MA 266 Lecture 13

Christian Moya, Ph.D.

Sec 3.1 Second Order Linear Equations

- Recall that a second-order differential equation in the (unknown) function $y(x)$ is:

$$G(x, y, y', y'') = 0.$$

- This differential equation is said to be *linear* provided that G is linear in the dependent variable y and its derivatives y' and y''.
- Thus a linear second-order equation takes the form:

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

- We assume that the (known) coefficient functions $A(x)$, $B(x)$, $C(x)$, and $F(x)$ are continuous on some open interval I. (x-axis)

Defintion 1. *A* homogeneous *linear equation takes the form:*

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

that is $F(x) = 0 \ \forall x \in I$.

- If $F(x) \neq 0$ on I, the linear equation is non-homogeneous.

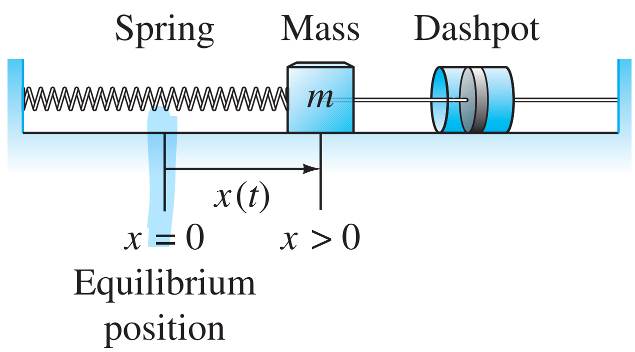
Example 1. *Homogeneous vs. Nonhomogeneous*

a) $x^2y'' + 2xy' + 3y \underbrace{- \cos x}_{F(x)} = 0$ NH.

b) $x^2y'' + 2xy' + 3y = 0$ Homogeneous

- In case the differential equation models a physical system, the nonhomogeneous term $F(x)$ frequently corresponds to some *external* influence on the system.

Example 2. *Model the following mass-spring-dashpot system using linear equations.*



- m. mass attached to a spring and a dashpot (shock absorber).
- F_S. force of the spring on the mass
- F_R. force of the dashpot on the mass
- Assume the restoring force F_S of the spring is proportional to the *displacement* x of the mass from its equilibrium position:

$$F_S = -kx \quad (k > 0)$$

- Assume the dashpot force F_R is proportional to the *velocity* $v = dx/dt$ of the mass and acts opposite to the direction of motion:

$$F_R = -c\frac{dx}{dt} \quad (c > 0)$$

- Newton's law $F = ma$ gives

$$m\frac{d^2x}{dt^2} = F = F_S + F_R.$$

- The *homogeneous* linear equation is then:

$$\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.}$$

$F(t) = 0$

- If, in addition to F_S and F_R, the mass m is acted on by an external force $F(t)$, the resulting equation is

$$\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)}$$

Homogeneous Second-Order Linear Equations

- Consider the general second-order linear equation

$$A(x)y'' + B(x)y' + C(x)y = F(x),$$

- If $A(x) \neq 0$ on I., we can write the above equation in the form:

$$y'' + p(x)\,y' + q(x)y = f(x)$$

- The corresponding *homogeneous* equation:

$$\boxed{y'' + p\,y' + q\,y = 0.} \quad (1)$$

Theorem 1 Principle of Superposition for Homogeneous Equations

- Let y_1 and y_2 be two solutions of the homogeneous linear equation (1)
- If c_1 and c_2 are constants, then the linear combination

$$y = c_1 \cdot y_1 + c_2 \cdot y_2.$$

is also a solution of this equation on I. of (1)

Why the Theorem 1 is true?

- Note that the linearity of differentiation gives

$$y' = c_1 y_1' + c_2 y_2' \qquad y'' = c_1 y_1'' + c_2 y_2''$$

- Then because y_1 and y_2 are solutions,

$$y'' + p y' + q y = (c_1 y_1'' + c_2 y_2'') + p\,(c_1 y_1' + c_2 y_2') + q\cdot(c_1 y_1 + c_2 y_2)$$

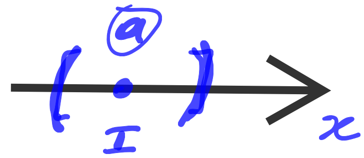
$$= c_1 \underbrace{(y_1'' + p y_1' + q y_1)}_{=0} + c_2 \underbrace{(y_2'' + p y_2' + q y_2)}_{=0}$$

- Thus $= 0$

$y = c_1 y_1 + c_2 y_2$ is a sol'n of (1) $\blacksquare$

Theorem 2 Existence and Uniqueness for Linear Equations

- Suppose that the functions p, q, and f are continuous on the open interval I containing the point a.
- Then, given any two numbers b_0 and b_1, the equation



$$y'' + p(x)y' + q(x)y = f(x)$$

IVP

has a unique (that is, one and only one) solution on the entire interval I that satisfies the initial conditions

$y(a) = b_0$ and $y'(a) = b_1$.

Remark

- The differential equation and the initial conditions in the theorem constitute a second-order linear *initial value problem.*
- Theorem 2 tells us that any such initial value problem has a unique solution on the *whole* interval I where the coefficient functions in the equation are continuous.

Example 3. *Consider the following homogeneous second-order linear equation:*

$$x^2y'' - 2xy' + 2y = 0. \Leftrightarrow y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0.$$

Let $y_1 = x$ *and* $y_2 = x^2$. *Find a solution of the form* $y = c_1y_1 + c_2y_2$ *that satisfies the following initial conditions:*

$a =$

$$y(1) = 3 \text{ and } y'(1) = 1.$$

- Where the *unique* solution exists? $I = \{x : x > 0\}$.

Find the largest I? I contains 1.

- Using the given initial conditions:

$y = c_1 y_1 + c_2 y_2 = c_1 x + c_2 x^2$.

$y(1) = 3$

$3 = c_1 \cdot 1 + c_2 \cdot 1$

$\Rightarrow \boxed{c_1 + c_2 = 3}$

$y'(1) = 1$

$y' = c_1 y_1' + c_2 y_2' = c_1 + 2c_2 x$.

$\boxed{1 = c_1 + 2c_2}$

$\begin{cases} c_1 = 5 \\ c_2 = -2 \end{cases}$ $y = 5x - 2x^2$.

$\forall x \in I := \{x : x > 0\}$.

Ensuring That the Equations Have a Solution

- In order for the procedure of the previous example to succeed, the two solutions y_1 and y_2 must have the property that the equations

$$\begin{cases} c_1 y_1(a) + c_2 y_2(a) = y(a) = b_0 \\ c_2 y_1'(a) + c_2 y_2'(a) = y'(a) = b_1 \end{cases}$$

  can always be solved for c_1 and c_2, no matter what the initial conditions b_0 and b_1 might be.

Defintion 2. *Two functions defined on an open interval I are said to be* linear independ. *on I provided that neither is a constant multiple of the other.*

Linear Dependence

- Two functions are said to be linear dep. on an open interval provided one of them is a constant multiple of the other.

$$f = \underset{\text{constant}}{c}\, g$$

- We can determine whether two given functions f and g are linearly dependent on an interval I by noting whether either of the quotients f/g or g/f is a constant-valued function on I.

$$f/g = h(x) \Rightarrow \text{linear ind.}$$

Example 4. *Determine if the following pair of functions are independent.*

a) $\sin x$ *and* $\cos x$; $f/g = \sin x/\cos x = \tan x$ "ind."

b) e^x *and* e^{-2x}; $f/g = e^{3x}$ "ind."

c) $\sin 2x$ *and* $\sin x \cos x$.

$$\frac{f}{g} = \frac{\sin 2x}{\sin x \cos x} = \frac{2\sin x \cdot \cancel{\cos x}}{\sin \cancel{x \cos x}} = 2.$$

"dependent"

General Solution

- We want to show, finally, that given *any* two linearly independent solutions y_1 and y_2 of the homogeneous equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

every solution y of the equation

$$y'' + py' + qy = 0$$

can be expressed as a linear combination

$$\boxed{y = c_1y_1 + c_2y_2}$$

of y_1 and y_2.

The Wronksian

- As suggested by the equations

$$c_1y_1(a) + c_2y_2(a) = b_0,$$
$$c_1y_1'(a) + c_2y_2'(a) = b_1,$$

the determination of the constants c_1 and c_2 in depends on a certain 2×2 determinant of values of y_1, y_2, and their derivatives.

- Given two functions f and g, the Wronksian. of f and g is:

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = f\cdot g' - g\cdot f'$$

Example 5. *Compute the Wroskian of $f(x) = \cos x$ and $g(x) = \sin x$.*

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1. \neq 0$$

f, g are linearly ind.

- The Wroskian of two linearly *dependent* functions is zero:

$$f = c\,g.$$

$$W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \cancel{cg.g'} - \cancel{c.g.g'} = 0.$$

Theorem 3 General Solutions of Homogeneous Equations

- Let y_1 and y_2 be two linearly independent solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

with p and q continuous on the open interval I.

- If Y is any solution of this equation on I, then there exist numbers c_1 and c_2 such that

$$\boxed{Y(x) = c_1 y_1(x) + c_2 y_2(x)}$$

for all x in I.

Linear Second-Order Equations with Constant Coefficients

- Consider the homogeneous second-order linear differential equation

$$\boxed{ay'' + by' + cy = 0.} \quad (2)$$

with *constant* coefficients a, b, c.

- Consider the *ansatz*: $y(x) = e^{rx}$.
- By noting:

$$y'(x) = re^{rx} \qquad y''(x) = r^2 e^{rx}.$$

- We conclude that $y(x) = e^{rx}$ satisfy

$$a\underbrace{r^2e^{rx}}_{=y''} + b\underbrace{re^{rx}}_{=y'} + c\underbrace{e^{rx}}_{y} = 0.$$

$$\Longleftrightarrow \quad \underbrace{(ar^2+br+c)}_{=0}\underbrace{e^{rx}}_{>0} = 0.$$

- when r is a *root* of the *algebraic* equation:

$$\boxed{ar^2 + br + c = 0}$$

- This quadratic equation is called the Characteristic eq'n of the homogeneous linear differential equation. (2)

Characteristic Roots

- If the algebraic equation

$$ar^2 + br + c = 0$$

has two distinct (real) roots r_1 and r_2, then the corresponding solutions:

$y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$.

if, $r_1 \neq r_2 \Rightarrow \frac{f}{g} = e^{(r_1 - r_2)x}$

are *linearly independent.*

- Theorem 3 then implies that

$$\boxed{y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.}$$

- is a *general* solution of (2).
- This leads to the following theorem.

Theorem 4 Distinct Real Roots

- If the roots r_1 and r_2 of the characteristic equation

$$ar^2 + br + c = 0$$

are real and distinct, then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is a general solution of

$$ay'' + by' + cy = 0.$$

Equal Roots

- If the characteristic equation

$$ar^2 + br + c = 0$$

has equal roots $r_1 = r_2$, we get (at first) only the single solution $y_1(x) = e^{r_1 x}$. of the differential equation

$$ay'' + by' + cy = 0.$$

- The problem in this case is to produce the "missing" second solution of the differential equation.

- A double root $r = r_1$ will occur precisely when the characteristic equation is a constant multiple of the equation

$$(r - r_1)^2 = r^2 - 2r_1 r + r_1^2 = 0.$$

- Any differential equation with this characteristic equation is equivalent to

$$\boxed{y'' - 2r_1 y' + r_1^2 y = 0} \quad (3)$$

- But it is easy to verify that $y_2(x) = xe^{r_1 x}$ is a second solution of (3).

- Moreover, it is easy to check that

$$y_1(x) = e^{r_1 x} \quad \text{and} \quad y_2(x) = xe^{r_1 x}$$

are linearly independent functions, so by Theorem 3, the general solution of the differential equation

$$y'' - 2r_1 y' + r_1^2 y = 0$$

is

$$\boxed{y(x) = c_1 e^{r_1 x} + c_2 \cdot x \cdot e^{r_1 x}}$$

Theorem 5 Repeated roots

- If the characteristic equation

$$ar^2 + br + c = 0$$

has equal (necessarily real) roots $r_1 = r_2$, then

$$y(x) = (c_1 + c_2 x)e^{r_1 x}$$

is a general solution of the differential equation

$$ay'' + by' + cy = 0.$$

Example 6. *Find the general solution of the differential equation:*

$$y'' + 2y' - 15y = 0$$

· characteristic eq'n:

$$r^2 + 2r - 15 = 0$$

$$(r+5)(r-3) = 0 \quad r_1 = -5, \quad r_2 = 3.$$

$$\boxed{y(x) = c_1 \cdot e^{-5x} + c_2 \cdot e^{3x}}$$

Example 7. *Find the general solution of the differential equation:*

$$9y'' - 12y' + 4y = 0$$

characteristic eq'n:

$$9r^2 - 12r + 4 = 0.$$

$$\Leftrightarrow \quad r^2 - \frac{4}{3}r + \frac{4}{9} = 0.$$

$$\left(r - \frac{2}{3}\right)^2 = 0. \qquad r_1 = \frac{2}{3}.$$

$$\boxed{y(x) = c_1 \cdot e^{\frac{2}{3}x} + c_2 \cdot x \cdot e^{\frac{2}{3}x}.}$$

MA 266 Lecture 14

Christian Moya, Ph.D.

Sec 3.2 General Solutions of Linear Equations

Review from last class:

- Consider the *homogeneous* ODE with constant coefficients ($a, b, c \in \mathbb{R}$):

$$ay'' + by' + cy = 0$$

- Look for a solution of the form: $y(x) = e^{rx}$. Then, we find that $(ar^2 + br + c)\underbrace{e^{rx}}_{>0} = 0$ results:

$$\boxed{ar^2 + br + c = 0}$$

- The above equation is called *characteristic equation* of the differential equation.

- By solving the characteristic equation, we find r (three possibilities):
  1) – roots $r_1 \neq r_2$ are real.
  2) – roots $r_1 = r_2$ is real.
  3) – roots r_1, r_2 are complex.

- (First case) Consider *distinct* real roots $r_1 \neq r_2$. (Theorem 3) The general solution of the homogeneous ODE is:

$$\boxed{y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}}$$

$y_1(x) = e^{r_1 x}$

$y_2(x) = e^{r_2 x}$

- Consider the *homogeneous* ODE with constant coefficients ($a, b, c \in \mathbb{R}$):

$$ay'' + by' + cy = 0$$

- (Second case) Consider *repeated* or equal real root $r_1 = r_2$. Here, we only one have solution

$$\underset{\sim}{y_1}(x) = e^{r_1 x}$$

$$\frac{f}{g} = \frac{e^{r_1 x}}{x e^{r_1 x}} = \frac{1}{x} \neq c.$$

- The problem is to produce the "missing" second solution.

- Note that the equal root $r = r_1$ occurs when the *characteristic equation* is a constant multiple of:

$$\boxed{(r - r_1)^2 = r^2 - 2r_1 r + r_1^2}$$

- Any differential equation with the above characteristic equation is equivalent to:

$$y'' - 2r_1 y' + r_1^2 \, y = 0 \tag{1}$$

- However, it is easy to verify that $y(x) = xe^{r_1 x}$ is a second (*linearly independent*) solution of (1).

- Thus, by Theorem 3, the general solution of (1) is: $\boxed{y(x) = C_1 e^{r_1 x} + C_2 x e^{r_1 x}}$

Example 1. *Find the general solution of the differential equation:*

$$9y'' - 12y' + 4y = 0$$

- Characteristic equation:

$$9r^2 - 12r + 4 = 0$$

$$\iff r^2 - \frac{4}{3} r + \frac{4}{9} = 0$$

- Solution:

$$\iff \left(r - \frac{2}{3}\right)^2 = 0 \implies r_1 = \frac{2}{3}.$$

$$\therefore \boxed{y(x) = C_1 e^{\frac{2}{3}x} + C_2 x e^{\frac{2}{3}x}}$$

Example 2. *Let* $y(x) = c_1 + c_2 e^{-10x}$ *be a general solution of a homogeneous second-order differential equation of the form*

$$ay'' + by' + c = 0,$$

with constant coefficients. Find such coefficients.

- Roots:

$$y(x) = c_1 + c_2 e^{-10x} = c_1 e^{0x} + c_2 e^{-10x}$$

$$r_1 = 0 \qquad r_2 = -10.$$

- Characteristic equation:

$$(r-0)\cdot(r+10) = 0$$

$$r^2 + 10r = 0.$$

- Homogeneous equation:

$$\boxed{y'' + 10y' = 0} \Rightarrow \begin{cases} a = 1 \\ b = 10 \\ c = 0. \end{cases}$$

General Linear Equations

- Consider the *nth-order linear* differential equation:

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \dots + P_{n-1}(x)y' + P_n(x)y = F(x)$$

- We assume $P_i(x)$ and $F(x)$ are continuous on some open interval I.

- If $P_0(x) \neq 0$ on I, we obtain:

$p_1(x) = \dfrac{P_1(x)}{P_0(x)}$

$$y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}(x)y' + p_n(x)y = f(x)$$

- The *homogeneous linear equation* associated with this differential equation is:

$$y^{(n)} + p_1 y^{(n-1)} + \dots + p_{n-1}y' + p_n y = 0.$$

$p_i \overset{\text{not}}{=} p_i(x)$

> **Theorem (Principle of Superposition for Homogeneous Equations)** If $y_1, y_2, \dots, y_n$ are n solutions of the linear equation on the interval I. If $c_1, c_2, \dots, c_n$ are constants, then the linear combination
>
> $$y = c_1y_1 + c_2y_2 + \dots + c_ny_n.$$
>
> is also a solution on I.

> **Theorem (Existence and Uniqueness of Linear equations)** Suppose that the functions $p_1, p_2, \ldots, p_n$, and f are continuous on the open interval I containing the point a. Then, given n numbers $b_0, b_1, \ldots, b_{n-1}$, the nth-order linear equation
>
> $$\boxed{y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)}$$
>
> has a unique (that is, one and only one) solution on the entire interval I that satisfies the n initial conditions
>
> $$y(a) = b_0, \quad y'(a) = b_1, \quad \ldots, \quad y^{(n-1)}(a) = b_{n-1}.$$

Example 3. *Without solving the ODE, find the existence and uniqueness interval I of the solution of the IVP:*

$a=1$

$$x(x-3)y'' + 2xy' - (x+1) = 0, \qquad y(1) = 1, \quad y'(1) = 2.$$

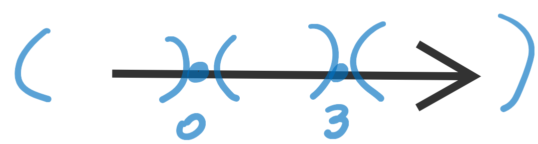
- Rewrite it in standard form: I must have $a=1$.

$$y'' + \frac{2}{(x-3)}y' - \frac{(x+1)}{x(x-3)} = 0.$$

- Use Theorem:

$x(x-3) \neq 0 \Rightarrow x \neq 0, 3.$

Since $a=1 \Rightarrow \boxed{I = (0,3)}$



Linear Independent Solutions

- Based on our knowledge of general solutions of second-order linear equations, we would expect that a general solution of the *homogeneous* nth-order linear equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$$

  will be a linear combination

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n,$$

  where $y_1, y_2, \ldots, y_n$ are particular solutions of

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0.$$

- However these n particular solutions must be “sufficiently independent” that we can always choose the coefficients $c_1, c_2, \ldots, c_n$ to satisfy arbitrary initial conditions of the form $y(a) = b_0$, $y'(a) = b_1$, $\ldots$, $y^{(n-1)}(a) = b_{n-1}$.

Linear Dependence of Two Functions

f_1, f_2

- Recall that *two* functions f_1 and f_2 are linearly *dependent* if one is a constant multiple of the other. That is, if either $f_1 = kf_2$ or $f_2 = kf_1$ for some constant k.

  $f_1 = k f_2$ or $f_2 = k f_1$

- If we write these equations as

  $(1)\cdot f_1 + (-k) f_2 = 0$ or $(k)\cdot f_1 + (1) f_2 = 0$

  we see that the linear dependence of f_1 and f_2 implies that there exist two constants c_1 and c_2 *not both zero* such that

  $$c_1 f_1 + c_2 f_2 = 0.$$

- By analogy, we say that n functions $f_1, f_2, \ldots, f_n$ are linearly dependent provided that some *nontrivial* linear combination of them vanishes identically.

  $$c_1 f_1 + c_2 f_2 + \ldots + c_n f_n = 0.$$

- *Nontrivial* means that *not all* of the coefficients $c_1, c_2, \ldots, c_n$ are zero (although some of them may be zero).

Defintion 1. *(Linear Dependence of Functions) The n functions $f_1, f_2, \ldots, f_n$ are said to be* ***linearly dependent*** *on the interval I provided that there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that*

$$\boxed{c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0}$$

on I, that is,

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all x in I.

Remarks:

- If not all the coefficients in

  $$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

  are zero, then clearly we can solve for at least one of the functions as a linear combination of the others, and conversely.

- Thus the functions $f_1, f_2, \ldots, f_n$ are linearly dependent if and only if at least one of them is a linear combination of the others.

for some $i \leq n$ $c_i \neq 0$. $f_i = \frac{-1}{c_i}\left(\sum_{j\neq i} c_j f_j\right)$

Example 4. *Show that the functions $f(x) = 0$, $g(x) = \sin(x)$ and $h(x) = e^x$ are linearly dependent on $\mathbb{R}$.*

find c_1, c_2, c_3 (not all zero):

$$c_1(0) + c_2 \sin(x) + c_3 \cdot e^x = 0.$$

Clearly $c_1 = k \neq 0$ and $c_2 = c_3 = 0$.

Defintion 2. *(Linear Independent Functions) The n functions $f_1, f_2, \ldots, f_n$ are called* linearly independent *on the interval I if they are not linearly dependent there. Equivalently, they are linearly independent on I provided that the identity*

$$\boxed{c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0}$$

holds on I only in the trivial case

$$c_1 = c_2 = \ldots = c_n = 0.$$

that is, no *nontrivial linear combination of these functions vanishes on I.*

- To show that n given functions are linearly independent, we use the *Wronksian Determinant.*

The Wronskian Determinant

- Suppose that the n functions $f_1, f_2, \ldots, f_n$ are each $n-1.$ times differentiable.

- Then their *Wronskian* is the $n \times n$ determinant

$$W = \begin{vmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)} \end{vmatrix}$$

- The Wronskian of n linearly dependent $f_1, f_2, \ldots, f_n$ *is identically zero.*

Example 5. *Use the Wronskian to show that the functions $y_1(x) = e^x$, $y_2(x) = \cos(x)$, and $y_3(x) = \sin(x)$ are linearly independent on $\mathbb{R}$.* $W \neq 0.$

$$W = \begin{vmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{vmatrix} = e^x \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} - \cos x \begin{vmatrix} e^x & \cos x \\ e^x & -\sin x \end{vmatrix} + \sin x \begin{vmatrix} e^x & -\sin x \\ e^x & -\cos x \end{vmatrix}$$

$$= 2\underbrace{e^x}_{>0}(\underbrace{\sin^2 x + \cos^2 x}_{=1}) \neq 0.$$

$\therefore f, g, \wedge h$ are lin. ind.

Wronksians of Solutions

- Provided that $W(y_1, y_2, \ldots, y_n) \neq 0$., it turns out (Theorem General Solutions of Homogeneous Equations) that we can always find values of the coefficients in the linear combination

$$\boxed{y = c_1y_1 + c_2y_2 + \cdots + c_ny_n}$$

that satisfy any given initial conditions of the form

$$y(a) = b_0 \quad , \ldots , \quad y^{(n-1)}(a) = b_{n-1}$$

Theorem (Wronksians of Solutions) Suppose that $y_1, y_2, \ldots, y_n$ are n solutions of the homogeneous nth-order linear equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$$

on an open interval I, where each p_i is continuous. Let

$$W := W(y_1, y_2, \ldots, y_n)$$

(a) If $y_1, y_2, \ldots, y_n$ are linearly dependent, then $W = 0$ on I.

(b) If $y_1, y_2, \ldots, y_n$ are linearly independent, then $W \neq 0$. on I.

Capturing All Solutions of a Homogeneous Equation

- Given any fixed set of n linearly independent solutions of a *homogeneous* nth-order equation, *every* (other) solution of the equation can be expressed as a linear combination of those n particular solutions.

Theorem (General Solutions of Homogeneous Equations)

- Let $y_1, y_2, \ldots, y_n$ be n $\boxed{\text{linearly independent}}$ solutions of the homogeneous equation

$\Leftarrow W(y_1, y_2, \ldots, y_n) \neq 0.$

$$(\Delta) \qquad y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$$

on an open interval I where the p_i are continuous.

- If Y is any solution whatsoever of this equation, then there exist numbers c_1, c_2, $\ldots$, c_n such that

$$\boxed{Y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n.}$$

for all x in I. General sol'n of (Δ)

Nonhomogeneous Equations

Example 6. *Solutions of nonhomogeneous equations.*

- Consider the *nonhomogeneous* nth-order linear differential equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)$$

  with associated homogeneous equation

$$\boxed{y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0.}$$

- Suppose that a single fixed particular solution y_p of the above nonhomogeneous equation is known

  • (1) $y_p^{(n)} + \dots + p_{n-1} y_p' + p_n y_p = f(x)$

- Let Y is any other solution of this equation.

  (2) $Y^{(n)} + \dots + p_{n-1} Y' + p_n Y = f(x)$

- Show that if $\boxed{y_c = Y - y_p}$, then y_c is the solution of the associated homogeneous Equation

$$y_c^{(n)} + p_1 y_c^{(n-1)} + \dots + p_{n-1} y_c' + p_n y_c = (Y^{(n)} - y_p^{(n)}) + \dots + p_{n-1}(Y' - y_p') + p_n(Y - y_p)$$

$$= \underbrace{(Y^{(n)} + p_1 Y^{(n-1)} + \dots + p_{n-1} Y' + p_n Y)}_{\text{L.H.S of (2)}} - \underbrace{(y_p^{(n)} + p_1 y_p^{(n-1)} + \dots + p_{n-1} y_p' + p_n y_p)}_{\text{L.H.S of (1)}}$$

$$= f(x) - f(x) = 0.$$

$\therefore$ $y_c = Y - y_p$ is a sol'n of the homog. eq'n.

$$\boxed{y_c = c_1 y_1 + \dots + c_n y_n}$$

- We call y_c a *complementary function* of the nonhomogeneous equation.

Theorem (Solutions Homogeneous Equations)

- Let y_p be a particular solution of the nonhomogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)$$

on an open interval I where the functions p_i and f are continuous.

- Let y_1, y_2, ..., y_n be linearly independent solutions of the associated homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0.$$

- If Y is any solution whatsoever of the equation nonhomogeneous equation on I, then there exist numbers c_1, c_2, ..., c_n such that

$$Y(x) = \underbrace{c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)}_{=:\, y_c} + y_p(x)$$

for all x in I.

Example 7. *We are given (i) the homogeneous IVP:*

$$y'' + y = 3x, \qquad y(0) = 2, \quad y'(0) = -2$$

(ii) the complementary solution: $y_c = C_1\cos(x) + C_2\sin(x)$, *and (iii) the particular solution:* $y_p = 3x$. *Find a solution for the IVP.*

By the theorem.

$$y = y_c + y_p$$

$$\boxed{y = C_1\cos x + C_2\sin x + 3x.}$$

$$y' = y_c' + y_p'$$

$$= -C_1\sin x + C_2\cos x + 3.$$

using the ICs:

$$2 = y(0) = C_1 \Rightarrow C_1 = 2.$$

$$-2 = y'(0) = C_2 + 3 \Rightarrow C_2 = -5$$

$$\boxed{y(x) = \underbrace{2\cos x - 5\sin x}_{y_c} + \underbrace{3x}_{y_p}.}$$

MA 266 Lecture 15

Christian Moya, Ph.D.

Sec 3.3-1 Homogeneous Eqs. Constant Coefficients

Solving nth-Order Equations

- A general solution of an nth-order homogeneous linear equation is a linear combination of n linearly independent particular solutions.
- Q: How to find a single solution?
- The solution of a linear differential equation with variable coefficients ordinarily requires numerical methods or infinite series methods.
- In this lecture, we show how to find linearly ind. sol'n's of a given nth-order linear equation if it has *constant* coefficients.
- Consider the homogeneous equation:

$$(1) \quad \boxed{a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0.}$$

where the coefficients $\underbrace{a_0, a_1, a_2, \dots, a_n}_{\in \mathbb{R}.}$ are real constants with $a_n \neq 0$.

Finding a single solution

- Consider the *ansatz*: $y(x) = e^{rx}.$
- and observe that any derivative is: $\frac{d^k}{dx^k}(e^{rx}) = r^k e^{rx}.$
- Substituting $y(x) = e^{rx}$ in (1) gives:

$$\underbrace{e^{rx}}_{>0}\left(\underbrace{a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0}_{r_i}\right) = 0.$$

$y(x) = e^{r_i x}$

- Because e^{rx} is never zero, we see that $y = e^{rx}$ will be a solution of ___(1)___ precisely when r is a root of the *algebraic equation*:

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0.$$

Defintion 1. *(The Characteristic Equation) The characteristic equation of*

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

is the algebraic equation

$$\boxed{a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0.}$$

- Fundamental theorem of algebra $\implies$ every nth-degree polynomial has n zeros, though not necessarily distinct and not necessarily real.
- Finding the exact values of these zeros may be difficult or even impossible.
- For equations of degree $n > 2$, we may need either to spot a fortuitous factorization or to apply a numerical technique such as Newton's method.

The case of *distinct* roots

- Assume (the simplest case) the characteristic equation has n distinct (no two equal) *real* roots:

  $r_1, r_2, \dots, r_n.$

- Then the functions

  $e^{r_1 x}, e^{r_2}, \dots, e^{r_n x}$

  are all solutions of (1).

- These n solutions are *linearly independent* on the entire real line.

> **Theorem (Distinct Real Roots)** If the roots $r_1, r_2, \dots, r_n$ of the characteristic equation
>
> $$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0$$
>
> are real and distinct, then
>
> $$\boxed{y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}}$$
>
> is a general solution of the differential equation
>
> $$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0.$$

Example 1. *Find the general solution of*

$$2y'' - 7y' + 3y = 0.$$

- Characteristic equation:

$$2r^2 - 7r + 3 = 0.$$

$$\Longleftrightarrow (2r-1)\cdot(r-3) = 0.$$

$$\Longrightarrow r_1 = \frac{1}{2} \text{ and } r_2 = 3.$$

$$\{e^{x/2}, e^{3x}\}. \text{ linearly ind. solu's.}$$

- General solution:

$$\boxed{y(x) = c_1 e^{x/2} + c_2 e^{3x}}$$

Example 2. *Find the general solution of*

$$y'' + 5y' + 5y = 0.$$

- Characteristic equation:

$$r^2 + 5r + 5 = 0.$$

$$r_{1,2} = \frac{-5 \pm \sqrt{25-20}}{2}.$$

$$r_1 = \frac{-5+\sqrt{5}}{2} \text{ and } r_2 = \frac{-5-\sqrt{5}}{2}.$$

$$\{e^{r_1 x}, e^{r_2 x}\}. \text{ linearly indep.}$$

- General solution: $\boxed{y(x) = c_1 e^{\left(\frac{-5+\sqrt{5}}{2}\right)x} + c_2 e^{\left(\frac{-5-\sqrt{5}}{2}\right)x}}$

Example 3. *Solve the initial value problem*

$$2y^{(3)} - 3y'' - 2y' = 0; \qquad y(0) = 1, y'(0) = -1, y''(0) = 3.$$

- Characteristic equation:

$$2r^3 - 3r^2 - 2r = 0.$$

$$\Leftrightarrow r\,(\underbrace{2r^2 - 3r - 2}) = 0$$

$$\Leftrightarrow r\left(r^2 - \tfrac{3}{2}r - 1\right) = 0.$$

$$\Leftrightarrow r\left(r + \tfrac{1}{2}\right)\cdot(r-2) = 0.$$

$$r_1 = 0, \quad r_2 = -\tfrac{1}{2}, \quad r_3 = 2.$$

- General solution:

$$\{e^{0x}, e^{-\frac{x}{2}}, e^{2x}\} \text{ linearly ind.}$$

$$\boxed{y(x) = c_1 + c_2 e^{-x/2} + c_3 e^{2x}}$$

- Particular solution:

$$y'(x) = -\frac{c_2}{2} e^{-x/2} + 2c_3 e^{2x}$$

$$y''(x) = \frac{c_2}{4} e^{-x/2} + 4c_3 e^{2x}$$

$$\left.\begin{array}{l} 1 = y(0) = c_1 + c_2 + c_3. \\ -1 = y'(0) = -\frac{c_2}{2} + 2c_3. \\ 3 = y''(0) = \frac{c_2}{4} + 4c_3 \end{array}\right. \Longrightarrow \left\{\begin{array}{l} c_1 = -\frac{7}{2}. \\ c_2 = 4 \\ c_3 = \frac{1}{2}. \end{array}\right.$$

$$\boxed{y(x) = -\frac{7}{2} + 4e^{-x/2} + \frac{1}{2}e^{2x}}$$

Polynomial Differential Operator

- If the roots of are *not* distinct $\implies$ there are repeated roots
- We cannot produce n linearly independent solutions of the differential equation

$$\boxed{a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0.}$$

  by the method of Theorem (Distinct Real Roots).

- The problem, then, is to produce the missing linearly independent solutions.
- For this purpose, it is convenient to adopt "operator notation" and write

$$L = a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1 \frac{d}{dx} + a_0.$$

  operates on the n-times differentiable function $y(x)$.

$$L(y) \overset{\text{not}}{=} Ly.$$

- The result is the linear combination

$$Ly. = a_n \underbrace{y^{(n)}}_{=: \frac{d^n}{dx^n} y(x)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y$$

  of y and its first n derivatives.

- We also denote by $D := d/dx$ the operation of differentiation with respect to x, so that

$$Dy = y'; \quad D^2 y \equiv D(\underbrace{Dy}_{=: y'}) = y'' \quad D^3 y = y^{(3)}$$

  and so on.

- In terms of D, the operator L may be written

$$L = a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0.$$

- We will find it useful to think of the right-hand side of this equation as a (formal) nth-degree polynomial in the "variable" D.

- $\boxed{\text{It is a } \textit{polynomial differential operator.}}$

Properties of Differential Operators

- A first-degree polynomial operator with leading coefficient 1 has the form $D - a$, where a is a real number.

- It operates on a function $y = y(x)$ to produce

$$(D-a)y = Dy - ay = y' - ay$$

- The important fact about such operators is that any two of them *commute*:

$$(D-a)\cdot(D-b)y = (D-b)\cdot(D-a)y.$$

  for any twice differentiable function $y = y(x)$.

- The proof of this formula is:

$a \in \mathbb{R}$. $b \in \mathbb{R}$

$$\begin{aligned}(D-a)\cdot(D-b)y &= (D-a)(y'-by)\\ &= D(y'-by) - a(y'-by)\\ &= y'' - by' - ay' + aby\\ &= (D-b)\cdot(y'-ay)\\ &= (D-b)\cdot(D-a)y \quad \blacksquare\end{aligned}$$

- We see here also that

$$(D-a)(D-b) = D^2 - (a+b)\cdot D + ab.$$

- Similarly, it can be shown by induction on the number of factors that an operator product of the form

$$(D-a_1)\cdot(D-a_2)\ldots(D-a_n)$$

  expands—by multiplying out and collecting coefficients—in the same way as does an ordinary product of linear factors, with x denoting a real variable.

$$(x-a_1)\cdot(x-a_2)\ldots(x-a_n)$$

The Operator Method and Repeated Real Roots

- Let us now consider the possibility that the characteristic equation

$$\boxed{a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0} \tag{3}$$

  has *repeated* roots.

- For example, suppose that this equation has only two distinct roots, $\boxed{r_0 \text{ of multiplicity}}$ 1 and $\boxed{r_1 \text{ of multiplicity } k = n - 1 > 1.}$

Two distinct real roots

- Then (after dividing by a_n) the characteristic equation can be rewritten in the form

$$(r - r_1)^k \cdot (r - r_0) = 0.$$

- Similarly, the corresponding operator L can be written as the order of the factors

$$L = (D - r_1)^k \cdot (D - r_0) = (D - r_0) \cdot (D - r_1)^k.$$

  making no difference because of the commutativity discussed earlier.

- Two solutions of the differential equation $Ly = 0$ are $y_0 = e^{r_0 x}$; $y_1 = e^{r_1 x}$.

- This is, however, not sufficient.

- We need $k + 1$ linearly independent solutions in order to construct a general solution, because the equation is of order $k + 1$.

- To find the missing $k - 1$ solutions, we note that

$$Ly = (D - r_0) \cdot \left[(D - r_1)^k y \right] = 0.$$

- Consequently, *every* solution of the kth-order equation

$$(D - r_1)^k \cdot y = 0.$$

  will also be a solution of the original equation $Ly = 0$.

- Hence our problem is reduced to that of finding a general solution of this differential equation.

- The fact that $y_1 = e^{r_1 x}$ is one solution of this equation suggests that we try the substitution

$$y(x) = u(x)\, e^{r_1 x}.$$

where $u(x)$ is a function yet to be determined.

- Observe that

$$(D-r_1)\left[u(x)e^{r_1 x}\right] = (Du)e^{r_1 x} + \cancel{u\underbrace{(r_1 e^{r_1 x})}_{D(e^{r_1 x})}} - \cancel{r_1 \cdot (u e^{r_1 x})}$$

- Upon k applications of this fact, it follows that

$$(D-r_1)^k\left[u e^{r_1 x}\right] = (D^k u)\underbrace{e^{r_1 x}}_{\neq 0} = (Du)e^{r_1 x}$$

for any sufficiently differentiable function $u(x)$.

- Hence $y = ue^{r_1 x}$ will be a solution of

$$(D-r_1)^k y = 0.$$

if and only if

$$D^k u(x) = u^{(k)} = 0.$$

- But this is so if and only if

$$\boxed{u(x) = c_1 + c_2 x + \ldots + c_k x^{k-1}}$$

a polynomial of degree at most $k-1$.

- Hence our desired solution of

$$(D-r_1)^k y = 0.$$

is

$$y(x) = u e^{r_1 x} = (c_1 + c_2 x + \ldots + c_k x^{k-1})e^{r_1 x}.$$

- In particular, we see here the additional solutions $xe^{r_1 x}, x^2 e^{r_1 x}, \ldots, x^{k-1}e^{r_1 x}$ of the original differential equation $Ly = 0$.

- The preceding analysis can be carried out with the operator $D - r_1$ replaced with an arbitrary polynomial operator, resulting in the following theorem.

Theorem (Repeated Roots) If the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0$$

has a repeated root r of multiplicity k, then the part of a general solution of the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

corresponding to r is of the form:

$$(c_1 + c_2 x + \cdots + c_k x^{k-1}) e^{rx}.$$

Root of Multiplicity k

- It is easy to verify that the k functions $e^{rx}, xe^{rx}, \ldots, x^{k-1}e^{rx}$ in the expression

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{rx}$$

  are linearly independent on the real line.

- Thus a root of multiplicity k corresponds to k linearly independent solutions of the differential equation.

Example 4. *Find the general solution of*

$$5y^{(4)} + 3y^{(3)} = 0.$$

- Characteristic equation:

$$5r^4 + 3r^3 = 0$$

$$\iff r^3(5r+3) = 0.$$

$$r_2 = 0 \text{ (with multiplicity } k=3)$$

$$r_1 = -\frac{3}{5}$$

- General solution:

$$r_1 \Longrightarrow \{e^{-\frac{3}{5}x}\}.$$

$$r_2 \Rightarrow \{e^{0x}, xe^{0x}, x^2e^{0x}\}$$

$$\boxed{y(x) = c_1 e^{-3/5 x} + c_2 + c_3 x + c_4 x^2}$$

Example 5. *Find the general solution of*

$$y^{(3)} + y'' - y' - y = 0.$$

- Characteristic equation:

$$r^3 + r^2 - r - 1 = 0.$$

$$\iff r(r^2-1) + (r^2-1) = 0.$$

$$(r^2-1)(r+1) = (r-1)(r+1)^2 = 0.$$

$$r_1 = 1 \quad r_2 = -1 \text{ (with } k=2)$$

- General solution:

$$\{e^x\} \quad \{e^{-x}, xe^{-x}\}$$

$$\boxed{y(x) = c_1 e^x + c_2 e^{-x} + c_3 x e^{-x}}$$

MA 266 Lecture 16

Christian Moya, Ph.D.

Sec 3.3-2 Homogeneous Eqs. Constant Coefficients

Theorem (Repeated Roots) If the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0$$

has a repeated root r of multiplicity k, then the part of a general solution of the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

corresponding to r is of the form:

$$\boxed{(c_1 + c_2 x + \ldots + c_k x^{k-1}) e^{rx}}$$

Example 1. *Find a function* $y(x)$ *such that* $y^{(4)}(x) = y^{(3)}(x)$ *for all* x *and* $y(0) = 18$, $y'(0) = 12$, $y''(0) = 13$, *and* $y^{(3)} = 7$.

- Characteristic equation:

$$r^4 - r^3 = 0 \iff r^3(r-1) = 0.$$

$$r_1 = 0 \ (k=3), \quad r_2 = 1 \ (k=1)$$

- General solution: $\{1, x, x^2\}$. $\{e^x\}$.

- Particular solution: $\boxed{y(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^x}$

$$18 = y(0) = c_1 + c_4$$
$$12 = y'(0) = c_2 + c_4$$
$$13 = y''(0) = 2c_3 + c_4$$
$$7 = y^{(3)}(0) = c_4$$

$$\Rightarrow \quad c_1 = 11, \ c_2 = 5, \ c_3 = 3, \ c_4 = 7$$

$$\boxed{y(x) = 11 + 5x + 3x^2 + 7e^x}$$

Complex-Valued Functions and Euler's Formula

Complex roots

- Any complex (nonreal) roots will occur in complex conjugate pairs:

$$r_{1,2} = \boxed{a \pm bi} \in \mathbb{C}, \quad a, b \in \mathbb{R} \qquad i := \sqrt{-1}$$

- This raises the question of what might be meant by an exponential such as

$$e^{(a \pm bi)x}.$$

- Recall from elementary calculus the Taylor series for the exponential function

$t = \alpha + \theta i. \quad \alpha, \theta \in \mathbb{R}.$

$e^{(\alpha+\theta i)} = e^{\alpha} e^{\theta i}$

$$e^{t} := \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$

- If we substitute $t = i\theta.$ in this series and recall that $i^2 = -1$, and so on, we get

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \dots$$

$$= \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right)}_{= \cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right)}_{= \sin\theta}$$

Euler's formula

- Because the two real series in the last line are the Taylor series for $\cos\theta$ and $\sin\theta$, respectively, this implies that

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

- This result is known as *Euler's formula*.

- Because of it, we *define* the exponential function e^z, for $z = x+iy$ an arbitrary complex number, to be

$$z \in \mathbb{C}: \quad z = x+iy \qquad e^z = e^{x+iy} = e^x e^{iy} = \boxed{e^x(\cos y + i\sin y)}$$

Complex-Valued functions

- Thus it appears that complex roots of the characteristic equation will lead to complex-valued solutions of the differential equation.

- A *complex-valued function* F of the real variable x associates with each real number x (in its domain of definition) the complex number

$$F(x) = \underbrace{f(x)}_{\text{real}} + i\,\underbrace{g(x)}_{\text{imaginary.}}$$

- The real-valued functions f and g are called the *real* and *imaginary* parts, respectively, of F.

$$F'(x) = f'(x) + i\,g'(x)$$

Complex Exponentials

- The particular complex-valued functions of interest here are of the form:

$$F(x) = e^{rx}, \quad r = a \pm bi.$$

- We note from Euler's formula that

$$e^{(a+bi)x} = e^{ax}(\cos bx + i \sin bx)$$

and

$$e^{(a-bi)x} = e^{ax}(\cos bx - i \sin bx)$$

- The most important property of e^{rx} is that

$$D_x(e^{rx}) = re^{rx}$$

if r is a complex number.

- The proof is straightforward:

$$D_x(e^{rx}) = D_x(e^{(a+bi)x}) = D_x(\underbrace{e^{ax}\cos bx}) + iD_x(e^{ax}\sin bx)$$

$$* \quad = (ae^{ax}\cos bx - be^{ax}\sin bx) + i(ae^{ax}\sin bx + be^{ax}\cos bx)$$

$$= (a+bi)e^{ax}(\cos bx + i \sin bx) = re^{rx}$$

Complex Characteristic Roots

- As a result, when r is complex (just as when r is real), e^{rx} will be a solution of the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

if and only if r is a root of its characteristic equation.

- If the complex conjugate pair of roots $r_1 = a + bi$ and $r_2 = a - bi$ are nonrepeated, then the corresponding part of a general solution of this differential equation is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{(a+bi)x} + c_2 e^{(a-bi)x}$$

$$= C_1 e^{ax}(\cos bx + i \sin bx) + C_2 e^{ax}(\cos bx - i \sin bx)$$

$$\boxed{y(x) = (C_1 + C_2)e^{ax}\cos bx + i(C_1 - C_2)e^{ax}\sin bx.}$$

where the arbitrary constants C_1 and C_2 can be complex.

- For instance, the choice $C_1 = C_2 = \dfrac{1}{2}$ gives the real-valued solution

$\cdot \quad y_1(x) = e^{ax} \cos bx.$

while the choice $C_1 = -\dfrac{1}{2}i$, $C_2 = \dfrac{1}{2}i$ gives the independent real-valued solution

$\cdot \quad y_2(x) = e^{ax} \sin bx$ $\qquad$ $y_1(x)$ and $y_2(x)$ are linearly indep.

- This yields the following result.

Theorem (Complex Roots) If the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0$$

$a \pm bi$ $\qquad$ $b \neq 0.$

has an unrepeated pair of complex conjugate roots $a \pm bi$ (with $b \neq 0$), then the corresponding part of a general solution of the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

has the form

$$\boxed{e^{ax}(C_1 \cos bx + C_2 \sin bx)}$$

Polar Form

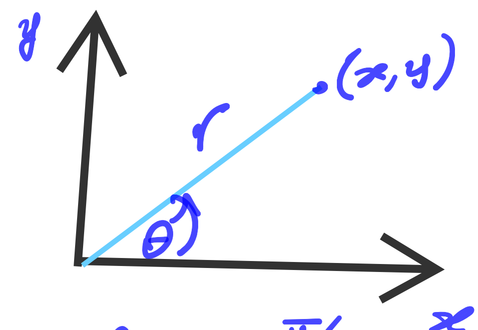
- We can employ the *polar form*

$$z = x + iy = re^{i\theta}$$

  of the complex number z.

- This form follows from Euler's formula upon writing

$$z = r\left(\frac{x}{r} + i\,\frac{y}{r}\right) = r(\cos\theta + i\sin\theta) =: re^{i\theta}.$$



- Here r is the *modulus*

$$r = \sqrt{x^2 + y^2}$$

  of the number z.

- The *argument* of z is the angle θ.

- For instance, the imaginary number i has modulus and argument $r = 1,\ \theta = \pi/2$.

- Similarly, $-i = e^{3\pi/2}$.

- Another consequence is the fact that the nonzero complex number $z = re^{i\theta}$ has the two square roots

$$\sqrt{z} = \pm\left(re^{i\theta}\right)^{1/2} = \pm\sqrt{r}\,e^{i\theta/2}$$

  where $\sqrt{r}$ denotes (as usual for a positive real number) the positive square root of the modulus of z.

Repeated Complex Roots

- Theorem can be extended for repeated complex roots.
- If the conjugate pair $a \pm bj$ has multiplicity k., then the corresponding part of the general solution has the form

$$(A_1 + A_2 x + \ldots + A_k x^{k-1}) e^{(a+bj)x} + (B_1 + B_2 x + \ldots + B_k x^{k-1}) e^{(a-bj)x}$$

$$= \sum_{p=0}^{k-1} x^p e^{ax} (c_p \cos bx + d_p \sin bx)$$

- It can be shown that the $2k$ functions

$$x^p e^{ax} \cos bx, \quad x^p e^{ax} \sin bx \quad (p = 0, \ldots, k-1)$$

appearing above are linearly independent.

Example 2. *Find the general solution of the differential equation:*

$$y'' - 6y' + 13y = 0.$$

- Characteristic equation:

$$r^2 - 6r + 13 = 0.$$

$$\Rightarrow r = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2j.$$

$$\{ e^{3x}\cos 2x, \; e^{3x}\sin 2x \}.$$

- General solution:

$$\boxed{y(x) = c_1 e^{3x}\cos 2x + c_2 e^{3x}\sin 2x}$$

Example 3. *Find the general solution of the differential equation:*

$$y^{(4)} + 18y'' + 81y = 0.$$

- Characteristic equation:

$$r^4 + 18r^2 + 81 = 0.$$

$$\Longleftrightarrow (r^2+9)^2 = 0.$$

roots. $r = \pm 3j \quad (k=2)$

$$\{ \cos 3x, \; x\cos 3x, \; \sin 3x, \; x\sin 3x \}$$

$3j$

- General solution:

$$\boxed{y(x) = (c_1 + c_2 x)\cos 3x + (c_3 + c_4 x)\sin 3x}$$

Example 4. *Solve the following initial value problem:*

$$9y'' + 6y' + 4y = 0, \qquad y(0) = 3, y'(0) = 4.$$

- Characteristic equation:

$$9r^2 + 6r + 4 = 0.$$

$$\Longleftrightarrow\ r = \frac{-6 \pm \sqrt{36 - 144}}{18} = \frac{-\cancel{6} \pm \cancel{6}\sqrt{3}}{\cancel{18}\,3}$$

$$r_{1,2} = -\frac{1}{3} \pm \frac{1}{\sqrt{3}} i$$

- General solution: $\left\{ e^{-x/3} \cdot \cos\left(\frac{x}{\sqrt{3}}\right),\ e^{-x/3} \sin\left(\frac{x}{\sqrt{3}}\right) \right\}$

- Particular solution: $\boxed{y(x) = c_1 e^{-x/3} \cos\left(\frac{x}{\sqrt{3}}\right) + c_2 e^{-x/3} \sin\left(\frac{x}{\sqrt{3}}\right)}$

$$3 = y(0) = c_1 \Rightarrow c_1 = 3.$$

$$4 = y'(0) = -\frac{c_1}{3} + \frac{c_2}{\sqrt{3}} \Rightarrow c_2 = 5\sqrt{3}.$$

$$\boxed{y(x) = 3e^{-x/3} \cos\left(\frac{x}{\sqrt{3}}\right) + 5\sqrt{3}\, e^{-x/3} \sin\left(\frac{x}{\sqrt{3}}\right)}$$

Example 5. *Find a linear homogeneous constant-coefficient equation with the given general solution:*

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \boxed{c_3 \cosh 2x + c_4 \sinh 2x.}$$

i) $\sinh(x) = \dfrac{e^{x} - e^{-x}}{2}.$

$\cosh(x) = \dfrac{e^{x} + e^{-x}}{2}.$

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{c_3}{2}\left(e^{2x} + e^{-2x}\right) + \frac{c_4}{2}\left(e^{2x} - e^{-2x}\right)$$

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{c_3}{2} e^{2x} + \frac{c_3}{2} e^{-2x} + \frac{c_4}{2} e^{2x} - \frac{c_4}{2} e^{-2x}$$

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \boxed{C e^{2x} + D e^{-2x}}$$

$r_1 = +2i$ $\quad$ $r_3 = +2$

$r_2 = -2i$ $\quad$ $r_4 = -2.$

Charact.

$$(r^2+4)(r^2-4) = 0$$

$$r^4 - 16 = 0.$$

$$\therefore \quad \boxed{y^{(4)} - 16y = 0}$$

MA 266 Lecture 17

Christian Moya, Ph.D.

Midterm Review and Sec 3.4 Mechanical Vibrations

Example 1. *Consider a pond that initially contains 10 million gal of water. Water containing a polluted chemical flows into the pond at the rate of 6 million gal/yr, and the mixture in the pond flows out at the rate of 5 million gal/yr. The concentration* $\boxed{\gamma(t)}$ *of chemical in the incoming water varies as* $\boxed{\gamma(t) = 2 + \sin 2t}$ *grams/gal. Let* $Q(t)$ *be the amount of chemical at time* t *measured by millions of grams. Derive the differential equation of the process.*

$$\frac{dQ}{dt} = r_i \cdot c_i - r_o \cdot c_o$$

$$= r_i \cdot \gamma_i - r_o \frac{Q(t)}{V(t)}$$

$V_0 = 10$

$r_i = 6$ mill gal/yr.

$c_i = \gamma(t) = 2 + \sin 2t$.

$r_o = 5$.

$$V(t) = V_0 + (r_i - r_o)t$$

$$= 10 + (6-5)t = 10 + t.$$

$$\boxed{\frac{dQ}{dt} = 6 \cdot (2 + \sin 2t) - \frac{5}{10+t} \cdot Q(t)}$$

Example 2. *Let $y(t)$ be the solution of the IVP:*

$$y''' + y' = 0, \quad y(0) = 2, \quad y'(0) = 1, \quad y''(0) = 1,$$

then $y(\pi) = ?$

1)

$$r^3 + r = 0$$

$$\Leftrightarrow r(r^2+1) = 0.$$

roots: $r_1 = 0$

$r_{2,3} = \pm i$

$$\{\underbrace{e^{0x}}_{=1}, \sin x, \cos x\}$$

2)

$$\boxed{y(x) = C_1 + C_2 \sin x + C_3 \cos x}$$

3)

$$y'(x) = C_2 \cos x - C_3 \sin x$$

$$y''(x) = -C_2 \sin x - C_3 \cos x.$$

$$\left.\begin{aligned} 2 &= y(0) = C_1 + C_3. \\ 1 &= y'(0) = C_2 \\ 1 &= y''(0) = -C_3 \end{aligned}\right\} \quad \begin{aligned} C_1 &= 3 \\ C_2 &= 1 \\ C_3 &= -1 \end{aligned}$$

$$\boxed{y(x) = 3 + \sin x - \cos x}$$

$$y(\pi) = 4$$

Example 3. *Find the particular solution of the IVP:*

$$y' = \frac{1-2x}{y}, \quad y(1) = -2,$$

in explicit form.

1)

$$\int y\,dy = \int (1-2x)\,dx + C.$$

$$y^2 = 2\cdot(x - x^2 + c)$$

$$\boxed{y(x) = \pm\sqrt{(2x-2x^2+c)},}$$

2) $y(1) = -2.$

$$-2 = y(1) = -\sqrt{(2x-2x^2+c)}$$

$$\boxed{C = 4}$$

$$\boxed{y(x) = -\sqrt{2x-2x^2+4},}$$

Example 4. *Find the solution of the IVP*

$$y'' + y' - 6y = 0, \quad y(0) = 0,\ y'(0) = 5.$$

1)

$$r^2 + r - 6 = 0.$$

$$(r+3)(r-2) = 0.$$

$$r_1 = -3, \quad r_2 = 2.$$

$$\{e^{-3x}, e^{2x}\}.$$

2)

$$\boxed{y(x) = c_1 e^{-3x} + c_2 e^{2x}}$$

3)

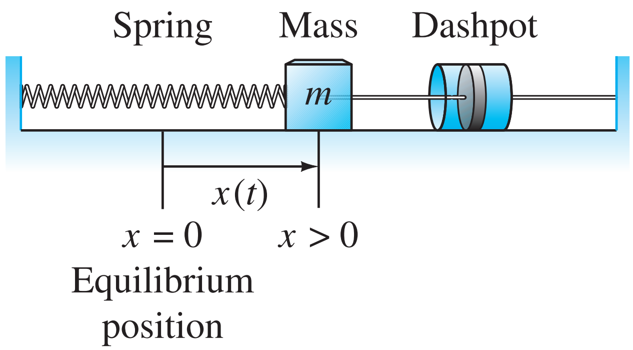
$$y'(x) = -3c_1 e^{-3x} + 2c_2 e^{2x}$$

$$\left.\begin{aligned} 0 &= y(0) = c_1 + c_2. \\ 5 &= y'(0) = -3c_1 + 2c_2. \end{aligned}\right. \Rightarrow \left\{\begin{aligned} c_1 &= -1 \\ c_2 &= +1. \end{aligned}\right.$$

$$\boxed{y(x) = -e^{-3x} + e^{2x}}$$

Mechanical Vibrations

In Sec 3.1, we considered the following mass connected to a spring and a dashpot.



We described the dynamics of this system using the linear equation:

$$(1) \quad \boxed{m \cdot x'' + cx' + kx = F(t)}$$

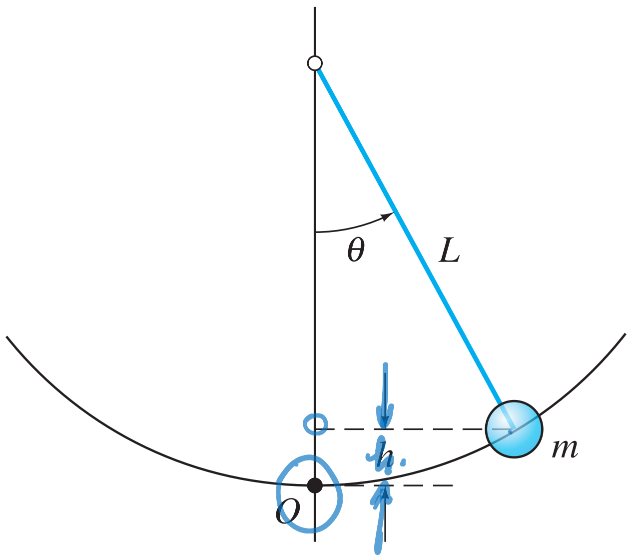
Here

- $F_S = -kx$: spring force, $k > 0$: spring constant
- $F_R = -cx'$: dashpot force, $c \geq 0.$: damping constant
- $F_E = F(t)$: external force.

Remarks

- undamped: no dashpot $\iff$ $c = 0.$
- damped: if $c > 0.$.
- Motion is FREE if $F(t) = 0.$.
- Motion is FORCED if $F(t) \neq 0.$.

The Simple Pendulum



We let

- $s = L\theta$: arc distance from O to m. lowest point / reference.
- $v = ds/dt = L\,d\theta/dt.$: velocity of m

Kinetic Energy:

$$\boxed{T = \frac{1}{2} m \cdot v^2 = \frac{1}{2} m L^2 \left(\frac{d\theta}{dt}\right)^2.}$$

Potential Energy:

$$\boxed{V = m \cdot g \cdot h = m \cdot g \cdot L(1-\cos\theta)}$$

The sum of the kinetic energy T and potential energy V:

$$\boxed{T + V = \frac{1}{2} \cdot m \cdot L^2 \left(\frac{d\theta}{dt}\right)^2 + m \cdot g \cdot L(1-\cos\theta) = C.}$$

Differentiating with respect to t both sides:

$$m \cdot l^2 \left(\frac{d\theta}{dt}\right)\left(\frac{d^2\theta}{dt^2}\right) + m \cdot g \cdot L \sin\theta \frac{d\theta}{dt} = 0.$$

$$\iff \boxed{\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.}$$

Note: We can also obtain the above differential equation using Newton's second law.

Going from nonlinear to linear

- Small angle approximation:

if θ is small $\Rightarrow$ $\sin\theta \approx \theta$.

$$\boxed{\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0.}$$

- Adding *frictional resistance:* $c\theta'$

$$\boxed{\theta'' + c\theta' + k\theta = 0.}$$

where

$$k := \frac{g}{L}.$$

FREE Undamped Motion $c = 0 \quad F(t) = 0.$

$$mx'' + kx = 0.$$

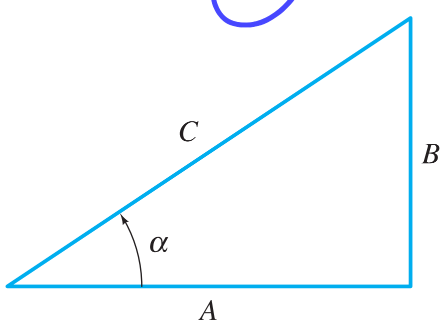
- Define: $\omega_0 := \sqrt{\frac{k}{m}} > 0.$

$$\Rightarrow \boxed{x'' + \omega_0^2 x = 0.}$$

- General solution: $r^2 + \omega_0^2 = 0 \qquad r_{1,2} = \pm \omega_0 i.$

$$\boxed{x(t) = A \cos \omega_0 t + B \sin \omega_0 t.}$$

Phase angle "α"



$$C = \sqrt{A^2 + B^2}$$
$$\cos \alpha = \frac{A}{C}$$
$$\sin \alpha = \frac{B}{C}.$$

- General solution:

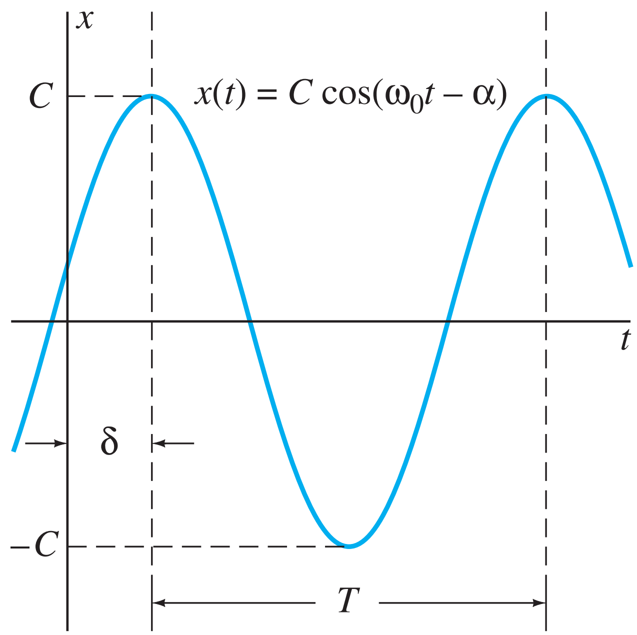
$$x(t) = C\left(\frac{A}{C} \cos \omega_0 t + \frac{B}{C} \sin \omega_0 t\right)$$
$$= C\left(\cos \alpha \cos \omega_0 t + \sin \alpha \sin \omega_0 t\right)$$

$$\Rightarrow \boxed{x(t) = C \cos(\omega_0 t - \alpha)} \quad *$$

Simple harmonic motion.

The mass oscillates with:

- Amplitude: C.
- Circular frequency: $\omega_0 \left(\frac{rad}{s}\right)$.
- Phase angle: α.



- Period:

$$T = \frac{2\pi}{\omega_0} \, (s).$$

- Frequency:

$$\nu = \frac{1}{T} = \frac{\omega_0}{2\pi} \, (Hz)$$

- Time lag: $\delta = \frac{\alpha}{\omega_0}$.

$$\Longrightarrow x(t) = C\cos\left(\omega_0\left(t - \frac{\alpha}{\omega_0}\right)\right) = C\cos(\omega_0(t-\delta))$$

FREE Damped Motion $c > 0, \quad F(t) = 0.$

$$mx'' + cx' + kx = 0.$$

$$\iff \boxed{x'' + 2p\,x' + \omega_0^2 x = 0}$$

$\omega_0 = \sqrt{k/m} > 0$: undamped circ. freq

- Characteristic equation: $p = \frac{c}{2m} > 0.$

$$r^2 + 2pr + \omega_0^2 = 0.$$

$$r_{1,2} = \frac{-\cancel{2}p \pm \sqrt{\cancel{4}p^2 - \cancel{4}\omega_0^2}}{\cancel{2}} = -p \pm (p^2 - \omega_0^2)^{1/2}.$$

- Sign depends on:

$$p^2 - \omega_0^2 = \frac{c^2}{4m^2} - \frac{k}{m} = \frac{c^2 - 4km}{4m^2}.$$

- **Critical damping:**

c_{cr} : $\quad c_{cr}^2 = 4km$

$$\iff c_{cr} = \sqrt{4km}.$$

$\Big\}$
- 2 distinct (real)
- repeated
- 2 complex conj

Overdamped case $c > c_r$

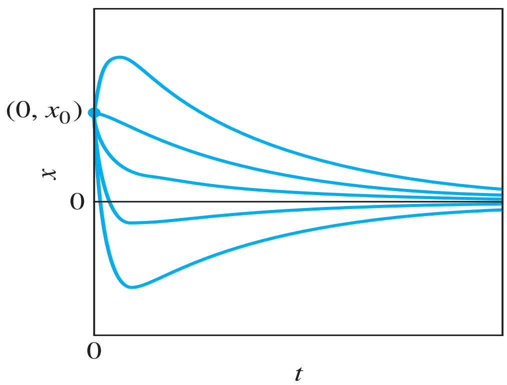
$\iff \boxed{c^2 > 4km.}$

- $\implies$ two *distinct* roots:

$r_1, r_2 = -p \pm (p^2 - \omega_0^2)^{1/2}$. <0

- General solution:

$$\boxed{x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$$



$\Rightarrow x(t) \to 0$ as $t \to +\infty$.

$r_1, r_2 = -p \pm (p^2 - \omega_0^2)^{1/2}$

$= \frac{c^2 - 4km}{4m^2}$.

Critically damped case $c = c_r$

$\iff \boxed{c^2 = 4km.}$

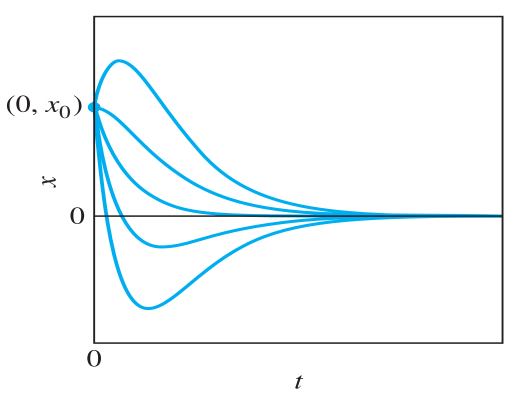
- $\implies$ *repeated* roots:

$r_{1,2} = -p$.

- General solution:

$\{e^{-pt}, te^{-pt}\}$.

$$\boxed{x(t) = (c_1 + tc_2)e^{-pt}.}$$



Since $p > 0$, as $t \to \infty$

$x(t) \to 0$.

at most one cross of zero.

$c_1 + c_2 t^* = 0$.

$t^* = -\frac{c_1}{c_2}$.

Underdamped case $c < c_r$

$\iff \boxed{c^2 < 4km.}$

- $\implies$ two *complex* roots: $r_1, r_2 = -p \pm (p^2 - \omega_0^2)^{1/2}$

$\frac{c^2 - 4km}{4m^2}.$

$$\boxed{r_1, r_2 = -p \pm i\sqrt{\omega_0^2 - p^2}}$$

- General solution:

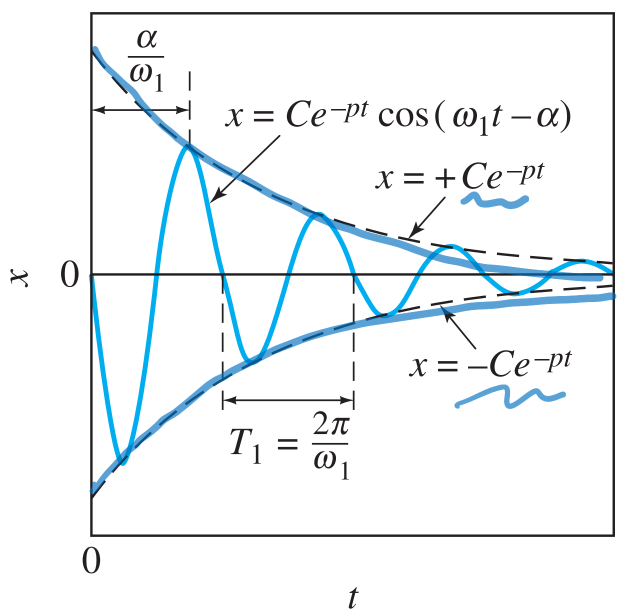
$\{e^{-pt}\cos(\sqrt{\omega_0^2 - p^2}\,t),\ e^{-pt}\sin(\sqrt{\omega_0^2 - p^2}\,t)\}$

$$\boxed{x(t) = e^{-pt}(A\cos\omega_1 t + B\sin\omega_1 t)}$$

$\omega_1 = (\omega_0^2 - p^2)^{1/2} = \frac{\sqrt{4km - c^2}}{2m}$

- Using derivation ______(□)______:

$x(t) = Ce^{-pt}\left(\underbrace{\frac{A}{C}}_{\cos\alpha}\cos\omega_1 t + \underbrace{\frac{B}{C}}_{\sin\alpha}\sin\omega_1 t\right)$



$C = \sqrt{A^2 + B^2}$

$\frac{A}{C} = \cos\alpha.$

$\frac{B}{C} = \sin\alpha.$

$$\boxed{x(t) = Ce^{-pt}(\cos(\omega_1 t - \alpha))}$$

Example 5. *Consider the differential equation of a spring-mass-(dashpot) system:*

$$mx'' + cx' + kx = 0.$$

Find the particular solution

a) with damping: $m = 1, c = 10, k = 125, x_0 = 6,$ *and* $v_0 = 50.$

b) without damping: $m = 1, k = 125, x_0 = 6,$ *and* $v_0 = 50.$

Solution a):

- Characteristic equation:

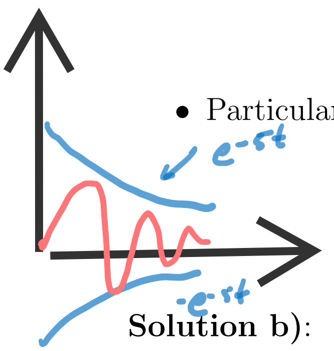
$$r^2 + 10r + 125 = 0.$$

$$\Rightarrow r_1, r_2 = \frac{-10 \pm \sqrt{100 - 500}}{2} = -5 \pm 10i.$$

- General solution:

$$\{e^{-5t}\cos 10t,\ e^{-5t}\sin 10t\}$$

$$\boxed{x(t) = e^{-5t}(A\cos 10t + B\sin 10t)}$$



- Particular solution:

$$6 = x(0) = A$$
$$50 = x'(0) = -5A + 10B \Rightarrow \begin{cases} A = 6 \\ B = 8. \end{cases}$$

$$\boxed{x(t) = e^{-5t}(6\cos 10t + 8\sin 10t)}$$

Solution b):

- Characteristic equation:

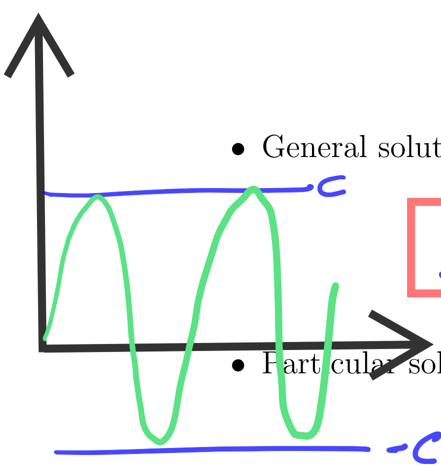
$$r^2 + 125 = 0.$$

$$\Rightarrow r = \pm 5\sqrt{5}\, i.$$

- General solution:

$$\{\cos(5\sqrt{5}t),\ \sin(5\sqrt{5}t)\}.$$

$$\boxed{x(t) = A\cos(5\sqrt{5}t) + B\sin(5\sqrt{5}t).}$$



- Particular solution:

$$6 = x(0) = A$$
$$50 = x'(0) = 5\sqrt{5}B \Rightarrow \begin{cases} A = 6. \\ B = 2\sqrt{5} \end{cases}$$

$$\boxed{x(t) = 6\cos(5\sqrt{5}t) + 2\sqrt{5}\sin(5\sqrt{5}t)}$$

$$C = \sqrt{6^2 + (2\sqrt{5})^2}$$

MA 266 Lecture 18

Christian Moya, Ph.D.

Sec 3.5-1 Nonhomogeneous Equations

- Consider the nonhomogeneous nth-order linear equation with constant coefficients:

$$\boxed{a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 = f(x)}$$

- Recall the following theorem:

Theorem (Solutions Nonhomogeneous Equations)

- Let y_p be a particular solution of the nonhomogeneous equation on an open interval I where the functions p_i and f are continuous.
- Let y_1, y_2, $\dots$, y_n be linearly independent solutions of the associated homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0.$$

- If Y is any solution whatsoever of the equation nonhomogeneous equation on I, then there exist numbers c_1, c_2, $\dots$, c_n such that

$$Y(x) = \underbrace{c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)}_{=:y_c(x)} + y_p(x)$$

  for all x in I.

- *Question:* How to find $y_p(x)$?

$$Ly = f(x)$$

Undetermined Coefficients

$$f(x) = P_m(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0.$$

- Suppose that $f(x)$ is a polynomial of degree m.
- Note that the derivatives of a polynomial are themselves polynomials of lower degree.
- Thus it is reasonable to *guess* a particular solution has the form

$$y_p(x) = A_m x^m + A_{m-1} x^{m-1} + \dots + A_1 x + A_0.$$

Example 1. *Find a particular solution of*

$$y'' + 3y' + 4y = \underbrace{3x + 2}_{f(x)}.$$

- Here $f(x) = 3x + 2$ is a polynomial of degree 1, so our *guess* is:

$$y_p(x) = Ax + B.$$

- Then

$$y_p' = A \quad ; \quad y_p'' = 0$$

$$y_p'' + 3y_p' + 4y = (0) + 3(A) + 4(Ax + B) = 3x + 2$$

$$(4A)x + (3A + 4B) = 3x + 2.$$

- Solve for the *undetermined coefficients* A and B.

$$\left.\begin{aligned} 4A &= 3 \\ 3A + 4B &= 2 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= \frac{3}{4} \\ B &= -\frac{1}{16} \end{aligned}$$

$$\boxed{y_p(x) = \frac{3}{4}x - \frac{1}{16}.}$$

$$\boxed{Y(x) = y_c(x) + y_p(x)}$$

- Similarly, suppose that $Ly = f(x)$

$$f(x) = a\cos kx + b\sin kx.$$

- Then it is reasonable to expect a particular solution of the same form:

$$y_p(x) = A\cos kx + B\sin kx$$

a linear combination with undetermined coefficients ___A___ and ___B.___.

- Any derivative of the linear combination of $\cos kx$ and $\sin kx$ has the same form.

Example 2. *Find a particular solution of*

$$3y'' + y' - 2y = 2\cos x.$$

- We try the *guess*: $y_p(x) = A\cos x$ No $\left(\begin{array}{l} y_p' \approx \alpha \sin x \\ LHS = RHS \end{array}\right)$

$$y_p(x) = A\cos x + B\sin x$$

- Then $y_p' = -A\sin x + B\cos x \qquad y_p'' = -A\cos x - B\sin x$

$$3y_p'' + y_p' - 2y_p = 2\cos x$$

$$(-5A + B)\cos x + (-A - 5B)\sin x = 2\cos x.$$

- Solve for the *undetermined coefficients* ___A___ and ___B.___.

$$\begin{array}{l} i) \ -5A + B = 2 \\ ii) \ -A - 5B = 0 \end{array} \Rightarrow \left\{ \begin{array}{l} A = -5/13 \\ B = 1/13. \end{array} \right.$$

$$\therefore \boxed{y_p(x) = -\frac{5}{13}\cos x + \frac{1}{13}\sin x.}$$

Eligible Functions

- The method of *undetermined coefficients* applies whenever the function $f(x)$ in

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x).$$

is a linear combination of (finite) products of functions of the following three types:

1. $P_m(x)$: polynomial in x.
2. $\cos kx$ or $\sin kx$
3. e^{rx}

e.g. $f(x) = x^2 \cos kx\, e^{-2x} + 5x \sin kx$.

Example 3. *Find a particular solution of*

$$y'' + y' + y = \sin^2 x. = \frac{1 - \cos 2x}{2}.$$

- We try the *guess*:

$$y_p(x) = A + B\cos 2x + C\sin 2x$$

- Then

$$y_p' = -2B\sin 2x + 2C\cos 2x$$
$$y_p'' = -4B\cos 2x - 4C\sin 2x$$

$$y_p'' + y_p' + y = \frac{1}{2} + (-4B + 2C + B)\cos 2x + (-4C - 2B + C)\sin 2x$$
$$= \frac{1}{2} - \frac{\cos 2x}{2}.$$

- Solve for the *undetermined coefficients* A, B and C.

$A = \frac{1}{2}$, $B = \frac{3}{26}$, $C = -\frac{1}{13}$.

$$\boxed{y_p(x) = \frac{1}{2} + \frac{3}{26}\cos 2x - \frac{1}{13}\sin 2x}$$

Caution!!!

Example 4. *Find a particular solution of*

$$y'' - 4y = \underbrace{2e^{2x}}_{f(x)}.$$

- We try the *guess*:

$$y_p(x) = Ae^{2x}$$

- Then $y_p' = 2Ae^{2x}$ $\quad y_p'' = 4Ae^{2x}$

$$y_p'' - 4y_p = 4Ae^{2x} - 4\left(Ae^{2x}\right) = 0 \neq \underbrace{2e^{2x}}_{\neq 0}.$$

$$y'' - 4y = 0 \Rightarrow r^2 - 4 = 0 \Rightarrow \begin{matrix} r_1 = 2 \\ r_2 = -2 \end{matrix} \quad \{e^{2x}, e^{-2x}\}.$$

- Solve for the *undetermined coefficient:* $\boxed{y_c(x) = C_1 e^{2x} + C_2 e^{-2x}}$

$\forall A$: $y_p = Ae^{2x}$ cannot satisfy $Ly = f(x)$.

because Ae^{2x} is duplicated in the complementary solution.

Rule 1 Method of Undetermined Coefficients

- Suppose that $\boxed{Ly = f(x)}$ is a nonhomogeneous linear equation with constant coefficients and that $f(x)$ is a linear combination of finite products of *eligible* functions.
- Also suppose that no term appearing either $\boxed{\text{in } f(x)}$ or in any of its derivatives satisfies the associated homogeneous equation $\underline{Ly = 0}$. $y_c(x)$ sol'n of $Ly=0$.
- Then take as a *guess/trial* solution for y_p a linear combination of all linearly independent such terms and their derivatives.
- Then determine the coefficients by substitution of this trial solution into the nonhomogeneous equation $Ly = f(x)$.

Example 5. *Find a particular solution of*

$$y'' + 9y = \underbrace{2x^2e^{3x}}_{f(x)} + 5.$$

i) $r^2+9=0$

$r_{1,2} = \pm 3i$

$y_c(x) = c_1\cos 3x + c_2 \sin 3x$

- We try the *guess*:

RULE 1 applies.

$y_p(x) = A + (B + Cx + Dx^2)e^{3x}$

$\boxed{\begin{array}{l} \text{ii)} \quad 5 . \; 0 \; 0 \ldots \\ x^2e^{3x} \; xe^{3x} \; e^{3x} \ldots \end{array}}$

- Then

$y_p' \; ; \; y_p''$

$y_p'' + 9y_p = 9A + (18B + 6C + 2D)e^{3x} + (18C + 12D)xe^{3x} + 18Dx^2e^{3x} = 2x^2e^{3x} + 5$

$\Rightarrow$ i) $9A = 5$
ii) $18B + 6C + 2D = 0$
iii) $18C + 12D = 0$.
iv) $18D = 2$.

$\Rightarrow$ $A = 5/9$, $B = \frac{1}{81}$, $C = -\frac{2}{27}$, $D = \frac{1}{9}$

- Solve for the *undetermined coefficients:*

$$\boxed{y_p(x) = \frac{5}{9} + \left(\frac{1}{81} - \frac{2}{27}x + \frac{1}{9}x^2\right)e^{3x}.}$$

Remarks on Rule 1

- In practice we check the supposition made in Rule 1 by first using the characteristic equation to find the complementary function y_c.
- Then we make a list of all the terms appearing in $f(x)$ and its successive derivatives.
- If none of the terms in the list duplicates a term in y_c, then we use Rule 1.

The Case of Duplication

- Consider the case in which *Rule* 1 does not apply.
- That is, some of the terms involved in $f(x)$ and its derivatives satisfy the associated homogeneous equation.
- For instance, suppose that we want to find a particular solution for:

$Ly = f(x)$ $\quad (D-r)^3 y = (2x-3)e^{rx}$

$\Updownarrow$ $\quad (r_0 - r)^3 = 0 \quad r_i = r \quad (k=3)$

- Proceeding as in Rule 1, our first guess would be $\{e^r, xe^r, x^2e^r\}$.

$f(x) = (2x-3)e^{rx}$.
xe^{rx}, e^{rx}.

$$y_p(x) = (Ax+B)e^{rx}$$

$$A: \quad y_{p_2}(x) = x^3(Ax+B)e^{rx}$$

- This form of $y_p(x)$ will not be adequate because the complementary function is

$$\boxed{y_c(x) = C_1 e^r + C_2 x e^{rx} + C_3 x^2 e^{rx}}$$

so substitution would yield zero rather than $(2x-3)e^{rx}$.

$$(D-r)^3\left[(Ax+B)e^{rx}\right] = 0$$

$$D = \frac{d}{dx}$$

Amending Our Initial Guess

- To amend our first guess, we observe that

$$(D-r)^2 f(x) = (D-r)^2\left[(2x-3)e^{rx}\right] = \left(\underbrace{D^2(2x+3)}_{\frac{d^2}{dx^2}}\right)e^{rx}$$
$$= 0.$$

  by our earlier discussion of differential operators.

- If $y(x)$ is *any* solution of our differential equation and we apply the operator $(D-r)^2$ to both sides, we see that $y(x)$:

$$(D-r)^2(D-r)^3 y = \underbrace{(D-r)^2(2x-3.)e^{rx}}_{=0}$$

$$L'y = 0 \implies (D-r)^5 y = 0.$$

- The general solution of this *homogeneous* equation is:

$$Y(x) = \underbrace{c_1 e^{rx} + c_2 x e^{rx} + c_3 x^2 e^{rx}}_{= y_c(x)} + \underbrace{Ax^3 e^{rx} + Bx^4 e^{rx}.}_{y_p(x).}$$

Form of the General Solution

- Thus *every* solution of our original equation is:

$$y_p(x) = Ax^3 e^{rx} + Bx^4 e^{rx}$$
$$= x^3(Ae^{rx} + Bxe^{rx}) = x^3 y_p(x)$$

- Note that the RHS can be obtained by multiplying each term of our first guess

$$y_i(x) = Ae^{rx} + Bxe^{rx}$$

  by

$$\times \quad x^3.$$

  that is, the least positive power of x (in this case, x^3) that eliminates duplication between the terms of the resulting trial solution $y_p(x)$ and the complementary function $y_c(x)$.

The General Case

- To simplify the general statement of *Rule 2*, we observe that to find a particular solution of the nonhomogeneous linear differential equation

$$Ly = \underbrace{f_1(x)}_{} + f_2(x)$$

it suffices to find *separately* particular solutions $Y_1(x)$ and $Y_2(x)$ of the two equations

$$Ly = f_1(x) \quad \text{and} \quad Ly = f_2(x)$$

respectively. $Y_1, Y_2.$

- Linearity then gives

$$L[Y_1 + Y_2] = LY_1 + LY_2 = f_1(x) + f_2(x)$$

and therefore $y_p = Y_1 + Y_2.$ is a particular solution of

- (This is a type of "superposition principle" for nonhomogeneous linear equations.)
- Now our problem is to find a particular solution of the equation $Ly = f(x)$, where $f(x)$ is a linear combination of products of the elementary functions listed earlier.
- *Notation:* $f(x)$ can be written as a sum of terms each of the form

where $P_m(x)$ is a polynomial in x of degree m.

MA 266 Lecture 19

Christian Moya, Ph.D.

Sec 3.5-2 Nonhomogeneous Equations

$L[y] = f(x)$

Rule 2 Method of Undetermined Coefficients

- If the function $f(x)$ is of the form

  $P_m(x)e^{rx}\cos kx$ or $P_m(x)e^{rx}\sin kx$

  take as the *guess/trial* solution

  $$y_p(x) = x^s\left[(A_0 + A_1x + \dots + A_mx^m)e^{rx}\cos kx + (B_0 + B_1x + \dots + B_mx^m)e^{rx}\sin kx\right].$$

  where s is the smallest nonnegative integer such that no term in the trial solution y_p duplicates a term in the complementary function y_c.

  $L[y_c] = 0.$

- Determine the coefficients in y_p by substituting y_p into the nonhomogeneous eq'n.

The next table lists the form of y_p in various common cases:

$f(x)$	y_p
$P_m(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$	$x^s(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)$
$a\cos kx + b\sin kx$	$x^s(A\cos kx + B\sin kx)$
$e^{rx}(a\cos kx + b\sin kx)$	$x^se^{rx}(A\cos kx + B\sin kx)$
$P_m(x)e^{rx}$	$x^s(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)e^{rx}$
$P_m(x)(a\cos kx + b\sin kx)$	$x^s[(A_0 + A_1x + \cdots + A_mx^m)\cos kx + (B_0 + B_1x + \cdots + B_mx^m)\sin kx]$

Example 1. *Find the particular solution of*

$$y^{(3)} + y'' = \underbrace{3e^x + 4x^2}_{f(x)}.$$

- Characteristic equation:

$$r^3 + r^2 = 0.$$

$$\Longleftrightarrow r^2(r+1) = 0 \longrightarrow r_1 = 0,\ r_2 = 0,\ r_3 = -1$$

- Complementary solution y_c:

$$y_c(x) = c_1 + c_2 x + c_3 e^{-x}.$$

- Initial trial y_p:

$$y_p(x) = (Ae^x) + (B + Cx + Dx^2)\,x^{s}$$

- Eliminate *duplication* terms $\iff$ $(s = ?)$

$x^0 \cdot y_p(x)$ $(s=0)$

$$\boxed{y_p = Ae^x + B + Cx + Dx^2}$$
$$y_p' = Ae^x + C + 2Dx$$
$$y_p'' = Ae^x + 2D$$
$$y_p''' = Ae^x$$

$$y_p^{(3)} + y_p'' = 2(Ae^x) + 2D = 3e^x + 4x^2 \quad ?$$

NO

$x^1 \cdot y_p(x)$ $(s=1)$

$$y_p = Ae^x + \boxed{Bx + Cx^2 + Dx^3} \leftarrow$$
$$y_p' = Ae^x + \cancel{B} + 2Cx + 3Dx^2$$
$$y_p'' = Ae^x + 2C + 6Dx$$
$$y_p''' = Ae^x + 6D$$

$$y_p^{(3)} + y_p'' = 2Ae^x + 6Dx + (2C + 6D) = 3e^x + 4x^2 \, ?$$

NO

$x^2 \times y_p(x)$ $(s=2)$

$$
\begin{aligned}
y_p &= Ae^x + \boxed{Bx^2 + Cx^3 + Dx^4} \\
y_p' &= Ae^x + 2Bx + 3Cx^2 + 4Dx^3 \\
y_p'' &= Ae^x + 2B + 6Cx + 12Dx^2 \\
y_p''' &= Ae^x + 6C + 24Dx
\end{aligned}
$$

$$y_p^{(3)} + y_p'' = 2Ae^x + (2B+6C) + (6C+24D)x + 12Dx^2 = 3e^x + 4x^2.$$

- The system of equations:

$$2A = 3 \qquad 6C + 24D = 0.$$

$$2B + 6C = 0. \qquad 12D = 4.$$

- Particular solution $y_p(x)$:

$$\boxed{y_p(x) = \frac{3}{2}e^x + 4x^2 - \frac{4}{3}x^3 + \frac{1}{3}x^4.}$$

Example 2. *Determine the appropriate form for a particular solution of*

$$y'' - 6y' + 13y = \underbrace{xe^{3x}\sin 2x}_{f(x)}.$$

- Characteristic equation:

$$r^2 - 6r + 13 = 0.$$

$$\Rightarrow \quad r_{1,2} = 3 \pm 2i$$

- The complementary solution is:

$$y_c(x) = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$$

- Initial trial y_p:

$$y_p(x) = \left[(A+Bx)e^{3x}\cos 2x + (C+Dx)e^{3x}\sin 2x\right]x^s$$

$$\boxed{(Ax+Bx^2)e^{3x}\cos 2x} + \cdots$$

- Eliminate duplication terms:

$$Ae^{3x}\cos 2x \quad ; \quad Ce^{3x}\sin 2x.$$

$$s = ? \quad \Longrightarrow \quad \underline{s = 1}$$

- Particular solution:

$$y_p(x) = x\left[(A+Bx)e^{3x}\cos 2x + (C+Dx)e^{3x}\sin 2x\right]$$

Example 3. *Determine the appropriate form for a particular solution of*

$$y^{(4)} + 5y'' + 4y = \underbrace{\sin x + \cos 2x}_{f(x)}.$$

- Characteristic equation:

$$r^4 + 5r^2 + 4 = 0.$$
$$(r^2+4)(r^2+1) = 0 \Longrightarrow \begin{cases} r_{1,2} = \pm i \\ r_{3,4} = \pm 2i \end{cases}$$

- The complementary solution is:

$$y_c(x) = (c_1 \cos x + c_2 \sin x) + (c_3 \cos 2x + c_4 \sin 2x)$$

- Initial trial y_p:

$$y_p(x) = (A\cos x + B\sin x) + (C\cos 2x + D\sin 2x)$$

- Eliminate duplication terms:

All duplication terms

$$x \cdot y_p(x)$$

- Particular solution:

$$y_p(x) = x\left[(A\cos x + B\sin x) + (C\cos 2x + D\sin 2x)\right]$$

Variation of Parameters

- Consider

$$y'' + y = \underbrace{\tan x}_{f(x)}$$

- Note:

$f(x) = \tan x$

∞-many linearly indep. der.

$\sec^2 x, \; 2\sec^2 x \tan x, \; \ldots\ldots,$

- Q: Can we find a *finite linear* combination to use as *trial* solution y_p?

NO !.

Method - Variation of Parameters

- Consider the nonhomogeneous equation:

$$(1) \qquad L[y] := y^{(n)} + p_{n_1} y^{(n-1)} + \ldots + p_1 y' + p_0 y = f(x).$$

- Assume we know the general solution

$$y_c(x) = c_1 y_1(x) + \ldots + c_n y_n(x)$$

- of the homogeneous equation:

$$L[y] := y^{(n)} + p_{n_1} y^{(n-1)} + \ldots + p_1 y' + p_0 y = 0.$$

- **Idea:** for y_p
  replace $c_1, c_2, \dots, c_n$.
  with variables/fu's $u_1(x), u_2(x), \dots, u_n(x)$

- *Objective:* Select *variables/functions* in such a way that:

$$y_p(x) = u_1(x)\,y_1(x) + \dots + u_n(x)\,y_n(x)$$

- is a solution of (1) $\Leftrightarrow L[y_p] = f(x)$.
- THIS IS ALWAYS POSSIBLE!!!

Example 4. *Consider the following second order nonhomogeneous equation:*

$$(2) \qquad L[y] := y'' + P(x)y' + Q(x)y = f(x).$$

where P, Q are continuous in some interval I. Assume the above has the complementary solution:

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x)$$

Find the particular solution $y_p(x)$.

- *Objective:* Find u_1, u_2. such that

$$y_p(x) = u_1 y_1 + u_2 y_2.$$

- is a *particular* solution of (2).
- **First** condition:

u_1, u_2 : $\boxed{L[y_p] = f(x)}$

$$y_p'' + P y_p' + Q y_p = f(x)$$

- We need **a second** condition (free of our choice).

$$y_p = u_1 y_1 + u_2 y_2.$$

$$y_p' = (u_1 y_1' + u_2 y_2') + \underbrace{(u_1' y_1 + u_2' y_2)}_{=0.} *$$

- *Idea:* to avoid u_1'', u_2'', we let our **second** condition:

$$\boxed{u_1' y_1 + u_2' y_2 = 0.}$$

$$y_p' = u_1 y_1' + u_2 y_2'$$

- Product rule gives:

$$(3) \quad y_p'' = (u_1 y_1'' + u_2 y_2'') + (u_1' y_1' + u_2' y_2')$$

- Note: $y_1, y_2 : \; L[y_1] = 0 \;\checkmark\; L[y_2] = 0.$

$$y_1'' + P y_1' + Q y_1 = 0.$$

$$y_1'' = -P y_1' - Q y_1$$

- Replace $y_1'', y_2'' \to (3)$.

$$y_p'' = (u_1' y_1' + u_2' y_2') - P\underbrace{(u_1 y_1' + u_2 y_2')}_{=: y_p'} - Q\underbrace{(u_1 y_1 + u_2 y_2)}_{=: y_p.}$$

$$\underbrace{y_p'' + P y_p' + Q y_p}_{L[y_p]} = \underbrace{(u_1' y_1' + u_2' y_2')}_{f(x)}.$$

- Thus, we obtain the system:

$$\boxed{\begin{aligned} u_1' y_1 + u_2' y_2 &= 0. \\ u_1' y_1' + u_2' y_2' &= f(x) \end{aligned}} \quad (\Delta)$$

Remarks

- Determinant of the coefficients $W(y_1, y_2) = W(x)$.
- After solving (Δ) for u_1' and u_2', we integrate to obtain:

$$u_1 = \int u_1'\,dx = -\int \frac{y_2(x)\,f(x)}{W(x)}\,dx$$

$$u_2 = \int u_2'\,dx = \int \frac{y_1(x)\,f(x)}{W(x)}\,dx$$

- We obtain the desired particular solution:

$$y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$$

Example 5. *Find the particular solution of*

$$y'' + y = \tan x.$$

- Complementary solution y_c:

$$y_c(x) = c_1 \cos x + c_2 \sin x.$$

- Setup (Δ):

$$y_1 = \cos x \qquad y_2 = \sin x$$
$$y_1' = -\sin x \qquad y_2' = \cos x.$$

- Hence (Δ) is:

$$(u_1')\cos x + (u_2')\sin x = 0.$$
$$(u_1')(-\sin x) + (u_2')\cos x = \tan x.$$

- Solve for u_1' and u_2':

$u_1' = \cos x - \sec x.$

$u_2' = \sin x.$

- Thus $u_1 = \int (\cos x - \sec x)\, dx = \sin x - \ln|\sec x + \tan x|$

$u_2 = \int \sin x \, dx = -\cos x$

$y_p = u_1 y_1 + u_2 y_2.$

Theorem - Variation of Parameters

- If the *nonhomogeneous* equation:

$$L[y] := y'' + P(x)y' + Q(x)y = f(x)$$

- has *complementary* function:

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x)$$

- Then a particular solution is given by:

$$y_p(x) = -y_1(x) \int \frac{y_2(x)f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x)f(x)}{W(x)} dx$$

,

- where $W(x) = W(y_1, y_)$ is the Wronskian of the two independent solutions y_1 and y_2 of the associated homogeneous equation $L[y] = 0$.

MA 266 Lecture 20

Christian Moya, Ph.D.

Sec 3.5 Nonhomogeneous Equations

Method - Variation of Parameters

- If the *nonhomogeneous* equation:

$$L[y] := y'' + P(x)y' + Q(x)y = f(x)$$

- has *complementary* function:

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x)$$

- The particular solution is :

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

- To find u_1 and u_2, we first solve the following system of equations for u_1' and u_2':

$$\boxed{\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= f(x). \end{aligned}} \qquad \begin{aligned}&(1a)\\&(1b)\end{aligned}$$

- We the find u_1 and u_2 via integration:

$$u_1(x) = \int u_1'(x)dx$$
$$u_2(x) = \int u_2'(x)dx.$$

- The determinant of (1) is the *Wronksian* of the two linear independent solutions y_1 and y_2: $W(y_1, y_2) = W(x)$.

Theorem - Variation of Parameters

- If the *nonhomogeneous* equation:

$$L[y] := y'' + P(x)y' + Q(x)y = f(x)$$

- has *complementary* function:

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x)$$

- Then a particular solution is given by:

$$y_p(x) = -y_1(x) \underbrace{\int \frac{y_2(x)f(x)}{W(x)}dx}_{=-u_1(x)} + y_2(x) \underbrace{\int \frac{y_1(x)f(x)}{W(x)}dx}_{=:\,u_2(x)}$$

- where $W(x) = W(y_1, y_2)$ is the Wronskian of the two independent solutions y_1 and y_2 of the associated homogeneous equation $L[y] = 0$.

Example 1. *Find the particular solution of*

$$y'' + 9y = \boxed{\underbrace{\sin 3x}_{=:f(x)}}.$$

- Complementary solution:

$$r^2 + 9 = 0 \implies r_{1,2} = \pm 3i$$

$$\implies \boxed{y_c(x) = c_1 \underbrace{\cos 3x}_{=:y_1(x)} + c_2 \underbrace{\sin 3x}_{=:y_2(x)}}$$

- The Wronksian $W(x)$ is

$y_1(x) = \cos 3x$

$y_2(x) = \sin 3x$

$$W(x) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3\cos^2 3x + 3\sin^2 x = 3.$$

- The desired functions are then

$$u_1(x) = -\int \frac{y_2(x) f(x)}{W(x)} dx = -\int \frac{\sin^2 3x}{3} dx = -\frac{1}{36}(6x - \sin 6x)$$

$$u_2(x) = \int \frac{y_1(x) \cdot f(x)}{W(x)} dx = \int \frac{\cos 3x \cdot \sin 3x}{3} dx = -\frac{1}{36}(1 + \cos 6x)$$

- Particular solution:

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$$

$$\vdots$$

$$\boxed{y_p(x) = -\frac{1}{6} x \cdot \cos 3x}$$

Example 2. *Find the particular solution of*

$$y'' - 4y = \boxed{xe^x}. =: f(x)$$

- Complementary solution:

$$r^2 - 4 = 0. \Rightarrow \begin{cases} r_1 = -2 \\ r_2 = +2. \end{cases}$$

$$\boxed{y_c(x) = c_1 \underbrace{e^{-2x}}_{=: y_1(x)} + c_2 \underbrace{e^{2x}}_{=: y_2(x)}.}$$

- The Wronksian $W(x)$ is

$$W(x) = \begin{vmatrix} e^{-2x} & e^{2x} \\ -2e^{-2x} & 2e^{2x} \end{vmatrix} = 4$$

- The desired functions are then

$$u_1(x) = -\int \frac{y_2(x) \cdot f(x)}{W(x)}\, dx = -\int \frac{e^{2x} x e^x}{4}\, dx = -\frac{1}{36}(3x-1)e^{3x}$$

$$u_2(x) = \int \frac{y_1(x) \cdot f(x)}{W(x)}\, dx = \int \frac{e^{-2x} x e^x}{4}\, dx = -\frac{1}{4}(x+1)e^{-x}$$

- Particular solution:

$$y_p(x) = u_1 y_1 + u_2 y_2$$

$$\vdots$$

$$\boxed{y_p(x) = -\frac{1}{9}(3x+2)e^{x}.}$$

Sec 3.6 Forced Oscillations

Forced Mass-Spring System

- In a previous lecture, we derived the differential equation

$$\boxed{mx'' + cx' + kx = F(t)}$$

$k > 0$

that models the motion of a mass m that is attached to a spring (with constant k) and a dashpot (with constant c) and is also acted on by an external force $F(t)$.

$c > 0$

- Machines with rotating components commonly involve mass-spring systems (or their equivalents) in which the external force is simple harmonic:

$$\boxed{F(t) = F_0 \cos \omega t} \quad \text{or} \quad \boxed{F(t) = F_0 \sin \omega t.}$$

Undamped Forced Oscillations

Undamped Forced Oscillations

$F(t) = F_0 \sin \omega t$

- Consider the external force $F(t) = F_0 \cos \omega t$ and let $c = 0$. Then, we have:

$$mx'' + kx = F_0 \cos \omega t,$$

- The *complementary* function is:

$$x_c(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

- The (circular) *natural frequency* of the mass–spring system is:

$$\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$$

- Assuming $\omega \neq \omega_0$, the *particular* solution is:

$$x_p(t) = \boxed{\frac{F_0/m}{\omega_0^2 - \omega^2}} \cos \omega t.$$

- The general solution $x = x_c + x_p$ is given by:

$$x(t) = \boxed{c_1 \cos \omega_0 t + c_2 \sin \omega_0 t} + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t,$$

  where the constants c_1 and c_2 are determined by the initial values $x(0)$ and $x'(0)$.

- As we saw earlier, this can be rewritten as

$$x(t) = C \cos(\omega_0 t - \alpha) + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t.$$

Example 3. *Use the method of undetermined coefficients to find the particular solution $x_p(t)$ of:*

$$L[x] = mx'' + kx = \underbrace{F_0 \cos \omega t}_{f(t)}. \quad (1)$$

- The *trial* particular solution is:

$$\boxed{x_p(t) = A\cos \omega t.}$$

$x_c(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$

$\omega \neq \omega_0.$

- **Note:** No sine term is needed in x_p because there is no term involving x' on the L.H.S.

  of (1).

- This gives

$$L[x_p] = m(-\omega^2 A\cos\omega t) + kA\cos\omega t = F_0 \cos \omega t.$$

- So,

$$-m\omega^2 A + kA = F_0.$$

$$A = \frac{F_0}{k - m\omega^2} = \frac{F_0/m}{\underbrace{\frac{k}{m}}_{=: \omega_0^2} - \omega^2}.$$

- Particular solution:

$$\boxed{x_p(t) = \frac{F_0/m}{\omega_0^2 - \omega} \cdot \cos \omega t.}$$

Example 4. *Find the solution $x(t) = x_c(t) + x_p(t)$ of the following initial value problem:*

$$L[x] := x'' + 4x = \underbrace{5\sin 3t}_{F(t)}, \qquad x(0) = 0, x'(0) = 0.$$

- The *complementary* solution:

$$r^2 + 4 = 0 \to r_{1,2} = \pm 2i$$

$$\boxed{x_c(t) = c_1 \cos 2t + c_2 \sin 2t.}$$

- The trial *particular* solution:

$$\boxed{x_p(t) = A \sin 3t.}$$

- $L[x_p] = 5\sin 3t$ gives:

$$-A9 \sin 3t + 4A \sin 3t = 5 \sin 3t$$

$$\Rightarrow -5A = 5 \Rightarrow A = -1.$$

$$\boxed{x_p(t) = -\sin 3t.}$$

- The *general* solution $x = x_c + x_p$ is:

$$\boxed{x(t) = c_1 \cos 2t + c_2 \sin 2t - \sin 3t.}$$

- Using the ICs, we find c_1 and c_2:

$x(0) = x'(0) = 0.$

$$0 = x(0) = c_1 \Rightarrow c_1 = 0.$$

$$0 = x'(0) = 2c_2 - 3 \Rightarrow c_2 = \frac{3}{2}.$$

$$\boxed{x(t) = \frac{3}{2} \sin 2t - \sin 3t.}$$

Damped Forced Oscillations

Damped Forced Oscillations

$F(t) = F_0 \sin \omega t$
$\Rightarrow$ analysis same.

- Consider the external force $\boxed{F(t) = F_0 \cos \omega t}$ and let $\boxed{c \neq 0.}$ Then, we have:

$$mx'' + cx' + kx = F_0 \cos \omega t,$$

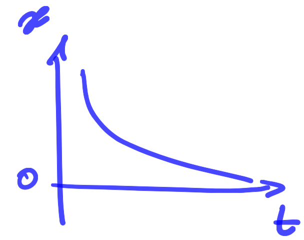
- The *complementary* function takes one of the three forms depending on:

$$c > \boxed{c_{cr} := \sqrt{4km},}\ c = c_{cr}, \text{ or } c < c_{cr}.$$

Transient solution

- In our previous lectures, we demonstrated that:

$$\boxed{x_c(t) \to 0 \text{ as } t \to +\infty.}$$



- Thus, $x_c(t)$ is the *transient* solution of the *damped forced motion.*
- $\implies$ $x_c(t)$ dies out with the passage of time.

Particular solution (steady oscillatory solution)

- The *particular* function is:

$$x(t) = A\cos \omega t + B \sin \omega t$$

where

$$A = \frac{(k - m\omega^2)\, F_0}{(k - m\omega^2)^2 + (c\omega)^2}, \quad B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}.$$

- We can show that the resulting $x_p(t)$ corresponds to the steady periodic oscillation:

$$x_p(t) = C \cos(\omega t - \alpha)$$

has amplitude $C = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}$

- Phase angle α:

$\alpha \in [0, \pi].$

$$\alpha = \begin{cases} \tan^{-1} \frac{c\omega}{k-m\omega^2} & \text{if } k > m\omega^2, \\ \pi + \tan^{-1} \frac{c\omega}{k-m\omega^2} & \text{if } k < m\omega^2 \end{cases}$$

$\tan^{-1} x \in [-\pi/2, \pi/2]$

Example 5. *Find the particular solution $x_p(t)$ of:*

$$mx'' + cx' + kx = F_0 \cos \omega t.$$

- The method of undetermined coefficients indicates $\Longrightarrow$ the *trial* particular function:

$$\boxed{x_p(t) = A\cos \omega t + B \sin \omega t}$$

- Replacing $L[x_p] = F_0 \cos \omega t$ gives:

$$\left((k - m\omega^2)A + c\omega B\right)\cos \omega t + \left(-c\omega A + (k - m\omega^2)B\right)\sin \omega t = F_0 . \cos \omega t.$$

- Two equations:

$$i)\ (k - m\omega^2)A + c\omega B = F_0.$$
$$ii)\ -c\omega A + (k - m\omega^2)B = 0.$$

- The undetermined coefficients:

$$A = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (c\omega)^2}, \quad B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2} > \underline{0}.$$

- If we write:

$$A\cos \omega t + B \sin \omega t = C(\cos \alpha \cos \omega t + \sin \alpha \sin \omega t)$$

$$\cos \alpha = \frac{A}{C}; \quad \sin \alpha := \frac{B}{C}$$

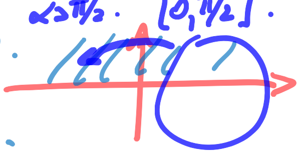
- Results in the *steady* periodic oscillation: $\boxed{x_p(t) = C\cos(\omega t - \alpha)}$

$$C = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} > 0.$$

$$\tan \alpha = \frac{B}{A} = \frac{c\omega}{k - \omega^2 m} > 0$$

$F_0 > 0 \quad c > 0 \quad \omega > 0.$

$\sin \alpha > 0.$ $\quad \alpha \geq \pi/2.$ $\quad [0, \pi/2].$



$\alpha \in [0, \pi].$

$$\Rightarrow \alpha = \begin{cases} \tan^{-1} \dfrac{c\omega}{k - m\omega^2} & k - \omega^2 m > 0. \\ \pi + \tan^{-1} \dfrac{c\omega}{k - \omega^2 m}. & k - \omega^2 m < 0. \end{cases}$$

$\tan^{-1} x \in [-\pi/2, \pi/2].$

MA 266 Lecture 21

Christian Moya, Ph.D.

Sec 3.6 Forced Oscillations

Damped Forced Oscillations

- Consider the external force $F(t) = F_0 \cos \omega t$ and let $c \neq 0$. Then, we have:

$$mx'' + cx' + kx = F_0 \cos \omega t,$$

Transient solution

- In our previous lectures, we demonstrated that:

$$x_c(t) \to 0 \text{ as } t \to +\infty.$$

- Thus, $x_c(t)$ is the *transient* solution of the *damped forced motion*.
- $\implies$ $x_c(t)$ dies out with the passage of time.

Particular solution

- The *particular* function is:

$$x(t) = A \cos \omega t + B \sin \omega t$$

where

$$A = \frac{(k - m\omega^2) F_0}{(k - m\omega^2)^2 + (c\omega)^2}, \quad B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}.$$

- We can show that the resulting $x_p(t)$ corresponds to the steady periodic oscillation:

$$x_p(t) = C \cos(\omega t - \alpha)$$

has amplitude $C = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$

- Phase angle α:

$$\alpha = \begin{cases} \tan^{-1} \frac{c\omega}{k - m\omega^2} & \text{if } k > m\omega^2, \\ \pi + \tan^{-1} \frac{c\omega}{k - m\omega^2} & \text{if } k < m\omega^2 \end{cases}$$

Example 1. *Find the steady state periodic solution of the differential equation:*

$$L[x] = x'' + 3x' + 5x = \underbrace{-4\cos 5t}_{F(t)}. \qquad \omega = 5 \neq \omega_0$$

- Trial particular solution:

$$\boxed{x_{sp}(t) = A\cos 5t + B\sin 5t.}$$

- Replacing into the differential equation gives: $L[x_{sp}] = -4\cos 5t$

$x_{sp}' = -5A\sin 5t + 5B\cos 5t.$

$x_{sp}'' = -25A\cos 5t - 25B\sin 5t.$

$(-25A + 15B + 5A)\cos 5t + (-25B - 15A + 5B)\sin 5t = -4\cos 5t.$

i) $-20A + 15B = -4$; ii) $20B + 15A = 0$

- The undetermined coefficients are:

$$A = \frac{16}{125}; \quad B = -\frac{12}{125}.$$

- Write as *steady* periodic solution:

$$x_{sp}(t) = C\cos(5t - \alpha) \qquad \alpha = \tan^{-1}\left(\frac{B}{A}\right)$$

$\dfrac{\cancel{-12}\ 3}{\cancel{125}} \Big/ \dfrac{\cancel{16}\ 4}{\cancel{125}}$

- Amplitude and angle: $C = \sqrt{A^2 + B^2} = \dfrac{4}{25} > 0.$

$\alpha = ??$

$\sin\alpha = \dfrac{B}{C} < 0$

$\cos\alpha = \dfrac{A}{C} > 0$

4th quadrant angle

$\alpha \in [3\pi/2, 2\pi].$

$\alpha = 2\pi - \tan^{-1}\left(\dfrac{3}{4}\right) \approx 5.6397$

$< 2\pi$

$> 3\pi/2.$

$$\boxed{x_{sp}(t) = \frac{4}{25}\cos(5t - 5.6397).}$$

Example 2. *Find the transient solution and the steady periodic solution of the initial value problem:*

$\omega = 1 \neq \omega_0$

$$L[x] := x'' + 8x' + 25x = \underbrace{200\cos t + 520\sin t}_{=: F(t)}, \qquad x(0) = -30, x'(0) = -10.$$

- Steady periodic solution:

$$\boxed{x_{sp}(t) = A\cos t + B\sin t.}$$

$L[x_{sp}] = 200\cos t + 520\sin t. \implies$

$(24A + 8B)\cos t + (-8A + 24B)\sin t = 200\cos t + 520\sin t.$

i) $24A + 8B = 200$

ii) $-8A + 24B = 520$

$\implies \begin{cases} A = 1 \\ B = 22. \end{cases}$

So,

$$x_{sp}(t) = \cos t + 22\sin t.$$
$$= C\cos(t - \alpha)$$

$C = \sqrt{A^2 + B^2} = \sqrt{485}.$

$\alpha = ??$ i) $\sin\alpha = B/C > 0$, $\cos\alpha = A/C > 0. \implies \alpha$: first quadrant angle. $\alpha \in [0, \pi/2].$

$$\alpha = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}(22) = 1.53. < \frac{\pi}{2}.$$

$$\boxed{x_{sp}(t) = \sqrt{485}\cos(t - 1.53).}$$

$$x'' + 8x' + 25x = F(t)$$

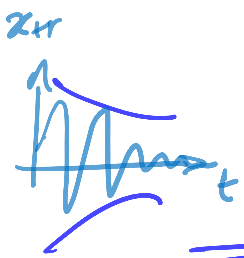
- Transient solution:

$$r^2 + 8r + 25 = 0.$$

$$r_{1,2} = -4 \pm 3i.$$

$$\Rightarrow \{e^{-4t}\cos 3t,\ e^{-4t}\sin 3t\}.$$

$$\boxed{x_c(t) = e^{-4t}\left(c_1\cos 3t + c_2 \sin 3t\right).}$$



$$\Longrightarrow \quad x(t) = x_{tr}(t) + x_{sp}(t)$$

$$x(t) = \underbrace{e^{-4t}\left(c_1\cos 3t + c_2 \sin 3t\right)}_{=:\, x_{tr}(t)} + \underbrace{\cos t + 22\sin t}_{=:\, x_{sp}(t)}$$

To find c_1 & c_2, we use ICs:

$$x(0) = -30 \quad ; \quad x'(0) = -10.$$

$$\Rightarrow \quad -30 = x(0) = c_1 + 1 \quad \Rightarrow \quad c_1 = -31.$$

$$-10 = x'(0) = -4c_1 + 3c_2 + 22 \quad \Rightarrow \quad c_2 = -52.$$

$$\Rightarrow \quad x_{tr}(t) = e^{-4t}(\underbrace{-31\cos 3t - 52\sin 3t}_{= D\cos(3t-\beta)}).$$

$$D = \sqrt{c_1^2 + c_2^2} = \sqrt{3665}$$

$\beta = ??$ i) $\sin\beta = \frac{c_2}{D} < 0$, $\cos\beta = \frac{c_1}{D} < 0$ $\Rightarrow$ β is 3rd quadrant angle $\beta \in [\pi, 3\pi/2]$.

$$\beta = \pi + \tan^{-1}\left(\frac{c_2}{c_1}\right) = \pi + \tan^{-1}\left(\frac{52}{31}\right) \approx 4.1748 > \pi$$

$$\boxed{x(t) = e^{-4t}\left(\sqrt{3665}\cos(3t - 4.1748)\right) + x_{sp}(t).}$$

Resonance - Forced Undamped Oscillations $(c = 0)$ $\quad \underline{\omega \neq \omega_0}$

- Allowing $\underline{\omega}$ to approach $\underline{\omega_0}$.
- Recall the particular solution:

$$\boxed{x_p(t) = \underbrace{\frac{F_0/m}{\omega_0^2 - \omega^2}}_{=: A.} \cos \omega t.}$$

- If ω becomes approximately equal to ω_0, the amplitude A of x_p becomes *large*.
- It is sometimes useful to rewrite $x_p(t)$ in the form:

$$A = \frac{F_0}{k - m\omega^2} = \frac{F_0/k}{1 - (\omega/\omega_0)^2} = \pm \rho \frac{F_0}{k}.$$

- $\underline{F_0/k}$ is the *static displacement* of a spring with k due to a *constant* force F_0.
- $\underline{\rho}$ is the *amplification factor* defined as:

$$\rho = \frac{1}{|1 - (\omega/\omega_0)^2|} \qquad \Rightarrow \rho \to \infty \text{ as } \omega \to \omega_0 \qquad \Rightarrow A \to \infty.$$

> The phenomenon of *resonance*—the increase without bound (as $\omega \to \omega_0$) in the amplitude of oscillations of an undamped system with natural frequency ω_0 in response to an external force with frequency $\omega \approx \omega_0$.

Example 3. ***Pure Resonance*** - *Find the particular solution $x_p(t)$ of the following undamped system:*

$$(1) \qquad x'' + \omega_0^2 x = \underbrace{\frac{F_0}{m}}_{=:F(t;\,\omega_0)} \cos \omega_0 t.$$

- *Complementary solution:*

$$r^2 + \omega_0^2 = 0 \implies r_{1,2} = \pm \omega_0 i.$$

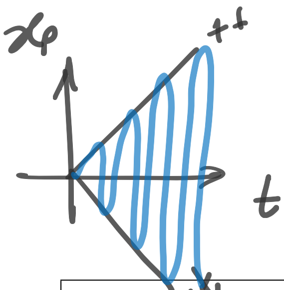
$$\boxed{x_c(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.}$$

- *Trial Particular solution:*

$$x_p(t) = (A \cos \omega_0 t.)\, t. \quad \underline{NO}$$

$$\boxed{x_p(t) = t\,(A \cos \omega_0 t + B \sin \omega_0 t)}.$$

$x_p \to (1) \iff \quad x_p'' + \omega_0 x_p = \dfrac{F_0}{m} \cos \omega_0 t.$



$A = 0.$

$B = \dfrac{F_0}{2m\omega_0}. \implies \boxed{x_p(t) = \dfrac{F_0}{2m\omega_0} t \sin \omega_0 t.}$

$t \to \infty.$

Bridge Crossings and Resonance

- *In practice, a mechanical system with very little damping can be destroyed by resonance vibrations.*
- *Any complicated structure such as a bridge has many natural frequencies of vibration.*
- *The resulting resonance vibrations can be of such large amplitude that the bridge will collapse.*

MA 266 Lecture 22

Christian Moya, Ph.D.

Practical Resonance - Forced Damped Oscillations

- Consider the damped system:

$$\boxed{mx'' + cx' + kx = F_0 \cos \omega t}$$

- Note that if $c > 0$, then the "forced amplitude" $C(\omega)$:

$$\boxed{C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} < +\infty,}$$

  always remains *finite*.

- However, the forced amplitude may attain a maximum for some value of ω, in which case we speak of *practical resonance*.

Example 1. *Show that if $c \geq \sqrt{2km}$ the amplitude $C(\omega)$ decreases for all $\omega > 0$; otherwise $C(\omega)$ attains a maximum value.*

$c < \sqrt{2km}$.

1. Use $C'(\omega)$:

$$\boxed{C'(\omega) = -\frac{\omega F_0}{2} \cdot \frac{(c^2 - 2km) + 2m\omega^2}{\underbrace{[(k - m\omega^2)^2 + (c\omega)^2]^{3/2}}_{>0}}}$$

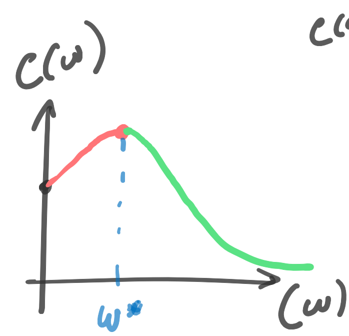
2. If $c \geq \sqrt{2km}$: $\Leftrightarrow$ $c^2 \geq 2km$.

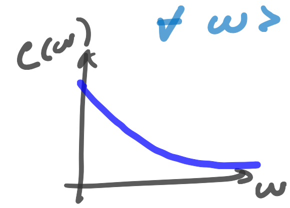
$C'(\omega) \gtreqless 0.$

$$-\underbrace{\omega F_0}_{>0} \cdot \left(\underbrace{(c^2 - 2km)}_{>0} + \underbrace{2m\omega^2}_{>0}\right) < 0 \Rightarrow C'(\omega) < 0 \ \forall\, \omega > 0.$$

3. But if $c < \sqrt{2km}$: $\Leftrightarrow$ $c^2 < 2k \cdot m$

$$-\underbrace{\omega F_0}_{>0} \left(\underbrace{\underbrace{(c^2 - 2km)}_{<0} + \underbrace{2m\omega^2}_{>0}}_{<0,\ >0}\right)$$





ω^*: practical resonance freq : $\boxed{C'(\omega^*) = 0.}$

Example 2. *Find the amplitude $C(\omega)$ and the practical resonance frequency ω of the following forced mass-spring-dashpot system:*

$$x'' + 10x' + 650x = 100\cos\omega t. \qquad (1)$$

- Particular solution:

$$\boxed{x_p(t) = A\cos\omega t + B\sin\omega t.}$$

$x_p, x_p', x_p'' \longrightarrow (1)$

- Solving for the coefficients A and B

$$\begin{cases} (650-\omega^2)A + 10\omega B = 100. \\ -10\omega A + (650-\omega^2)B = 0. \end{cases}$$

$$\Rightarrow \quad \boxed{A = \frac{100(650-\omega^2)}{422500 - 1200\omega^2 + \omega^4}} \qquad \boxed{B = \frac{1000\omega}{422500 - 1200\omega^2 + \omega^4}}$$

- The amplitude $C(\omega)$ of the steady periodic forced oscillations with freq. ω :

$$C(\omega) = \sqrt{A(\omega)^2 + B(\omega)^2}$$

$$\boxed{C(\omega) = \frac{100}{(422500 - 1200\omega^2 + \omega^4)^{1/2}}.}$$

- Find the practical resonance by solving $C'(\omega) = 0$:

$$\boxed{C'(\omega) = \frac{-200\,\omega(-600+\omega^2)}{(\underbrace{422500 - 1200\omega^2 + \omega^4}_{>0})^{3/2}}.}$$

Hence,

practical res. frequency ω^*: $C'(\omega^*) = 0$

$\Longleftrightarrow \quad -600 + \omega^{*2} = 0.$

$\omega^* = \sqrt{600}$

Max. amplitude $C(\omega^*)$ for prac. resonance