherwise, the critical point is *unstable*.



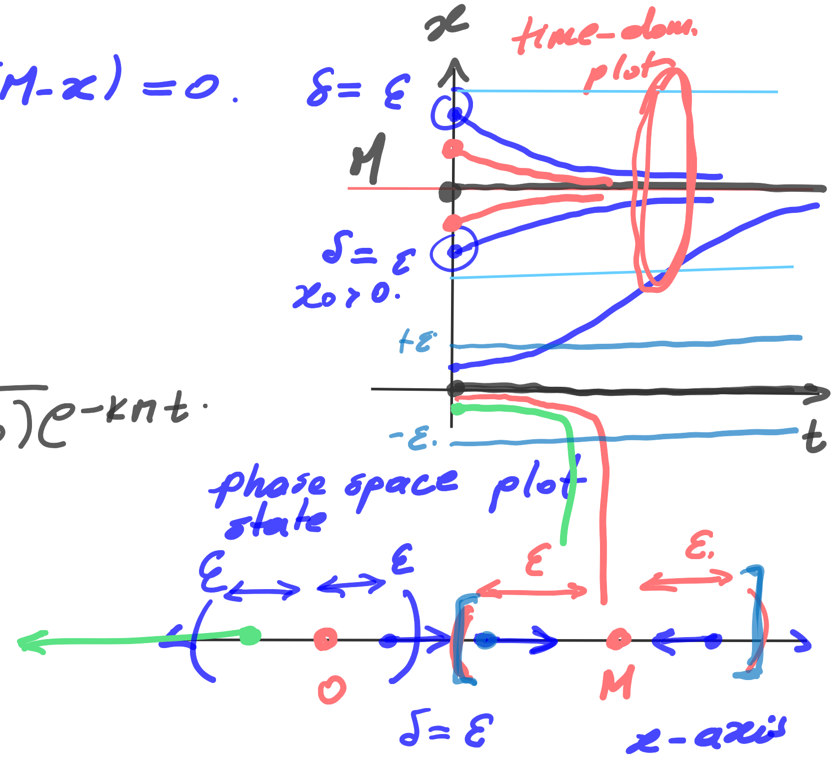
Example 3. *Consider the logistic initial value problem:*

$$\frac{dx}{dt} = kx\cdot(M-x)\,; \quad x(0) = x_0.$$

Critical points:

$$f(x) = kx(M-x) = 0.$$

$$\Rightarrow \quad c_1 = 0, \quad c_2 = M.$$



The particular solution:

$$x(t) = \frac{Mx_0}{x_0 + (M-x_0)e^{-kMt}}.$$

Equilibrium solutions:

i) $x(t) = 0.$

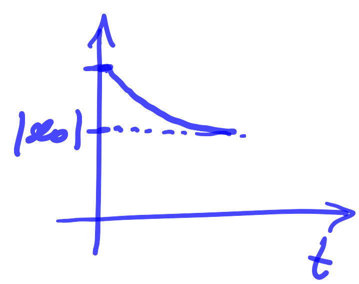
ii) $x(t) = M$

We observed (in previous lecture):

- if $x_0 > 0 \Rightarrow x(t) \to M$ as $t \to \infty$

But if $x_0 < 0.$:

$$x(t) = \frac{\overbrace{Mx_0}^{<0}}{\underbrace{\underbrace{x_0}_{<0} + \underbrace{(M-x_0)e^{-kMt}}_{>0}}}.$$



Note for $t \in [0, t^*)$ denom > 0.

$$t^* : \quad -x_0 = (M-x_0)e^{-kMt^*}$$

$$e^{kMt^*} = \frac{(M-x_0)}{-x_0}$$

$$\Rightarrow t^* = \frac{1}{kM}\cdot \ln\left(\frac{M-x_0}{\underbrace{-x_0}_{>0}}\right) > 0.$$

Example 4. *Consider:*

$$\frac{dx}{dt} = kx - x^3$$

a) *Let $k \le 0$. Show that the only critical point is stable.*

Solution: Note that if $k \le 0$, then we can let $k := -a^2$. The **first** step is to find *critical points*. To this end, we solve the following algebraic equation:

$$f(x) = 0 \iff (k - x^2)x = -\underbrace{(a^2 + x^2)}_{>0}x = 0.$$

So, the only *critical point* is:

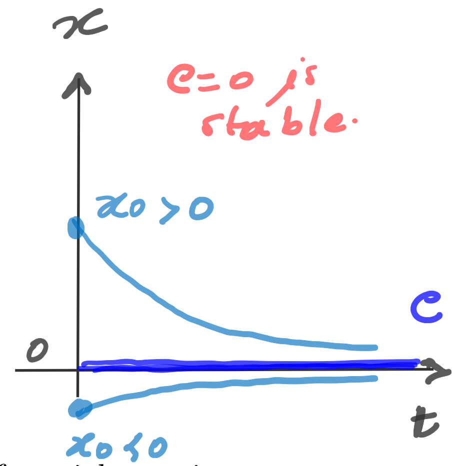
$$c = 0.$$

The **second** step is to compute the solution of the above differential equation. Observe that this equation is *separable*. Let $a > 0$. Then by separating variables, we obtain:

$$\begin{aligned} \int \frac{dx}{(a^2+x^2)x} &= \int -dt + C \\ \iff \int \frac{a^2}{(a^2+x^2)x} dx &= \int -a^2 dt + C \end{aligned}$$

The *partial fractions* method yields:

$$\begin{aligned} \int \frac{1}{x} - \frac{x}{a^2+x^2} dx &= \int -dt + C \\ \iff \ln(x) - \frac{1}{2}\ln(a^2+x^2) &= -a^2 t + C \\ \iff \frac{x}{(a^2+x^2)^{1/2}} &= Ce^{-a^2 t} \\ \iff \frac{x^2}{(a^2+x^2)} &= Ce^{-2a^2 t} \end{aligned}$$



By solving the above for x^2, we obtain the general solution of the differential equation:

$$x^2 = \frac{a^2 Ce^{-2a^2 t}}{1 - Ce^{-2a^2 t}}$$

In the **third** step, we use the above general solution to determine the stability of the critical point $c = 0$. To this end, we check what happens with the solution as $t \to \infty$. Clearly, if we let $t \to \infty$, $Ce^{-2a^2 t} \to 0$. As a result:

$$x(t) \to 0 \equiv c \text{ as } t \to \infty.$$

Thus, we conclude that the critical point $c = 0$ is *stable*.

b) Let $k > 0$. Analyze the stability of critical point(s).

Solution: Note that if $k > 0$, then we can let $k = a^2$ with $a \neq 0$. As before, the **first** step is to find the critical points by solving the following algebraic equation:

$$f(x) = 0 \iff (k - x^2)x = \underbrace{(a^2 - x^2)}_{>0} x = 0.$$

So, when $k > 0$, we have *three* critical point:

$$\begin{cases} c_1 = 0 \\ c_2 = +a = +\sqrt{k} \\ c_3 = -a = -\sqrt{k}. \end{cases}$$

The **second** step is to compute the solution of the differential equation. Note that we can write the differential equation as follows:

$$\frac{dx}{dt} = -x(x-a)(x+a).$$

This is a separable equation. Thus, by separating variables, we obtain:

$$\int \frac{2a^2}{x(x-a)(x+a)} dx = -\int 2a^2 dt + C.$$

The partial fractions method yields:

$$\begin{aligned} &\int -\frac{2}{x} + \frac{1}{x-a} + \frac{1}{x+a} dx = -\int 2a^2 dt + C \\ \iff\ & -2\ln(x) + \ln(x-a) + \ln(x+a) = -2a^2 t + C \\ \iff\ & \frac{x^2 - a^2}{x^2} = Ce^{-2a^2 t}. \end{aligned}$$

By solving for x, we obtain the general solution:

$$x(t) = \frac{\pm\sqrt{k}}{\sqrt{1 - Ce^{-2a^2 t}}}$$

For this example, it is convenient to also compute the particular solution for the initial condition $x(0) = x_o$. Using this initial condition, we find that is $C = 1 - k/x_o^2$. As a result, the particular solution is:

$$x(t) = \frac{\pm\sqrt{k}}{\sqrt{1 - (1 - k/x_o^2)e^{-2a^2 t}}}.$$

In the **third step**, we use the above particular solution to determine the stability of the three critical points. To this end, we analyze four cases.

(Case 1) Let $x_o \in (0, \sqrt{k})$ and use the particular solution:

$$x(t) = \frac{+\sqrt{k}}{\sqrt{1 - (1 - k/x_o^2)e^{-2a^2 t}}}.$$

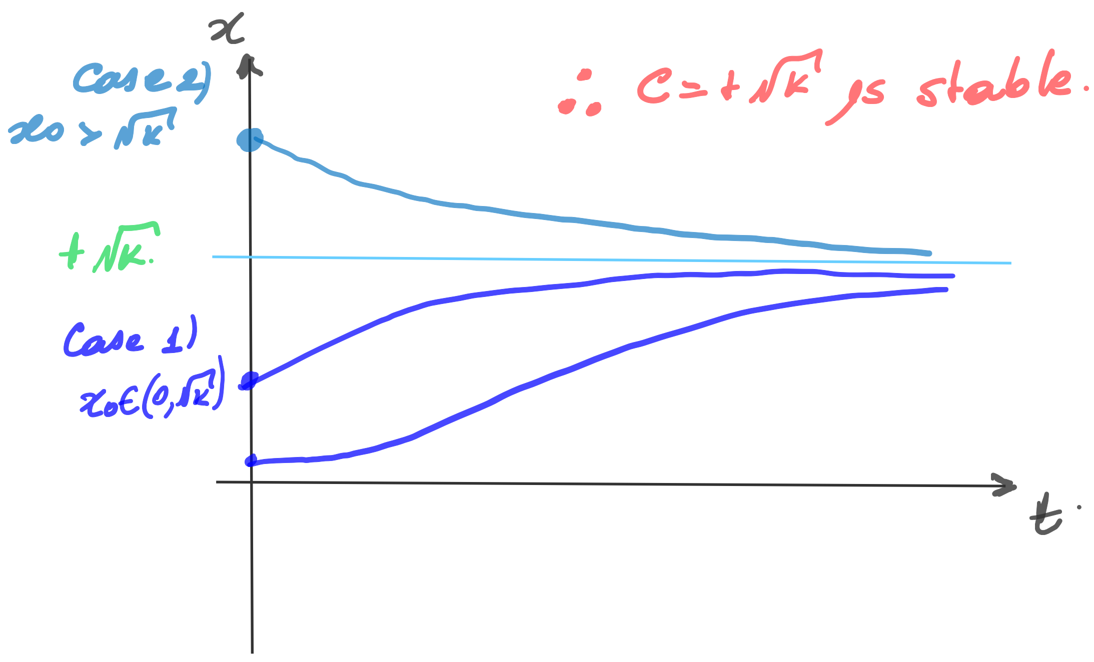
- Note that for this case $(1 - k/x_o^2) < 0$.
- Hence the denominator satisfies: $\sqrt{1 - (1 - k/x_o^2)e^{-2a^2 t}} \searrow 1$.
- As a result, $x(t)$ increases towards $+\sqrt{k}$, *i.e.*, $x(t) \nearrow +\sqrt{k}$ as $t \to \infty$.

(Case 2) Let $x_o > \sqrt{k}$ and use the particular solution:

$$x(t) = \frac{+\sqrt{k}}{\sqrt{1 - (1 - k/x_o^2)e^{-2a^2 t}}}.$$

- Note that for this case $(1 - k/x_o^2) > 0$.
- Hence the denominator satisfies: $\sqrt{1 - (1 - k/x_o^2)e^{-2a^2 t}} \nearrow 1$.
- As a result, $x(t)$ decreases towards $+\sqrt{k}$, *i.e.*, $x(t) \searrow +\sqrt{k}$ as $t \to \infty$.

Case 1) and 2) show that the critical point $c_2 = +\sqrt{k}$ is *stable*.



(Case 3) Let $x_o \in (-\sqrt{k}, 0)$ and use the particular solution:

$$x(t) = \frac{-\sqrt{k}}{\sqrt{1-(1-k/x_o^2)e^{-2a^2t}}}.$$

- Note that for this case $(1-k/x_o^2) < 00$.
- Hence the denominator satisfies: $\sqrt{1-(1-k/x_o^2)e^{-2a^2t}} \searrow 1$.
- As a result, $x(t)$ becomes more negative ; $x(t)$ decreases towards $-\sqrt{k}$, *i.e.*, $x(t) \searrow -\sqrt{k}$ as $t \to \infty$.

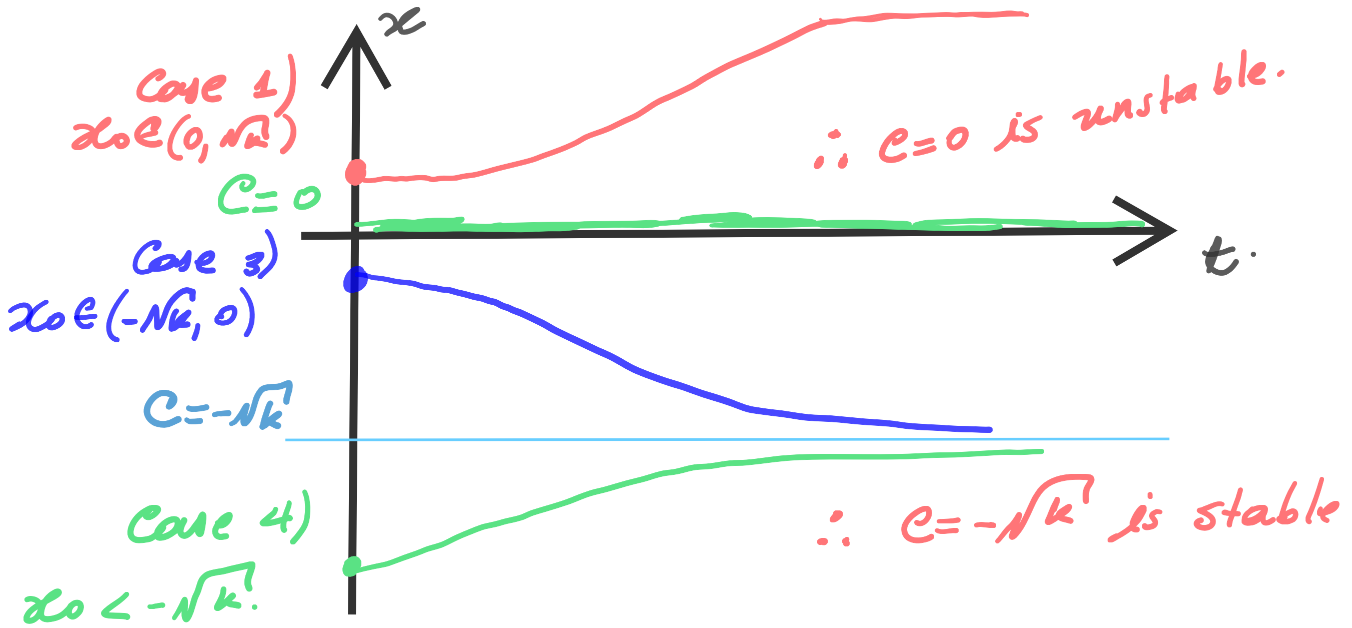
(Case 4) Let $x_o < -\sqrt{k}$ and use the particular solution:

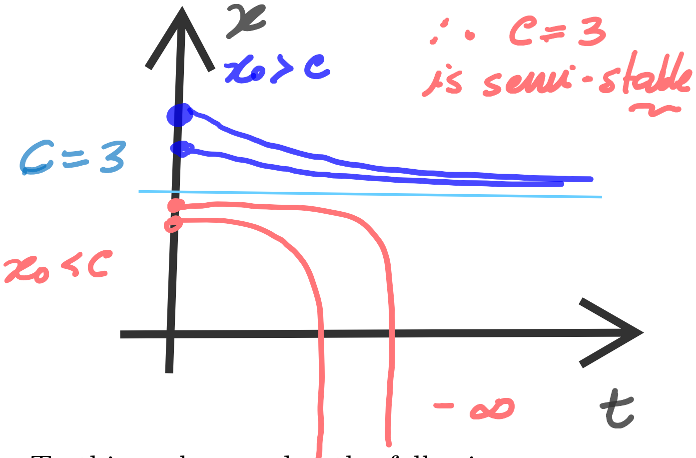
$$x(t) = \frac{-\sqrt{k}}{\sqrt{1-(1-k/x_o^2)e^{-2a^2t}}}.$$

- Note that for this case $(1-k/x_o^2) > 0$.
- Hence the denominator satisfies: $\sqrt{1-(1-k/x_o^2)e^{-2a^2t}} \nearrow 1$.
- As a result, $x(t)$ becomes less negative; $x(t)$ increases towards $-\sqrt{k}$, *i.e.*, $x(t) \nearrow -\sqrt{k}$ as $t \to \infty$.

Case 3) and 4) show that the critical point $c_3 = -\sqrt{k}$ is *stable*.

Finally, **Case 1)** and **Case 3)** show that the critical point $c_1 = 0$ is *unstable*.





Example 6. *Given the differential equation:*

$$\frac{dx}{dt} = -(3-x)^2$$

Analyze the stability of the critical point(s).

Solution. The **first** step is to find the critical points. To this end, we solve the following algebraic equation:

$$f(x) = 0 \iff -(3-x)^2 = 0.$$

So, the only *critical point* is:

$$c = 3.$$

The **second** step is to solve the differential equation. Observe that the equation is separable. Thus, by separating variables, we obtain:

$$\int \frac{1}{(x-3)^2}dx = -\int dt + C.$$

So, the *general* solution satisfies:

$$\frac{1}{x-3} = t + C.$$

For this example, it is convenient to compute the *particular* solution. Let $x(0) = x_o$ be the initial condition. Then we have $C = \frac{1}{x_o+3}$. The particular solution is then:

$$x(t) = 3 + \frac{x_o - 3}{1 + t(x_o - 3)}.$$

In the **third** step, we use the above particular solution to determine the stability of the critical point.

- If we let $x_o > c = 3$. Then, it is easy to verify that $x(t) \to 3 \equiv c$ as $t \to \infty$.
- Now, let $x_o < c = 3$ and observe that there exists some t^* such that for $t \in [0, t^*)$ the denominator is positive $1 + t(x_o - 3) > 0$ and decreases towards 0. At t^*, the denominator vanishes. This t^* is computed as follows:

$$t^* : -1 = t^* \underbrace{(x_o - 3)}_{<0} \iff t^* = -\frac{1}{x_o - 3} > 0.$$

  Since the numerator is always negative, the solution $x(t)$ diverges to $-\infty$.

As a result, solutions nearby the critical point behave as depicted in the figure above. Thus, we say that the critical point $c = 3$ is *semi-stable*.

MA 266 Lecture 11

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Sec 2.2-b Bifurcation Points

Sec 2.3 Acceleration-Velocity Models

Review example from Lecture 10

Example 1. *Consider:*

depend. var parameter of DE

$$\frac{dx}{dt} = kx - x^3$$

indep. var.

a) *Let $k \leq 0$. Show that the only critical point is stable.*

Solution: Note that if $k \leq 0$, then we can let $k := -a^2$. The **first** step is to find *critical points*. To this end, we solve the following algebraic equation:

$$f(x) = 0 \iff (k - x^2)x = -\underbrace{(a^2 + x^2)}_{>0} x = 0.$$

So, the only *critical point* is:

$$c = 0.$$

The **second** step is to compute the solution of the above differential equation. Observe that this equation is *separable*. Let $a > 0$. Then by separating variables, we obtain:

$$\int \frac{dx}{(a^2 + x^2)x} = \int -dt + C$$

$$\iff \frac{x^2}{(a^2 + x^2)} = Ce^{-2a^2t}$$

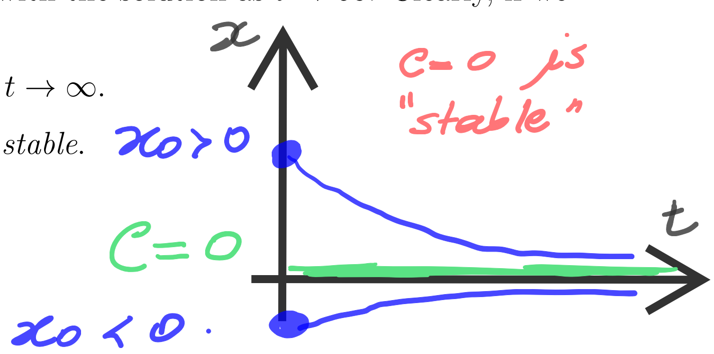
By solving the above for x^2, we obtain the general solution of the differential equation:

$$x^2 = \frac{a^2 Ce^{-2a^2t}}{1 - Ce^{-2a^2t}}$$

In the **third** step, we use the above general solution to determine the stability of the critical point $c = 0$. To this end, we check what happens with the solution as $t \to \infty$. Clearly, if we let $t \to \infty$, $Ce^{-2a^2t} \to 0$. As a result:

$$x(t) \to 0 \equiv c \text{ as } t \to \infty.$$

Thus, we conclude that the critical point $c = 0$ is *stable*.



$\frac{dx}{dt} = \underbrace{kx - x^3}_{=: f(x)}$

b) Let $k > 0$. Analyze the stability of critical point(s).

Solution: Note that if $k > 0$, then we can let $k = a^2$ with $a \neq 0$. As before, the **first** step is to find the critical points by solving the following algebraic equation:

$$f(x) = 0 \iff (k - x^2)x = \underbrace{(a^2 - x^2)}_{\geq 0} x = 0.$$

So, when $k > 0$, we have *three* critical point:

$$\begin{cases} c_1 = 0 \\ c_2 = +a = +\sqrt{k} \\ c_3 = -a = -\sqrt{k}. \end{cases}$$

The **second** step is to compute the solution of the differential equation. Note that we can write the differential equation as follows:

$$\frac{dx}{dt} = -x(x-a)(x+a).$$

This is a separable equation. Thus, by separating variables, we obtain:

$$\int \frac{2a^2}{x(x-a)(x+a)} dx = -\int 2a^2 dt + C.$$

The partial fractions method yields:

$$\begin{aligned} \int -\frac{2}{x} + \frac{1}{x-a} + \frac{1}{x+a} dx &= -\int 2a^2 dt + C \\ \iff -2\ln(x) + \ln(x-a) + \ln(x+a) &= -2a^2 t + C \\ \iff \frac{x^2 - a^2}{x^2} &= Ce^{-2a^2 t}. \end{aligned}$$

By solving for x, we obtain the general solution:

$$x(t) = \frac{\pm\sqrt{k}}{\sqrt{1 - Ce^{-2a^2 t}}}$$

For this example, it is convenient to also compute the particular solution for the initial condition $x(0) = x_o$. Using this initial condition, we find that is $C = 1 - k/x_o^2$. As a result, the particular solution is:

$$x(t) = \frac{\pm\sqrt{k}}{\sqrt{1 - (1 - k/x_o^2)e^{-2a^2 t}}}.$$

In the **third step**, we use the above particular solution to determine the stability of the three critical points. To this end, we analyze four cases.

(Case 1) Let $x_o \in (0, \sqrt{k})$ and use the particular solution:

$$x(t) = \frac{+\sqrt{k}}{\sqrt{1-(1-k/x_o^2)e^{-2a^2t}}}.$$

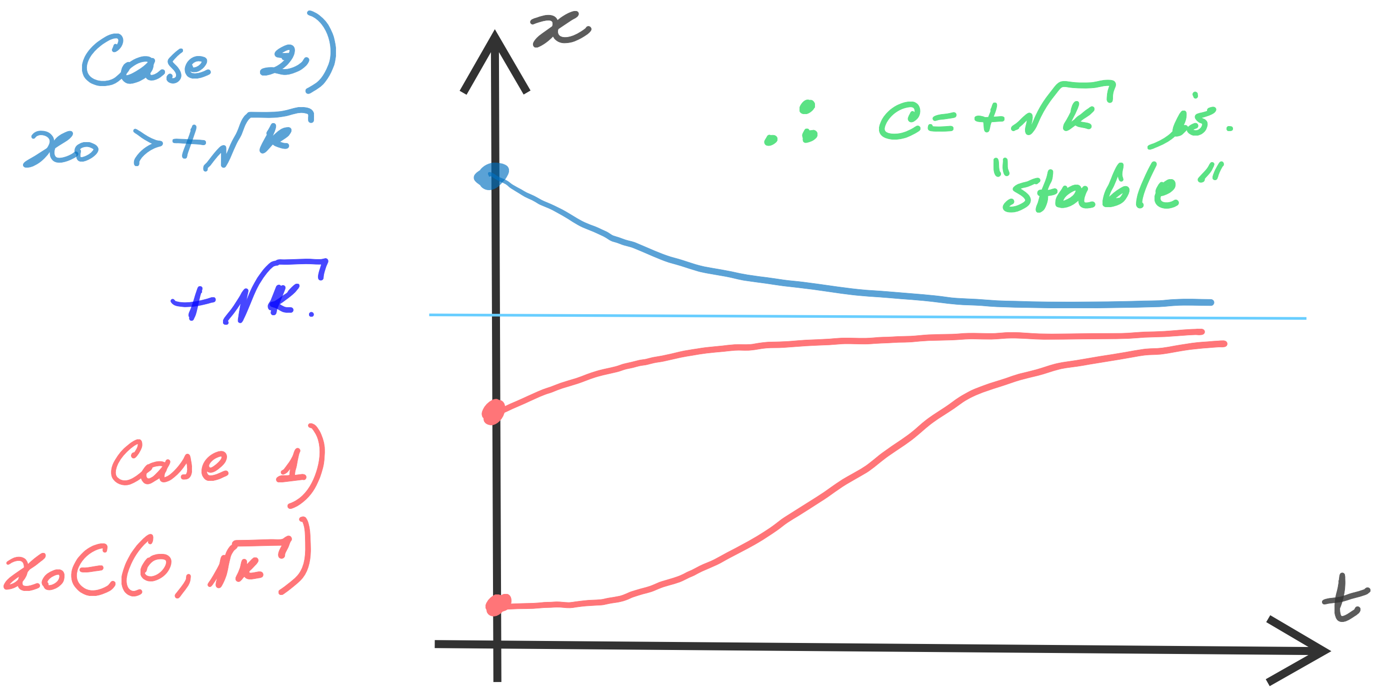
- Note that for this case $(1 - k/x_o^2) < 0$.
- Hence the denominator satisfies: $\sqrt{1-(1-k/x_o^2)e^{-2a^2t}} \searrow 1$.
- As a result, $x(t)$ increases towards $+\sqrt{k}$, *i.e.*, $x(t) \nearrow +\sqrt{k}$ as $t \to \infty$.

(Case 2) Let $x_o > \sqrt{k}$ and use the particular solution:

$$x(t) = \frac{+\sqrt{k}}{\sqrt{1-(1-k/x_o^2)e^{-2a^2t}}}.$$

- Note that for this case $(1 - k/x_o^2) > 0$.
- Hence the denominator satisfies: $\sqrt{1-(1-k/x_o^2)e^{-2a^2t}} \nearrow 1$.
- As a result, $x(t)$ decreases towards $+\sqrt{k}$, *i.e.*, $x(t) \searrow +\sqrt{k}$ as $t \to \infty$.

Case 1) and 2) show that the critical point $c_2 = +\sqrt{k}$ is *stable*.



(Case 3) Let $x_o \in (-\sqrt{k}, 0)$ and use the particular solution:

$$x(t) = \frac{-\sqrt{k}}{\sqrt{1-(1-k/x_o^2)e^{-2a^2t}}}.$$

- Note that for this case $(1 - k/x_o^2) < 0$.
- Hence the denominator satisfies: $\sqrt{1-(1-k/x_o^2)e^{-2a^2t}} \searrow 1$.
- As a result, $x(t)$ becomes more negative ; $x(t)$ decreases towards $-\sqrt{k}$, *i.e.*, $x(t) \searrow -\sqrt{k}$ as $t \to \infty$.

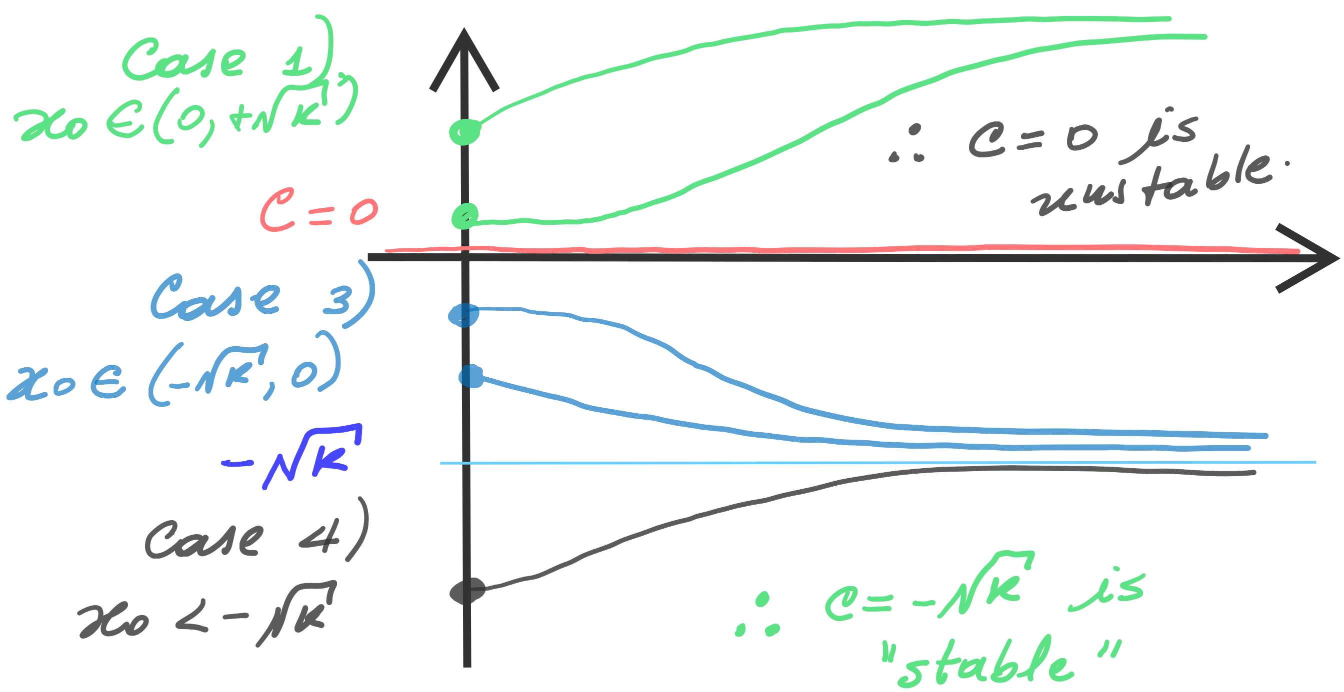
(Case 4) Let $x_o < -\sqrt{k}$ and use the particular solution:

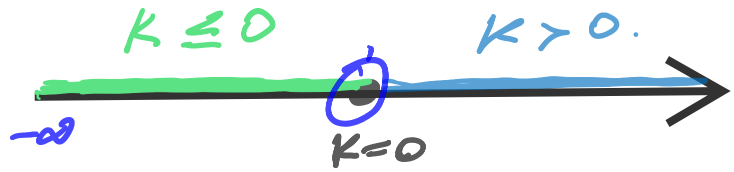
$$x(t) = \frac{-\sqrt{k}}{\sqrt{1-(1-k/x_o^2)e^{-2a^2t}}}.$$

- Note that for this case $(1 - k/x_o^2) > 0$.
- Hence the denominator satisfies: $\sqrt{1-(1-k/x_o^2)e^{-2a^2t}} \nearrow 1$.
- As a result, $x(t)$ becomes less negative; $x(t)$ increases towards $-\sqrt{k}$, *i.e.*, $x(t) \nearrow -\sqrt{k}$ as $t \to \infty$.

Case 3) and 4) show that the critical point $c_3 = -\sqrt{k}$ is *stable*.

Finally, **Case 1)** and **Case 3)** show that the critical point $c_1 = 0$ is *unstable*.





Bifurcation points

- As we gradually increase the value of the parameter $K.$.
- We have seen that the differential equation has

$$\frac{dx}{dt} = Kx - x^3 \quad : \begin{cases} \text{if } K \leq 0 \Rightarrow 1 \text{ critical point} \\ \text{if } K > 0 \Rightarrow 3 \text{ critical points} \end{cases}$$

critical points.

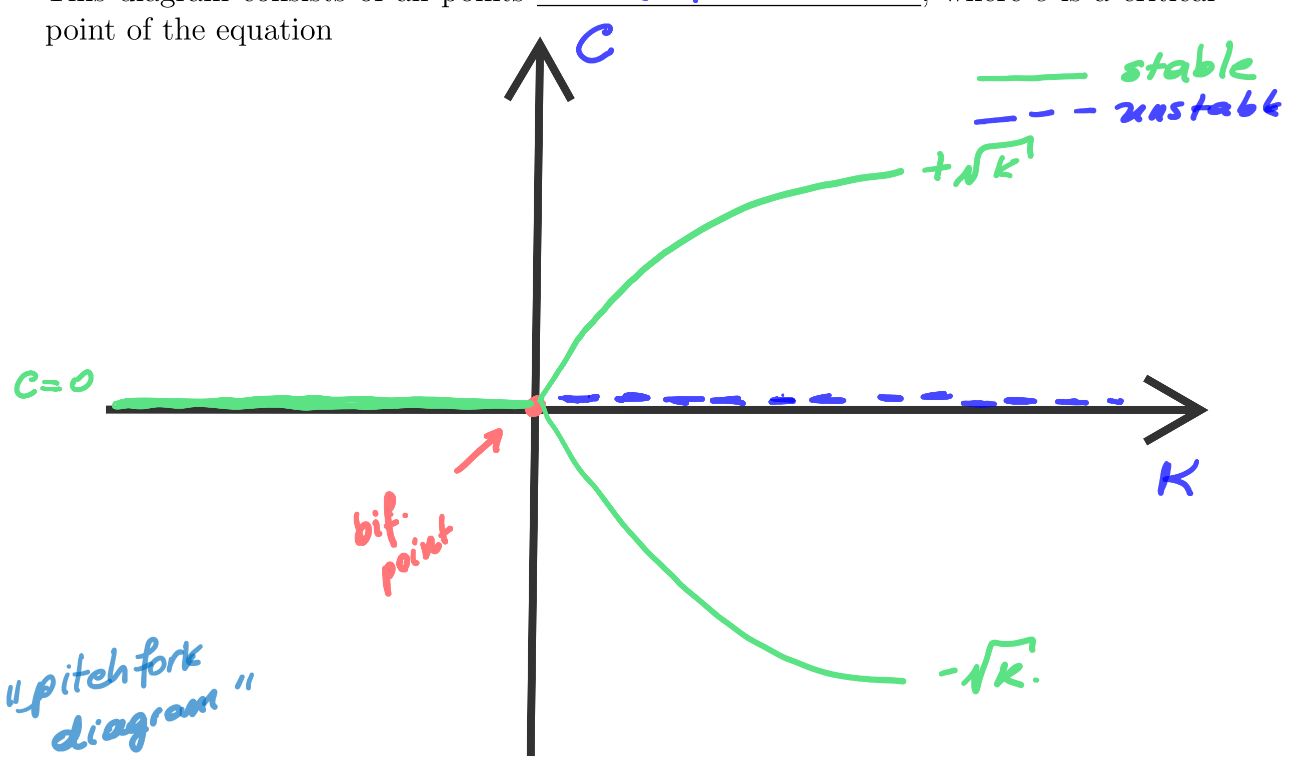
- The value $K=0$, for which the qualitative nature of the solutions changes

  $K \nearrow$
  as the parameter increases, is called a bifurcation points. for the differential equation containing the parameter.

Bifurcation diagram

- A common way to visualize the corresponding “bifurcation” in the solutions is to plot the *bifurcation diagram* for the equation.
- This diagram consists of all points (K, c), where c is a critical point of the equation

  parameters, critical points



Example 2. *Construct the bifurcation diagram of the following logistic equation with harvesting:*

$$\frac{dx}{dt} = \underbrace{x(4-x)-h}_{=: f(x)}$$

"harvesting"

Critical points:

$$f(x)=0 \iff x(4-x)-h=0.$$

$$\iff x^2-4x+h=0.$$

$$c_{1,2} = \frac{4 \pm \sqrt{16-4h}}{2} = 2 \pm \sqrt{4-h}.$$

bifurcation point

$\boxed{\text{i) if } h=4 \Rightarrow 1 \text{ critical point } c=2.}$

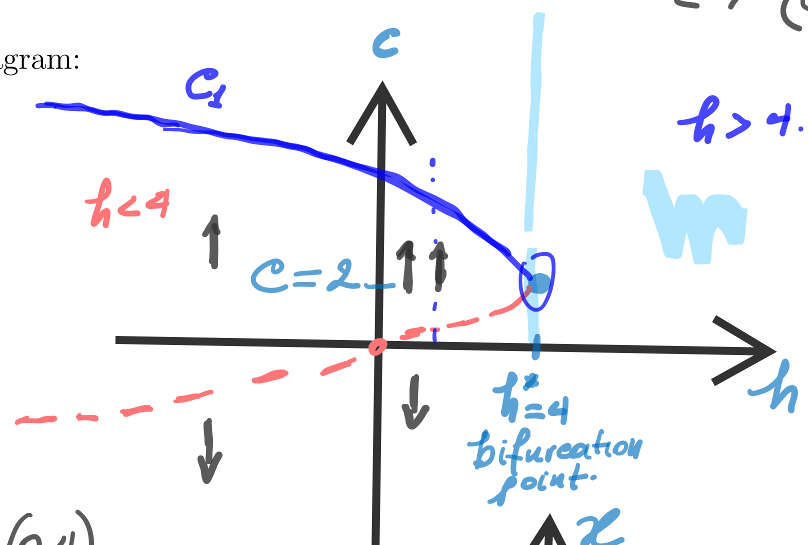
ii) if $h>4 \Rightarrow 0$ critical points

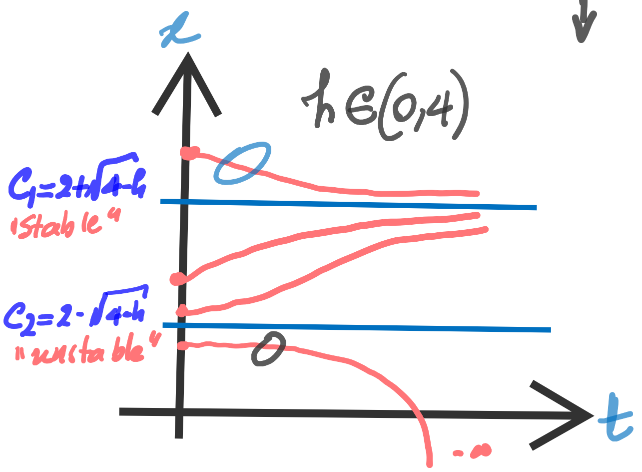
iii) if $h<4 \Rightarrow 2$ critical points $c_{1,2} = 2 \pm \sqrt{4-h}$.

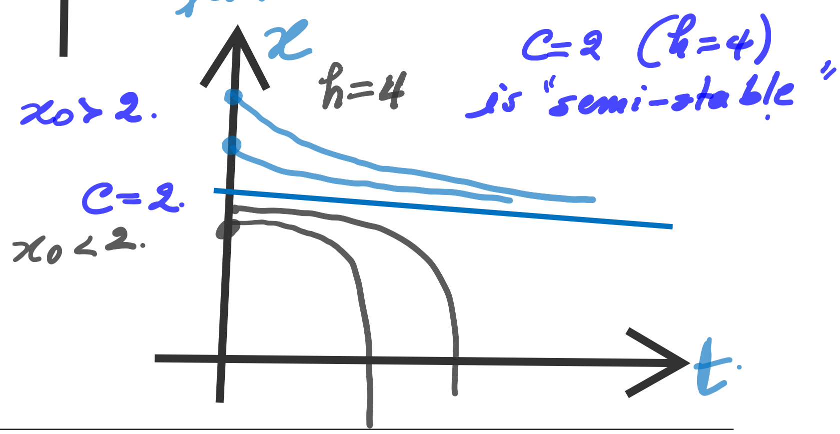
$\iff (c-2)^2 = 4-h.$

Bifurcation diagram:

(Ex 5/6 book)







Sec. 2.3

Acceleration-velocity models

- In chapter 1, we studied vertical motion of a mass m without considering air resistance.
- Newton's second law:

$$m \cdot \underbrace{\frac{dv}{dt}}_{a} = F_G.$$

- $F_G = -mg.$ is the (downward-direction) force of gravity.

In section 2.3, we want to take into account air resistance.

- $F_R.$: force exerted by air resistance on the moving mass m.
- Newton's second law:

$$m \cdot a = m \cdot \frac{dv}{dt} = F = F_G + F_R.$$

- For many problems, it suffices to model the force as:

$$F_R = k \cdot v^{\underset{\sim}{p}}$$

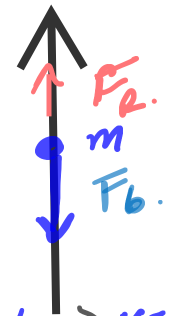
where $p \in [1,2]$ and k depends on the size and shape of the body, as well as the density and viscosity of the air.

Generally speaking, we have:

- $p = 1$ for relatively low speeds. $F_R = k \cdot \underset{\sim}{v}$
- $p = 2$ for high speeds. $F_R = k \cdot v^2.$ $v \in (0,1)$
- $p \in (1,2)$ for intermediate speeds.

falling object

Example 3. *Consider the vertical motion of an object near the surface of the earth.*



- m: *mass.*

Subject to two forces:

- $F_G = -mg$: *downward gravitational force.*
- $F_R = -kv$ $(p=1)$: *air resistance force, where* $v = \frac{dy}{dt}$.

Falling obj $\Rightarrow$ $v<0$.

Find the particular solutions $v(t)$ and $y(t)$ for the initial conditions $v(0) = v_o$ and $y(0) = y_o$.

- Newton's law of motion:

Net force $F = F_G + F_R = -mg - kv$.

$F = m.a. \Rightarrow m\frac{dv}{dt} = -mg - kv.$

let $\rho \triangleq \frac{k}{m} \Rightarrow \boxed{\frac{dv}{dt} = -g - \rho v.}$

- Separating variables:

$$\int \frac{dv}{g+\rho v} = -\int dt + C \Rightarrow \frac{1}{\rho}\ln(g+\rho.v) = -t + C.$$

$$\Leftrightarrow g + \rho v = Ce^{-\rho t}$$

- Velocity equation: $\boxed{v(t) = Ce^{-\rho t} - \frac{g}{\rho}.}$

using $v(0) = v_0. \Rightarrow C = v_0 + \frac{g}{\rho}.$

particular sol'n: $\boxed{v(t) = \left(v_0 + \frac{g}{\rho}\right)\cdot e^{-\rho t} - \frac{g}{\rho}.}$

- Q: $\lim_{t\to\infty} v(t)$?

$$v_\tau = \lim_{t\to\infty} v(t) = -\frac{g}{\rho}.$$

$$|v_\tau| = \frac{g}{\rho} < +\infty \quad \text{"terminal speed"}$$

- Position equation:

$$v(t) = (v_0 - v_\tau)e^{-\rho t} + v_\tau$$

$$\frac{dy}{dt} = (v_0 - v_\tau)e^{-\rho t} + v_\tau.$$

$$\boxed{y(t) = -\frac{1}{\rho}(v_0 - v_\tau)e^{-\rho t} + v_\tau \cdot t + C.}$$

using $y(0) = y_0$. $\Rightarrow$ $C = y_0 + \frac{1}{\rho}(v_0 - v_\tau)$

particular sol'n:

$$\boxed{y(t) = y_0 + v_\tau \cdot t + \frac{1}{\rho}(v_0 - v_\tau)(1 - e^{-\rho t})}$$

$p=2$.

Example 4. *Consider a body that moves horizontally with resistance $-kv^2$ such that:*

$$\frac{dv}{dt} = -kv^2.$$

Show that the velocity and position equations are:

$$v(t) = \frac{v_o}{1+v_o kt} \qquad \textit{and} \qquad x(t) = x_o + \frac{1}{k}\ln(1+v_o kt).$$

where $x(0) = x_o$ and $v(0) = v_o$.

- Separating variables:

$$\int \frac{dv}{v^2} = \int -k\,dt + C.$$

$$\iff -\frac{1}{v} = -kt + C.$$

- Velocity equation:

$$\boxed{v(t) = \frac{1}{kt+C}.}$$

using $v(0)=v_0 \Rightarrow C = \frac{1}{v_0}$.

$$\therefore \boxed{v(t) = \frac{v_0}{1+v_0 kt}}$$

- Position equation:

$$\frac{dx}{dt} = \frac{v_0}{1+v_0 \cdot kt}. \Rightarrow \int dx = \int \frac{v_0}{1+v_0 kt}\,dt + C.$$

$$\iff \boxed{x(t) = \frac{1}{k}\ln(1+v_0 kt) + C}$$

using $x(0) = x_0 \Rightarrow C = x_0$

$$\therefore \boxed{x(t) = x_0 + \frac{1}{k}\ln(1+v_0 kt)}$$

- Q: $\lim_{t\to\infty} v(t), x(t)$?

i) as $t\to\infty \Rightarrow v(t)\to 0$.

ii) as $t\to\infty \Rightarrow x(t)\to\infty$.

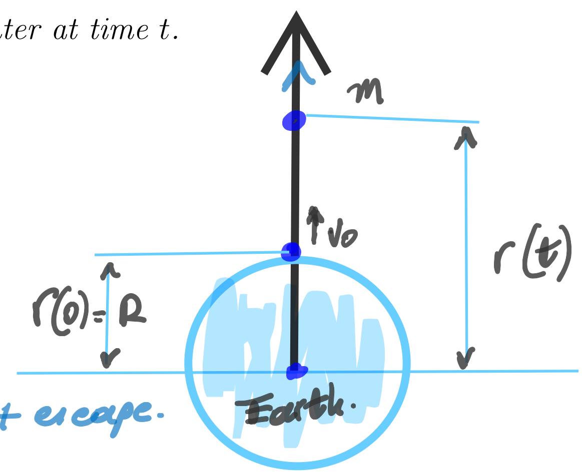
Variable Gravitational Acceleration

- Consider a body (m) in vertical motion.
- Unless the body remains in the immediate vicinity of the earth's surface, the gravitational acceleration acting on the body is *not constant*.
- According to Newton's law of gravitation,
  - the gravitational force between two point masses m and M.
  - located at a distance r is:

$$F_G = \frac{GMm}{r^2}.$$

Example 5. ***Escape velocity.*** *Consider a body with mass m and let*

- $r(t)$: *body's distance from earth's center at time t.*
- $r(0) = R$.: *earth's radius.*
- M: *earth's mass.*



Find $v_0 = v(R)$ *such that $v(t) > 0$ for all t.*

if $v(t)$ becomes 0, object cannot escape.

$$\frac{dv}{dt} = \frac{d^2r}{dt^2} = -\frac{GM}{r^2}$$

$$v = \alpha(t, r)$$

Chain rule:

$$v = \frac{dr}{dt} \; ; \qquad \frac{d^2r}{dt^2} = \frac{dv}{dt} = \frac{dv}{dr} \cdot \underbrace{\frac{dr}{dt}}_{=v.}$$

$$\boxed{v \cdot \frac{dv}{dt} = -\frac{GM}{r^2}.}$$

- Velocity equation:

$$\int v\,dv = -\int \frac{GM}{r^2}\,dr + C.$$

$$\frac{v^2}{2} = +\frac{GM}{r} + C.$$

using $v(0) = v_0$. $\Rightarrow$ $C = \frac{v_0^2}{2} - \frac{GM}{\cancel{r(0)} \to R}$.

$$\boxed{v^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{R}\right)}$$

- Escape velocity $v_o = v(R)$: such that $v(t) > 0 \quad \forall t \geq 0$.

$$v^2 = \underbrace{v_0^2}_{>0} + \underbrace{\frac{2GM}{r}}_{>0} \underbrace{- \frac{2GM}{R}}_{<0}.$$

- $v^2 > v_0^2 - \frac{2GM}{R} \overset{?}{>} 0.$

So, $v(t) > 0$ only if $v_0^2 - \frac{2GM}{R} > 0.$

$$\Rightarrow \quad v_0 > \sqrt{\frac{2GM}{R}}$$

As a result, the escape velocity is:

$$v_0^* = \sqrt{\frac{2GM}{R}}. \quad \blacksquare$$

MA 266 Lecture 12

Christian Moya, Ph.D.

Sec 2.4 Numerical Approximation: Euler's method

Explicit solutions

- It is the exception rather than the rule when a differential equation of the general form

$$\frac{dy}{dx} = f(x, y)$$

  can be solved exactly and explicitly by elementary methods.

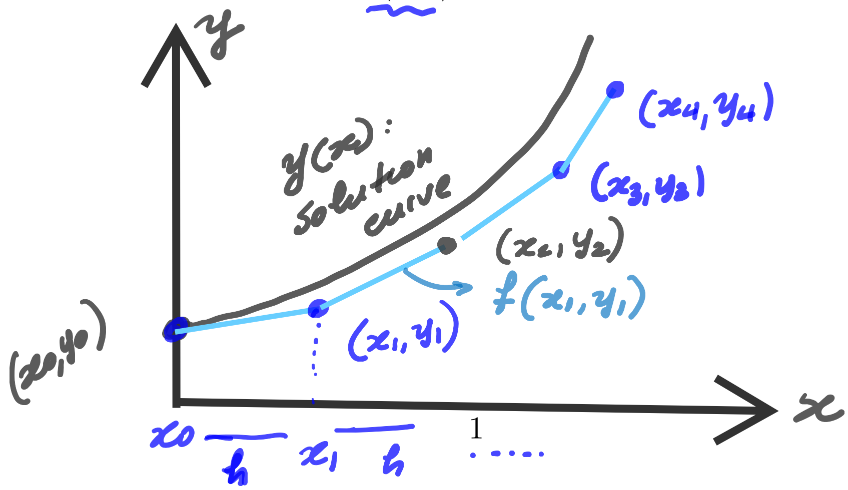
- For example, consider the simple equation

$$\frac{dy}{dx} = e^{-x^2}.$$

- A solution of this equation is just an antiderivative of $f(x) = e^{-x^2}$. However, every antiderivative of this function is known to be a *nonelementary* function—one that cannot be expressed as a finite combination of the familiar functions of elementary calculus.

Alternative approach

- Construct a solution curve that starts (x_0, y_0) and follows the slope field of the given differential equation $y' = f(x, y)$.



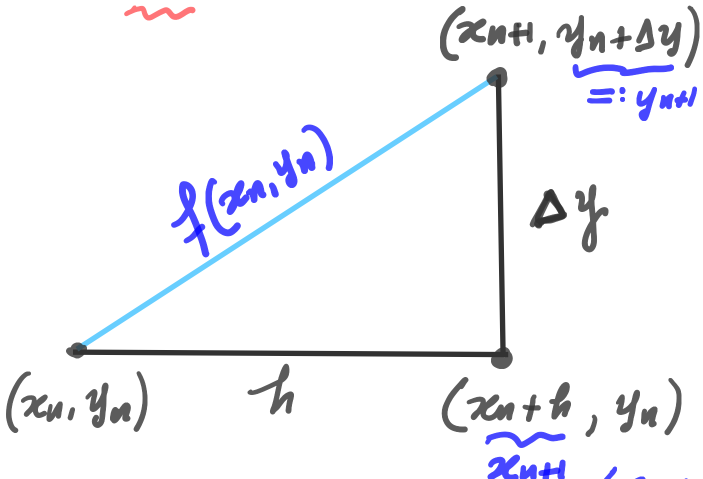
Euler's Method

- To approximate the solution of the initial value problem:

$$\frac{dy}{dx} = f(x,y) \quad ; \quad y(x_0) = y_0.$$

- We first select a fixed (horizontal) *step size* h. to use in making each step from one point to the next.

- Suppose we've started at the initial point (x_0, y_0) and after n steps have reached the point (x_n, y_n). How do we compute the coordinates of the new point (x_{n+1}, y_{n+1})?



$$x_{n+1} := x_n + h.$$

$$\Delta y = ?$$

$$m = f(x_n, y_n) = \frac{\Delta y}{h}.$$

$$\Delta y = h \cdot f(x_n, y_n)$$

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

- Thus, the coordinates of the new point (x_{n+1}, y_{n+1}) are given in terms of the old coordinates by

$$x_{n+1} = x_n + h. \qquad y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

- Given the above initial value problem, *Euler's method* with step size h consists of starting with the initial point (x_0, y_0) and applying the above iterative formulas

$x_1 =$

$$\begin{aligned}
x_1 &= x_0 + h & y_1 &= y_0 + h \cdot f(x_0, y_0) \\
x_2 &= x_1 + h & y_2 &= y_1 + h \cdot f(x_1, y_1) \\
x_3 &= x_2 + h & y_3 &= y_2 + h \cdot f(x_2, y_2) \\
&\vdots & &\vdots \\
&\vdots & &\vdots
\end{aligned}$$

Algorithm: Euler's method

- Given the initial value problem

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0,$$

- *Inputs - Euler's method:* the step size h and the initial condition (x_0, y_0).

- Apply the iterative formula

sequence

$$y_{n+1} = y_n + h \cdot f(x_n, y_n) \quad (n \geq 0)$$

- Results successive approximations:

$$y_1, y_2, \ldots, y_n, \ldots$$

to the [true] values:

$$y(x_1), y(x_2), \ldots, y(x_n), \ldots$$

$$y_i \approx y(x_i)$$

of the [exact] solution $y(x)$ at the points:

$$x_1, x_2, \ldots, x_n, \ldots$$

Example 1. *Apply Euler's method to approximate the solution of the following IVP on the interval* $[0, 1/2]$*:*

$$\frac{dy}{dx} = y + 1, \qquad y(0) = 1,$$

a) first with $h = 0.25$.

b) then with $h = 0.1$.

Note that the particular solution of this IVP is: $y(x) = 2e^x - 1$.

Solution a) With $x_0 = 0$ and $y_0 = 1$, $f(x, y) = y + 1$, and $h = 0.25$ the Euler's iterative formula yields the approximate values at the points $x_1 = 0.25$ and $x_2 = 0.5$:

$x_1 = x_0 + h$
$= 0 + 0.25 = 0.25$

$y_{n+1} = y_n + h \cdot (y_n + 1)$

$$y_1 = y_0 + h \cdot [y_0 + 1] = (1) + (0.25)\,[1 + 1] = 1.5$$

$$y_2 = y_1 + h \cdot [y_1 + 1] = (1.5) + (0.25)\,[1.5 + 1] = 2.125$$

- Note how the result of each calculation feeds into the next one.
- The resulting table of approximate values is

x	0	0.25	0.5
Approx. y	1	1.5	2.125

Solution b) With $x_0 = 0$ and $y_0 = 1$, and $h = 0.1$ the Euler's iterative formula yields the approximate values at the points $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = 0.4$, and $x_5 = 0.5$:

$y_{n+1} = y_n + h \cdot (y_n + 1)$

$$y_1 = y_0 + h \cdot [y_0 + 1] = (1) + (0.1)\,[1 + 1] = 1.2$$

$$y_2 = y_1 + h \cdot [y_1 + 1] = (1.2) + (0.1)\,[1.2 + 1] = 1.42$$

$$y_3 = y_2 + h \cdot [y_2 + 1] = (1.42) + (0.1)\,[1.42 + 1] = 1.662$$

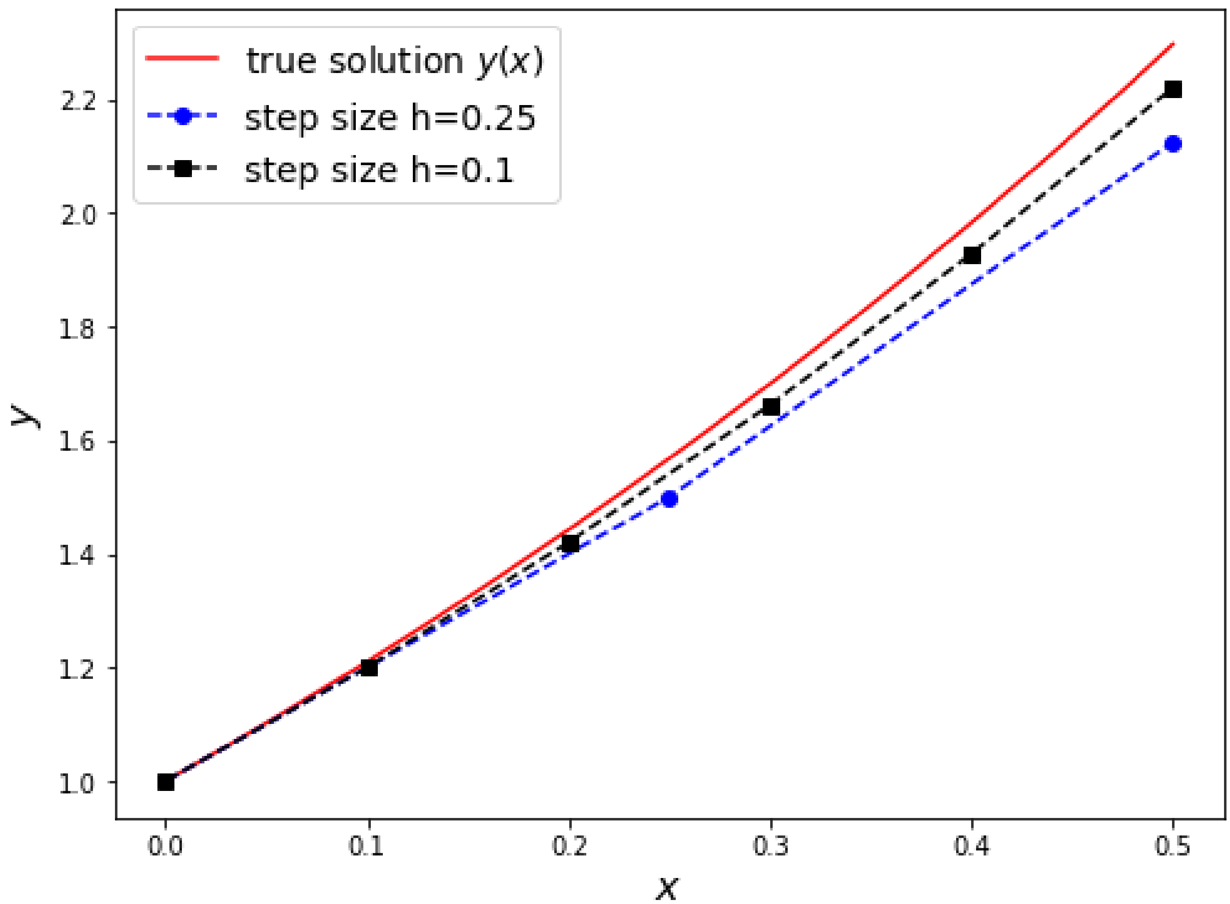
$$y_4 = y_3 + h \cdot [y_3 + 1] = (1.662) + (0.1)\,[1.662 + 1] = 1.9282$$

$$y_5 = y_4 + h \cdot [y_4 + 1] = (1.9282) + (0.1)\,[1.9282 + 1] = 2.221$$

- The resulting table of approximate values is

x	0	0.1	0.2	0.3	0.4	0.5
Approx. y	1	1.2	1.42	1.662	1.9282	2.221

- The next figure shows the graph of the true solution $y(x) = 2e^x - 1$, together with the graphs of the Euler approximations obtained with step sizes $h = 0.25$ and 0.1.



Remarks:

- decrease ↘ the step size h increase ↗ the accuracy.
- decrease ↘ the step size h increase ↗ the number of operations.
  ⟺ decreases efficiency
- Yet with any single approximation, the accuracy decreases with distance from the initial point.

Local and Cumulative Errors

- There are several sources of error in Euler's method that may make the approximation:

$$y_n \approx y(x_n)$$

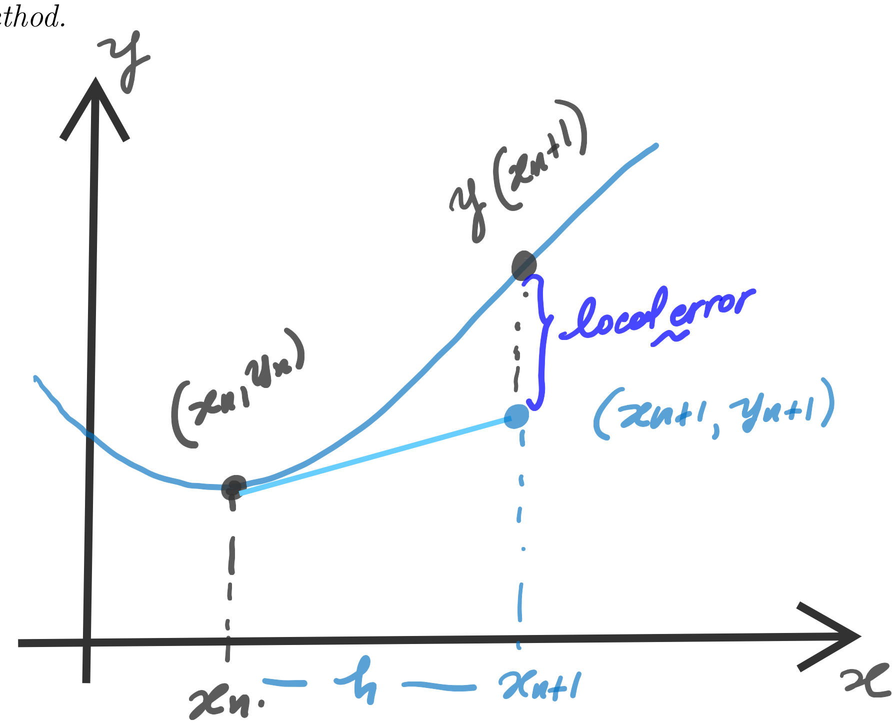
  unreliable for large values of n, those for which x_n is not sufficiently close to x_0.

- The error in the linear approximation formula:

$$y(x_{n+1}) \approx y_n + h \cdot f(x_n, y_n) =: y_{n+1}$$

  is the amount by which the tangent line at (x_n, y_n) departs from the solution curve through (x_n, y_n).

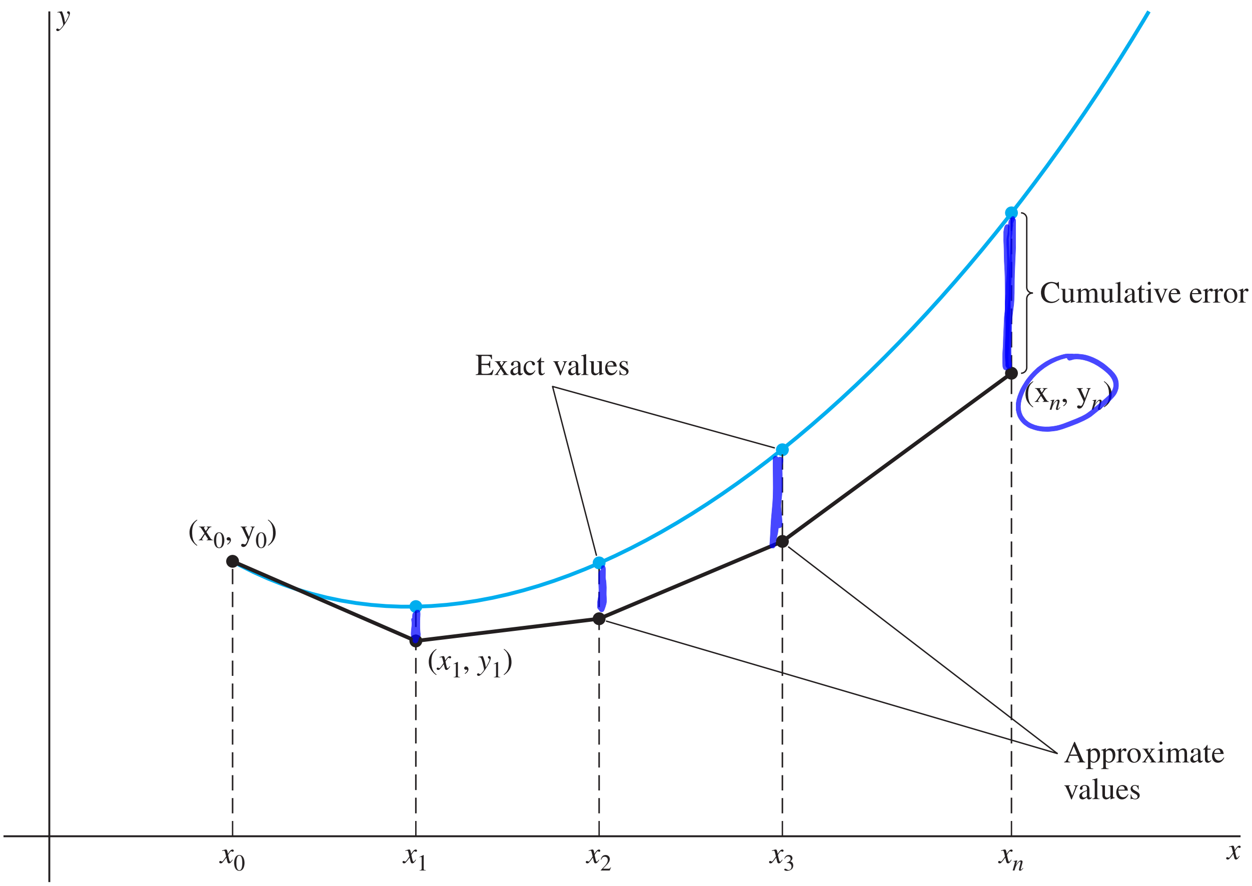
Defintion 1. *The error introduced at each step in the process, is called the* local error. *in Euler's method.*



- Note that y_n itself is merely an approximation to the actual value $y(x_n)$.

$$y_n \approx y(x_n)$$

Defintion 2. *The* ___cummulative error___ *at* y_n *is a measure of all the accumulated effects of all the local errors introduced at the previous steps*



Reducing Cumulative Error

- The usual way of attempting to reduce the cumulative error in Euler's method is to decrease the step size h.
- However, if h is too small, then (i) the number of operations may be too large, (ii) we may have to deal with computer precision/roundoff errors.

Example 2. *Consider the following logistic initial value problem:*

$$\frac{dy}{dx} = y\cos x, \quad y(0) = 1$$

The exact solution of the above equation is the ***periodic*** *function:* $y(x) = e^{\sin x}$. *Use Euler's method to approximate the solution in the interval* $0 \leqq x \leqq 6\pi$ *and using* $n \in \{50, 100, 200, 400\}$ *subintervals.*

- Euler's iterative formula for this examples is:

$$x_{n+1} = x_n + h \quad ; \quad y_{n+1} = y_n + h\cdot(y_n \cdot \cos x_n)$$

$n=50$

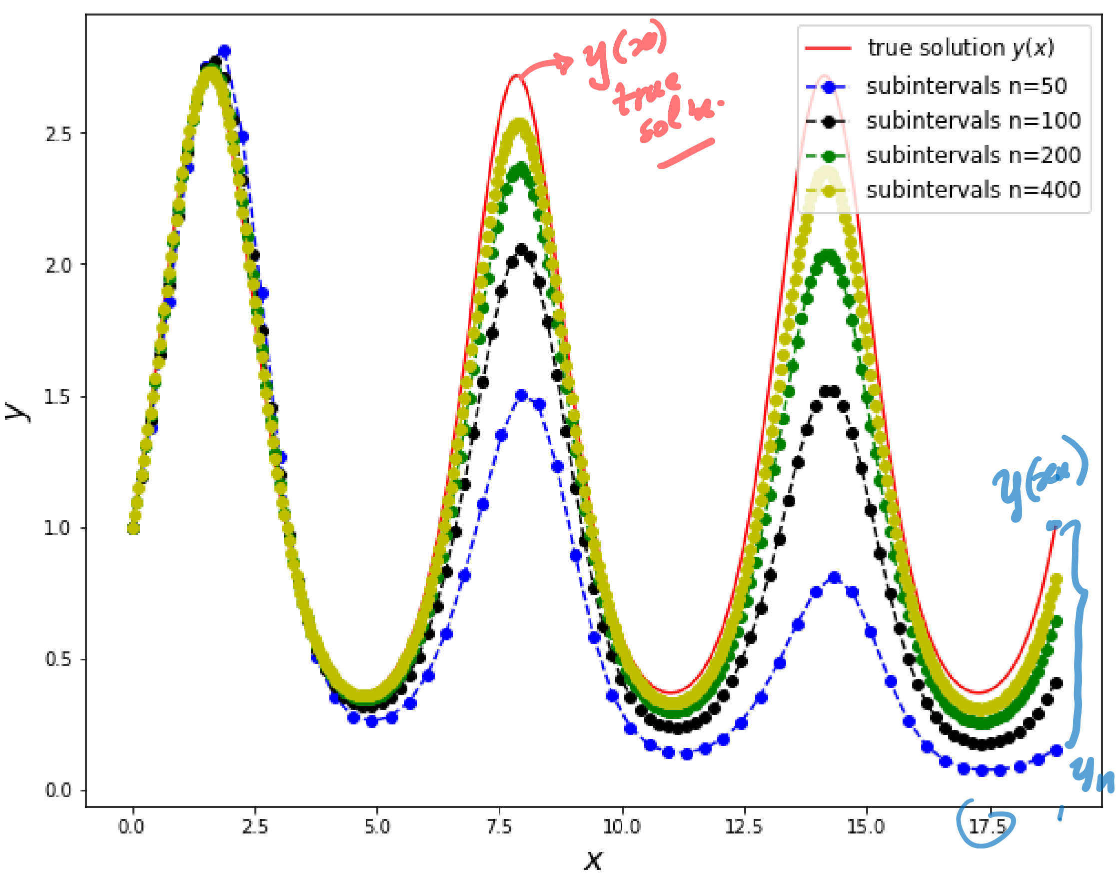
- Computing the step size h from n:

$h.$ $\quad 0 \quad 6\pi$

$$h = \frac{\Delta x}{n} = \frac{6\pi}{n}. \qquad n\nearrow \Rightarrow h\searrow$$

$$h_{50} > h_{100} > \dots > h_{400}.$$

- Next figure shows the exact solution curve and approximate solution curves obtained by applying Euler's method.



A Common Strategy

- The computations in the preceding example illustrate the common strategy of applying a numerical algorithm, such as Euler's method, several times in succession.

  $n=50.$

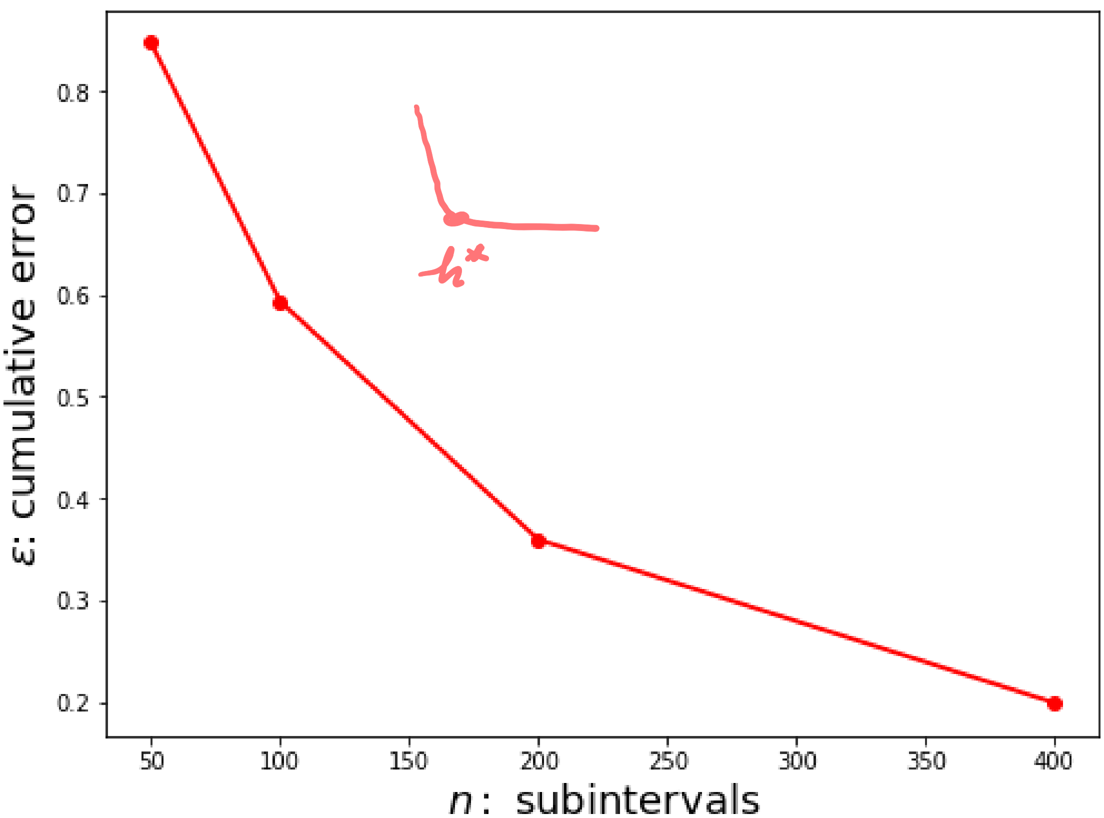
- We begin with a selected number n of subintervals for the first application, then double n for each succeeding application of the method.

  $n=100$, $n=200$, .. $n=400$.

- Visual comparison of successive results often can provide an "intuitive feel" for their accuracy.

Cumulative Error vs. Number of Intervals

- Next figure illustrates a graph comparing the cumulative error ϵ with the number of subintervals n.



A Word of Caution

Example 3. *Use Euler's method to approximate the solution of the initial value problem*

$$\boxed{\frac{dy}{dx} = x^2 + y^2, \quad y(0) = 1}$$

on the interval $[0, 1]$.

- The iterative formula of Euler's method:

  $x_{n+1} = x_n + h \quad ; \quad y_{n+1} = y_n + h\cdot(x_n^2 + y_n^2)$

- With step size $h = 0.1$ we obtain

$$y_1 = 1 + (0.1) \cdot [(0)^2 + (1)^2] = 1.1,$$

$$y_2 = 1.1 + (0.1) \cdot \left[(0.1)^2 + (1.1)^2\right] = 1.222,$$

$$y_3 = 1.222 + (0.1) \cdot \left[(0.2)^2 + (1.222)^2\right] \approx 1.3753,$$

  and so forth.

- Rounded to four decimal places, the first ten values obtained in this manner are

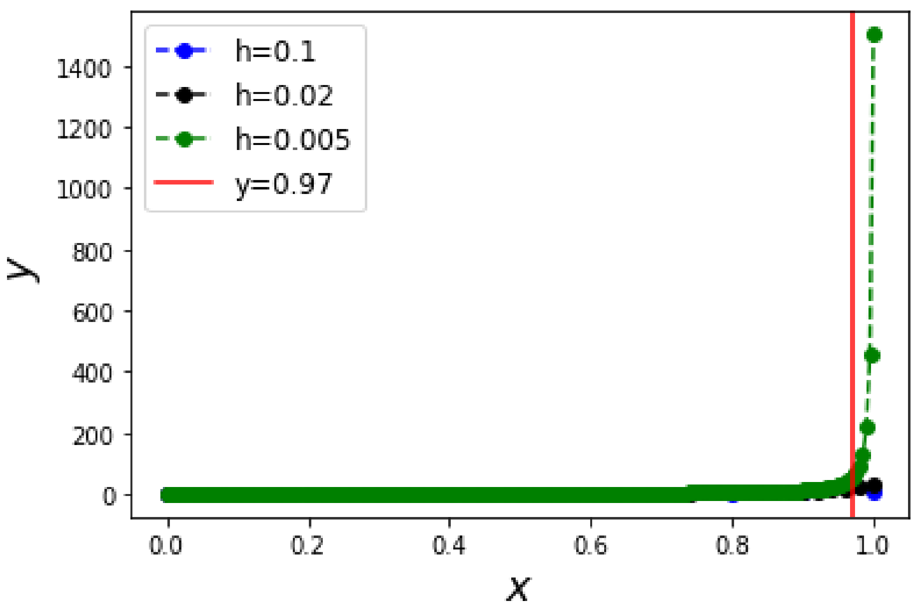
$y_1 = 1.1000$	$y_6 = 2.1995$
$y_2 = 1.2220$	$y_7 = 2.7193$
$y_3 = 1.3753$	$y_8 = 3.5078$
$y_4 = 1.5735$	$y_9 = 4.8023$
$y_5 = 1.8371$	$y_{10} = 7.1895$ $\approx y(x_{10}) = y(1).$

$[0, 1]$

- We could naively accept these results as accurate approxmations.
- We instead can use a computer to repeat the computations with smaller values of h.
- The table on the next page shows the results obtained with step sizes $h = 0.1$, $h = 0.02$, and $h = 0.005$.

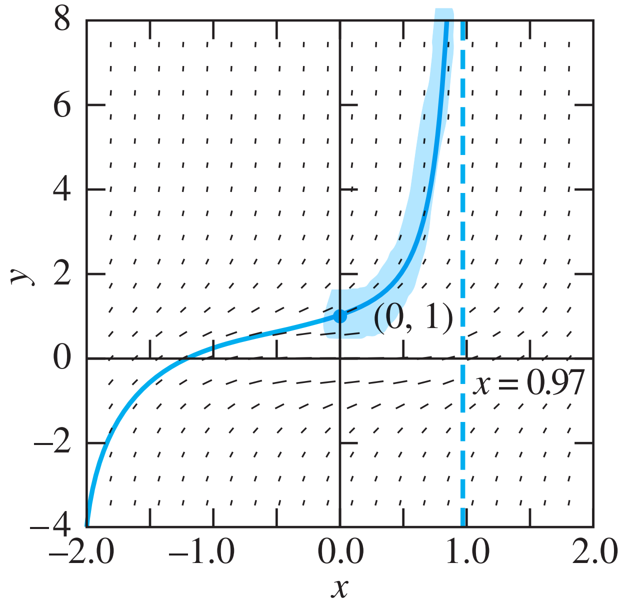
x	y with $h = 0.1$	y with $h = 0.02$	y with $h = 0.005$
0.1	1.1000	1.1088	1.1108
0.2	1.2220	1.2458	1.2512
0.3	1.3753	1.4243	1.4357
0.4	1.5735	1.6658	1.6882
0.5	1.8371	2.0074	2.0512
0.6	2.1995	2.5201	2.6104
0.7	2.7193	3.3612	3.5706
0.8	3.5078	4.9601	5.5763
0.9	4.8023	9.0000	12.2061
1.0	7.1895	30.9167	1502.2090

"unstable behavior"



- Observe that now the "stability" of the numerical procedure is missing.
- Indeed, it seems obvious that something is going wrong near $x = 1$.

- Next figure provides a clue to the difficulty for approximating the solution.



$x \to 0.97$

$y(x) \to +\infty$.

- It appears that this solution curve may have a vertical asymptote near $x = 0.97$.

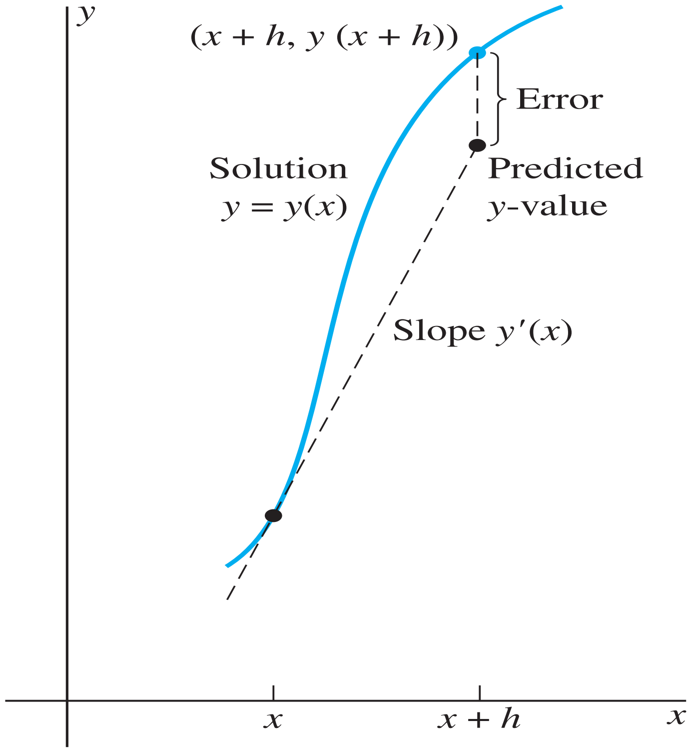
- Indeed, an exact solution using Bessel functions can be used to show that $y(x) \to +\infty$ as $x \to 0.969811$ (approximately).

- Although Euler's method gives values (albeit spurious ones) at $x = 1$, the actual solution does not exist on the entire interval $[0, 1]$.

- Moreover, Euler's method is unable to "keep up" with the rapid changes in $y(x)$ that occur as x approaches the infinite discontinuity near 0.969811.

"stiff differential eq's"

- As the figure shows, Euler's method is rather unsymmetrical.



- It uses the predicted slope $k = f(x_n, y_n)$ of the graph of the solution at the left-hand endpoint of the interval $[x_n, x_n + h]$ as if it were the actual slope of the solution over that entire interval.
- To increase the accuracy of our approximation, we can use the *improved Euler Method*.

Improved Euler Method (2 estimates of the slope)

- Given the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0,$$

  suppose that after carrying out n steps with step size h we have computed the approximation y_n to the actual value $y(x_n)$ of the solution at $x_n = x_0 + nh$.

- We can use the Euler method to obtain a first estimate—which we now call u_{n+1} rather than y_{n+1}—of the value of the solution at $x_{n+1} = x_n + h$:

$$u_{n+1} = y_n + h \cdot f(x_n, y_n) \approx y(x_{n+1})$$

- Now that $u_{n+1} \approx y(x_{n+1})$ has been computed, we can take

$$k_2 = f(x_{n+1}, u_{n+1})$$

  as a second estimate of the slope of the solution curve $y = y(x)$ at $x = x_{n+1}$.

- Note that, the approximate slope

$$k_1 = f(x_n, y_n)$$

  has already been calculated.

- Why not *average* these two slopes to obtain a more accurate estimate of the average slope of the solution curve over the entire subinterval $[x_n, x_{n+1}]$?

$$y_{n+1} = y_n + h \cdot \hat{f} \quad \therefore \quad y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$$

- This idea is the essence of the *improved* Euler method.

- The algorithm for this method is presented next.

Algorithm: The Improved Euler Method

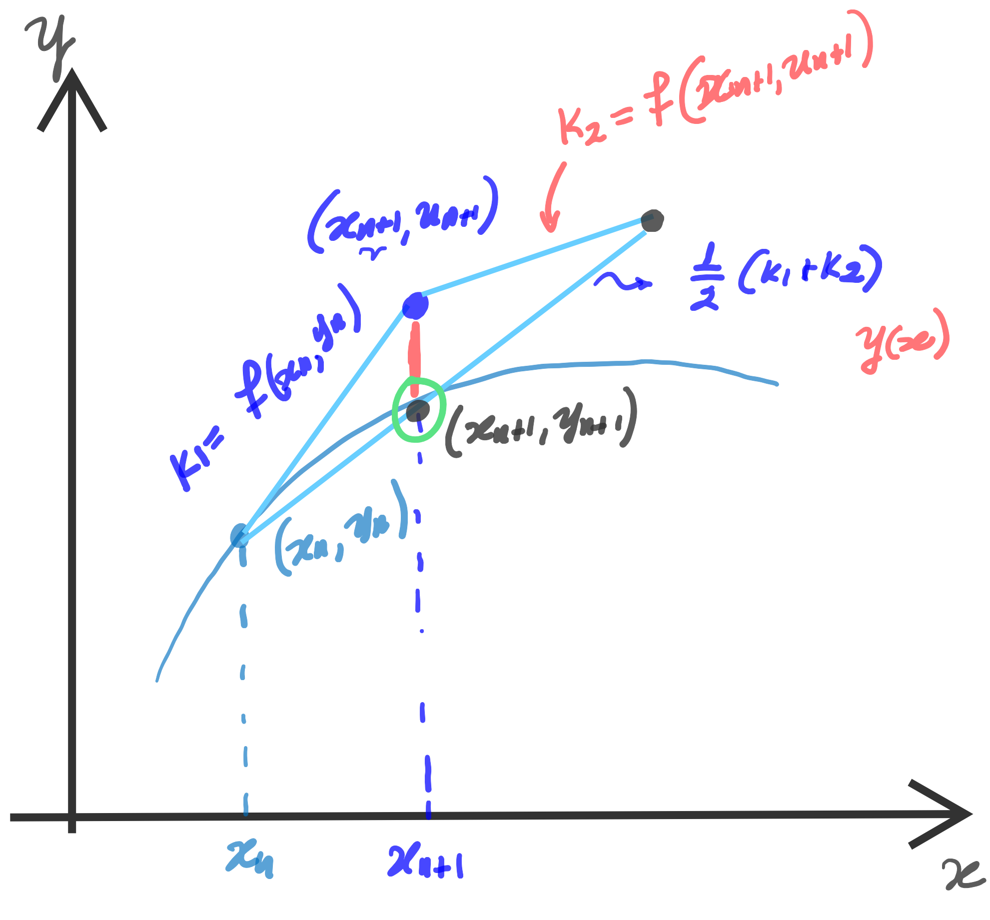
- Given the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0,$$

  the *improved Euler method with step size h* consists in applying the iterative formulas:

$$\begin{aligned} k_1 &= f(x_n, y_n), \\ u_{n+1} &= y_n + h \cdot k_1, \\ k_2 &= f(x_{n+1}, u_{n+1}), \\ y_{n+1} &= y_n + h \cdot \frac{1}{2}(k_1 + k_2). \end{aligned}$$

- These formulas compute successive approximations y_1, y_2, $y_3, \ldots$ to the [true] values $y(x_1)$, $y(x_2)$, $y(x_3), \ldots$ of the [exact] solution $y = y(x)$ at the points $x_1, x_2, x_3, \ldots$, respectively.



Example 4. *Consider the following logistic initial value problem:*

$$\frac{dy}{dx} = y\cos x, \quad y(0) = 1$$

The exact solution of the above equation is the ***periodic*** *function:* $y(x) = e^{\sin x}$. *Use the improved Euler's method to approximate the solution in the interval* $0 \leqq x \leqq 6\pi$ *and using* $n \in \{50, 100, 200, 400\}$ *subintervals.*

- Improved Euler's iterative formula for this example is:

$$k_1 = y_n \cdot \cos x_n.$$

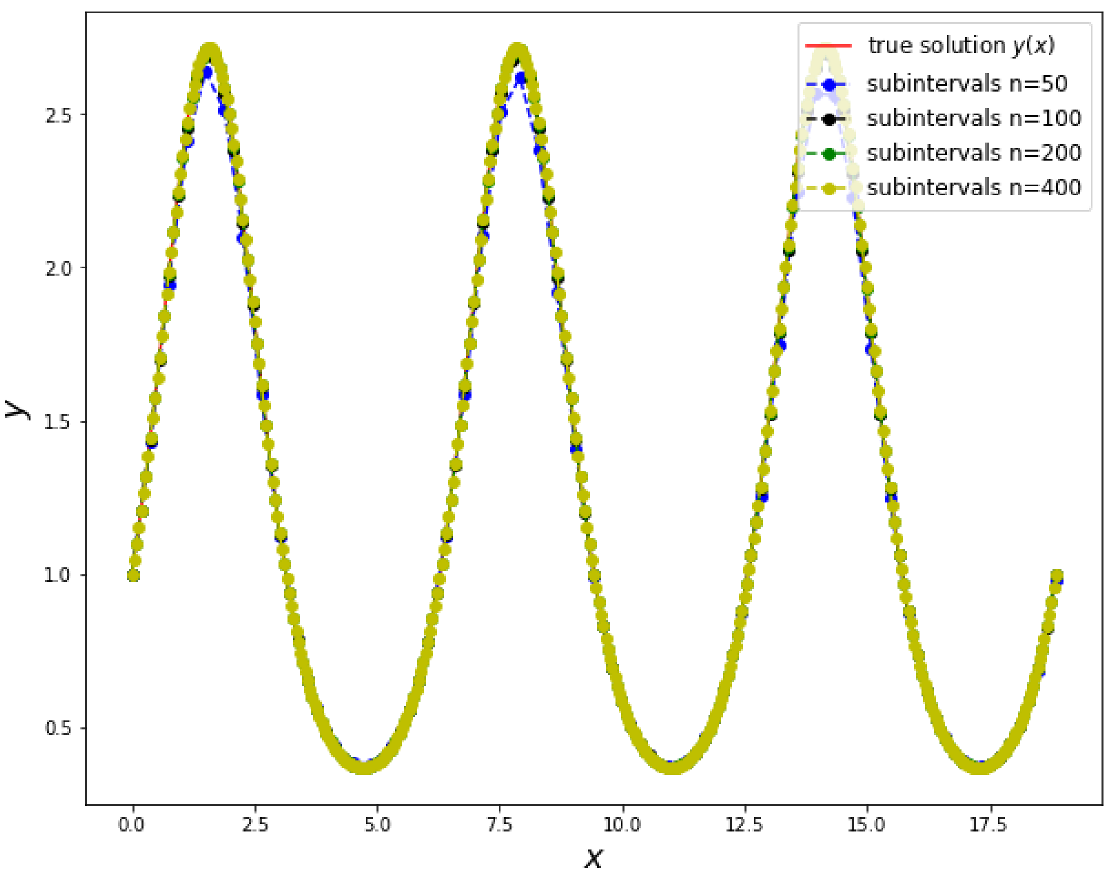
$$u_{n+1} = y_n + h \cdot k_1$$

$$k_2 = u_{n+1} \cdot \cos x_{n+1}.$$

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$$

predictor
corrector

- Next figure shows the exact solution curve and approximate solution curves obtained by applying the Improved Euler's method.



Cumulative Error vs. Number of Intervals

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