MA 266 Lecture 24

Christian Moya, Ph.D.

Sec 5.1 Matrices and Linear Systems

Review of Matrices

- A matrix A with m rows and n columns can be written as:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \overset{\text{notation}}{=} [a_{ij}].$$

- The **transpose** of A $\triangleq A^T$ $(n \times m)$

A is $(m \times n)$

$$A^T := [a_{ji}].$$

Example 1. *Let*

$$A = \begin{pmatrix} 3 & 2-i \\ 4+3i & -5+2i \end{pmatrix}$$ (2×2) complex matrix

Then $A^\top = ?$

Q. dim A^T? (2×2)

$$A^T = \begin{pmatrix} 3 & 4+3i \\ 2-i & -5+2i \end{pmatrix}$$

Properties of Matrices

- **Equality**: $A = B$ if and only if

$(m \times n)$

$$[a_{ij}] = [b_{ij}] \quad \forall i, j.$$

- **Addition**: $A + B =$ $[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] = C$

$$= [c_{ij}].$$

- **Scalar Multiplication**: $\alpha A =$ $\alpha [a_{ij}] = [\alpha a_{ij}] = C$

$$= [c_{ij}].$$

$\alpha \in \mathbb{R}.$

Vectors

- A **row** vector u $(1 \times n)$- matrix

$$u = (u_1, u_2, \ldots, u_n)$$

- A **column** vector v $(n \times 1)$- matrix

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = (v_1, v_2, \ldots, v_n)^T$$

- Convenient to describe an $m \times n$ matrix in terms of either its m row vectors or its n column vectors:

$$A = \begin{pmatrix} — a_1 — \\ — a_2 — \\ \vdots \\ — a_m — \end{pmatrix} \qquad B = \begin{pmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{pmatrix}$$

$A (m \times n)$ $\qquad$ $(m \times n)$

a_i : row vector. $\qquad$ b_j : is a column vector

Q: size of a_i? $(1 \times n)$ $\qquad$ Q: dim. of b_j? $(m \times 1)$

Scalar or Dot Product

- If

$$a = (a_1, a_2, \ldots, a_p) \quad (1 \times p) \quad \wedge \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{pmatrix} \quad (p \times 1)$$

then

$$\boxed{\underset{(1,1)}{c} = \underset{(1\times p)}{a} \cdot \underset{(p\times 1)}{b} = \sum_{j=1}^{p} a_j b_j.}$$

Product of Matrices

- If **A** is an $m \times p$ matrix and **B** is a $p \times n$ matrix, then their *product* is the $m \times n$ matrix:

$$\underset{(m\times n)}{C} = \underset{(m\times p)}{A}\,\underset{(p\times n)}{B} = [a_i \cdot b_j] = [c_{ij}].$$

- Visualizing **AB** :

$$\mathbf{a}_i \longrightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix}.$$

$$\uparrow \mathbf{b}_j$$

$$c_{ij} = a_i \cdot b_j$$

Example 2. *Let* $A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & -2 & -1 \end{pmatrix}$ *and* $y = \begin{pmatrix} e^t \\ \cos 3t \\ \sin 4t \end{pmatrix}$. *Find* $Ay = ?$

$(2\times 3) \quad (3\times 1)$

$$\underset{(2,1)}{C} = \begin{pmatrix} 1 & 0 & -2 \\ 3 & -2 & -1 \end{pmatrix} \begin{pmatrix} e^t \\ \cos 3t \\ \sin 4t \end{pmatrix} = \begin{pmatrix} e^t - 2\sin 4t \\ 3e^t - 2\cos 3t - \sin 4t \end{pmatrix}$$

- **Identity**: The **identity** matrix I is defined as (of order n) $(n \times n)$

$$I := \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

- **Inverse**: The matrix A is called **nonsingular** or **invertible** if $\underset{(n \times n)}{A}$.

$\exists!$ (unique) A^{-1}. s.t $\boxed{A^{-1}A = I = AA^{-1}}$.

Note if $A^{-1} \Rightarrow (A^{-1})^{-1} = A$. ■

- **Determinant:** The **determinant** of a (2×2) matrix is defined as $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\boxed{|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \cdot d - b \cdot c}$$

if $|A| \neq 0$: $A^{-1} = \frac{1}{|A|} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Matrix Functions

We consider vectors or matrices whose elements are functions of real variable t, i.e.,

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad \underset{(m \times n)}{A(t)} = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{m1}(t) & \cdots & a_{mn}(t) \end{pmatrix} =: [a_{ij}(t)].$$

The derivative of a matrix function is defined by

$A(m \times n)$ $$A'(t) \triangleq \frac{dA}{dt} = \left[\frac{da_{ij}}{dt}\right]$$ $\frac{dA}{dt}$ dim: $(m \times n)$

If A, B are matrix functions, and C is a constant matrix, then

$$\frac{d}{dt}(CA) = C \cdot \frac{dA}{dt}. \qquad \frac{d}{dt}(A + B) = \frac{dA}{dt} + \frac{dB}{dt}. \qquad \frac{d}{dt}(AB) = A \cdot \frac{dB}{dt} + \frac{dA}{dt} B.$$

$(m \times n)$ $(m \times p)$ $(p \times n)$

First-Order Linear Systems

- Consider a general system of n first-order linear equations

$$\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + f_1(t),\\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + f_2(t),\\
x_3' &= p_{31}(t)x_1 + p_{32}(t)x_2 + \cdots + p_{3n}(t)x_n + f_3(t),\\
&\vdots\\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + f_n(t).
\end{aligned}$$

- We introduce the *coefficient* matrix:

$$P(t) = [p_{ij}(t)].$$

and the column vectors:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

- Then the above system takes the form of a *single* matrix equation:

$$\boxed{\underset{(n\times 1)}{\frac{dx}{dt}} = \underset{(n\times n)}{P(t)} \cdot \underset{(n\times 1)}{x} + \underset{(n\times 1)}{f(t)}}$$

Example 3. *Write the given system in the form* $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$,

$$x_1' = 3x_1 - 4x_2 + x_3 + t, x_2' = x_1 - 3x_3 + t^2, x_3' = 6x_2 - 7x_3 + t^3.$$

Sol'n

$$\underset{(3\times 1)}{x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} \qquad P = \begin{pmatrix} 3 & -4 & 1 \\ 1 & 0 & -3 \\ 0 & 6 & -7 \end{pmatrix} \qquad f(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}.$$

■

$$\frac{dx}{dt} = P(t)x + f(t).$$

- A **solution** of our system on some open interval I is a column vector function:

$$x(t) = [x_i(t)] = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

s.t $x(t)$ satisfies: $x' = P(t)x + f(t)$.

Example 4. *Verify that the given vector solutions satisfy the given linear system:*

$$\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -3 & 4 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}_1 = \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} 2e^{-2t} \\ e^{-2t} \end{pmatrix}.$$

$x' = P\, x$

x_1:

$$\checkmark \begin{pmatrix} 3e^{3t} \\ 9e^{3t} \end{pmatrix} = x_1' = \begin{pmatrix} -3 & 2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} = \begin{pmatrix} 3e^{3t} \\ 9e^{3t} \end{pmatrix} \checkmark$$

Yes!

x_1 is a sol'n.

x_2:

$$\checkmark \begin{pmatrix} -4e^{-2t} \\ -2e^{-2t} \end{pmatrix} = x_2' = \begin{pmatrix} -3 & 2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 2e^{-2t} \\ e^{-2t} \end{pmatrix} = \begin{pmatrix} -4e^{-2t} \\ -2e^{-2t} \end{pmatrix} \checkmark$$

Yes!

x_2 is a sol'n of L.S.

Associated Homogeneous Equation

- To investigate the general solutions of the (linear) matrix differential equation

$$\boxed{\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)x + \mathbf{f}(t),}$$

we consider first the *associated homogeneous equation*

$$\boxed{\frac{dx}{dt} = P(t)\,x.}$$

in which $f(t) = 0$.

Principle of Superposition

- Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n solutions of the homogeneous linear equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x}$$

$$\frac{dx_i}{dt} = P(t)x_i \quad \text{for } i = 1, \ldots, n.$$

on the open interval I.

- If $c_1, c_2, \ldots, c_n$ are constants, then the linear combination

$$(1) \quad \boxed{x(t) = c_1 x_1(t) + c_2 x_2(t) + \ldots + c_n x_n(t)}$$

is also a solution of the homogeneous linear equation on I.

Why the principle of superposition is true?

Show (1) $x(t)$ is a sol'n of $x' = P(t)\,x$.

$$x'(t) = c_1 x_1'(t) + \ldots + c_n x_n'(t)$$
$$= c_1 P(t) x_1(t) + c_2 P(t) x_2(t) + \ldots + c_n P(t) x_n(t)$$
$$= P(t) \cdot \underbrace{\left(c_1 x_1(t) + \cdots + c_n x_n(t)\right)}_{x(t)}$$

$$\boxed{x'(t) = P(t)\,x.} \Rightarrow x \text{ is a sol'n.}$$

Linear Independence

- The vector-valued functions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are *linearly dependent* on the interval I provided that there exist constants $c_1, c_2, \ldots, c_n$ *not all zero* such that

$$\boxed{c_1 x_1(t) + \ldots + c_n x_n(t) = 0.}$$

  for all t in I.

- Otherwise, they are *linearly independent*.

$$c_1 = c_2 = \ldots = c_n = 0.$$

To tell whether or not n given solution s of the associated homogeneous equation are linearly dependent, we can use the *Wronksian Determinant*.

Wronksian Determinant

- If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are such solutions, then their *Wronskian* is the $n \times n$ determinant

$$\boxed{W(t) \overset{\text{not}}{=} W(x_1, \ldots, x_n) = \begin{vmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{vmatrix}}$$

Theorem - Wronksian of Solutions

- Suppose that $\mathbf{x}_1$, $\mathbf{x}_2$, $\ldots$, $\mathbf{x}_n$ are n solutions of the homogeneous linear equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an open interval I.
- Let
$$W = W(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n).$$
- Then:
  - If $\mathbf{x}_1$, $\mathbf{x}_2$, $\ldots$, $\mathbf{x}_n$ are linearly dependent on I, then $W=0 \ \forall t \in I$.
  - If $\mathbf{x}_1$, $\mathbf{x}_2$, $\ldots$, $\mathbf{x}_n$ are linearly independent on I, then $W \neq 0 \ \forall t \in I$.

General Solution of Homogeneous Systems

Theorem - General Solutions of Homogeneous Systems

- Let $\mathbf{x}_1$, $\mathbf{x}_2$, $\ldots$, $\mathbf{x}_n$ be n linearly independent solutions of the homogeneous linear equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an open interval I, where $\mathbf{P}(t)$ is continuous.
- If $\mathbf{x}(t)$ is any solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on I, then there exist numbers c_1, c_2, $\ldots$, c_n such that

$$\boxed{x(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_n x_n(t)}$$

  for all t in I.

Example 5. *Use the Wronksian to show that the following solutions are linearly independent. Then, write the general solution.*

$$\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -3 & 4 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}_1 = \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} 2e^{-2t} \\ e^{-2t} \end{pmatrix}.$$

a) $W(t) = ?$

$$W(t) = \begin{vmatrix} e^{3t} & 2e^{-2t} \\ 3e^{3t} & e^{-2t} \end{vmatrix} = e^{t} - 6e^{t}$$

$$= -5\underbrace{e^{t}}_{>0} \neq 0.$$

$\Rightarrow$ x_1 and x_2 are linearly indep.

b) general solution:

$$x(t) = c_1 x_1(t) + c_2 x_2(t)$$

$$= c_1 \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} + c_2 \begin{pmatrix} 2e^{-2t} \\ e^{-2t} \end{pmatrix}$$

MA 266 Lecture 25

Christian Moya, Ph.D.

Sec 5.1 Linear Systems

Conclusion - Theorem: General Solutions of Homogeneous Linear Systems

- It suffices to find ___n___ *linearly independent* solution vectors:

$$x_1, x_2, \ldots, x_n.$$

- The linear combination

$$\boxed{x(t) = c_1 x_1(t) + c_2 x_2(t) + \ldots + c_n x_n(t)}$$

with arbitrary coefficients will then be a general solution of our system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

Example 1. *Write a general solution of the following Linear System:*

$$\mathbf{x}' = \underbrace{\begin{pmatrix} -3 & 2 \\ -3 & 4 \end{pmatrix}}_{=:A} \mathbf{x}.$$

- *The general solution obtained was:*

$$x(t) = c_1 \underbrace{\begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix}}_{x_1} + c_2 \underbrace{\begin{pmatrix} 2e^{-2t} \\ e^{-2t} \end{pmatrix}}_{x_2.}.$$

Example 2. *Find a particular solution of the following Linear System Initial Value Problem (IVP):*

$$\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -3 & 4 \end{pmatrix} \mathbf{x}, \qquad x_1(0) = 0, x_2(0) = 5.$$

- *Recall the general solution is:*

$$x(t) = c_1 \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} + c_2 \begin{pmatrix} 2e^{-2t} \\ e^{-2t} \end{pmatrix}.$$

$$\Leftrightarrow x(t) = \begin{pmatrix} c_1 e^{3t} + c_2 2e^{-2t} \\ c_1 3e^{3t} + c_2 e^{-2t} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} e^{3t} & 2e^{-2t} \\ 3e^{3t} & e^{-2t} \end{pmatrix}}_{=: X(t)} \underbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}_{c.}$$

$$\Rightarrow \boxed{x(t) = X(t) \cdot c}$$

- *Use the initial conditions:* $x_1(0) = 0$ and $x_2(0) = 5$

$$\begin{pmatrix} 0 \\ 5 \end{pmatrix} = x(0) = X(0)c$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} c_1 = 2. \\ c_2 = -1 \end{cases}$$

soln IVP: $\boxed{x(t) = 2 \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} - 1 \begin{pmatrix} 2e^{-2t} \\ e^{-2t} \end{pmatrix}}$

Sec 5.2 Eigenvalue Method for Homogeneous Systems

with matrix of constant coeff.

Q: How to find the n needed *linearly independent* solution vectors?

- To find these *linearly independent* solution vectors, we proceed by analogy with the characteristic root method for solving a single homogeneous equation with constant coefficients.

e^{rt}

Form of the Solution Vectors

- It is reasonable to anticipate solution vectors of the form

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \\ \vdots \\ v_n e^{\lambda t} \end{pmatrix} = \underbrace{\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}}_{=: v} e^{\lambda t} = v e^{\lambda t}$$

where $\lambda, v_1, v_2, \ldots, v_n$. are appropriate scalar constants.

Matrix Form

- Consider the Homogeneous System in matrix form:

$$\underset{(n \times n)}{x' = A}\, \underset{(n\times 1)}{x}.$$

- Verify the trial solution: $x(t) = v e^{\lambda t}$ $\quad x'(t) = \lambda v e^{\lambda t}$

$$\lambda v \underbrace{\cancel{e^{\lambda t}}}_{\neq 0} = A v \underbrace{\cancel{e^{\lambda t}}}_{\neq 0}.$$

- We cancel the nonzero scalar factor $e^{\lambda t}$ to get

$$\boxed{Av = \lambda v.} \qquad (1)$$

Conclusion:

our trial soln.

- $x(t) = v e^{\lambda t}$ will be a nontrivial solution of

$$x' = Ax.$$

provided that $\mathbf{v}$ is a *nonzero* vector and λ is a constant such that the *matrix product* $\mathbf{Av}$ *is a scalar multiple of the vector* $\mathbf{v}$. (1)

- The question now is this: How do we find $\mathbf{v}$ and λ?

Finding v and λ

- To answer this question, we rewrite the equation $\mathbf{Av} = \lambda\mathbf{v}$ in the form

$$\boxed{(A - \lambda I)v = 0}$$

- By *a standard theorem of linear algebra*, it has a nontrivial solution if and only if the determinant of its coefficient matrix vanishes; that is, if and only if

$$\boxed{|A - \lambda I| \overset{\text{not.}}{=} \det(A - \lambda I) = 0.}$$

The Eigenvalue Method

The Eigenvalue Method

- For solving the homogeneous system:

$$x' = Ax$$

- consists of finding $\underline{\quad\lambda\quad}$ so that

$$\det(A - \lambda I) = |A - \lambda I| = 0.$$

and next solving

$$(A - \lambda I)v = 0$$

with this value of $\underline{\quad\lambda\quad}$ to obtain $\underline{\quad v \quad}$.

- Then $\underline{x(t) = ve^{\lambda t}.}$ will be a solution vector.
- The name of the method comes from the following definition.

Definition Eigenvalues and Eigenvectors

- The number λ (either zero or nonzero) is called an **eigenvalue** of the $n \times n$ matrix $\mathbf{A}$ provided that

$$\det(A - \lambda I) = |\mathbf{A} - \lambda \mathbf{I}| = 0.$$

- An **eigenvector** associated with the eigenvalue λ is a *nonzero* vector $\mathbf{v}$ such that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, so that

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}.$$

Example 3. *Find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$$

(2×2)

- Eigenvalues:

$$\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix}$$

$$= (3-\lambda)(-2-\lambda) + 4$$

$$= \boxed{\lambda^2 - \lambda - 2 = 0.}$$

$$\boxed{\begin{cases} \lambda_1 = 2 \\ \lambda_2 = -1. \end{cases}}$$

$$\lambda I = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \\ 0 & 0 & \ddots & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

- Eigenvectors:

$$(A-\lambda I)v = 0$$

Case 1. $\lambda = 2$.

$$(A-2I)v = 0. \Leftrightarrow \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{array}{l} v_1 - v_2 = 0. \\ 4v_1 - 4v_2 = 0. \end{array}$$

Q:

singular system.
∞-solutions.

Simplest one. select $v_1 = 1 \Rightarrow v_2 = 1$

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$v = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Case 2: $\lambda = -1$.

$$v_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Characteristic Equation

- The equation

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0.$$

  is called the **characteristic equation** of the matrix **A**.

- Its roots are the eigenvalues of **A**.
- Expanding this determinant, we evidently get an nth-degree polynomial of the form

$$\boxed{(-1)^n\lambda^n + b_{n-1}\lambda^{n-1} + \cdots + b_1\lambda + b_0 = 0.}$$

- By the fundamental theorem of algebra, this equation has n roots.
- Possibly some are complex, possibly some are repeated.

- Thus an $(n \times n)$ A matrix has n eigenvalues (counting repetitions).

Outline of the Eigenvalue Method

Steps for the Eigenvalue Method

- In outline, this method for solving the $n \times n$ homogeneous constant-coefficient system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ proceeds as follows:

  $\det(A - \lambda I) = 0.$

  1. Solve the characteristic equation for the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix $\mathbf{A}$.
  2. Attempt to find n *linearly independent* eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ associated with these eigenvalues.
  3. Step 2 is not always possible, but when it is, we get n linearly independent solutions:

  $x_1(t) = v_1 e^{\lambda_1 t}, \quad x_2(t) = v_2 e^{\lambda_2 t}, \quad \ldots, \quad x_n(t) = v_n e^{\lambda_n t}.$

- In this case the *general solution* of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is a linear combination

  $$\boxed{x(t) = c_1 x_1(t) + \ldots + c_n x_n(t).}$$

  of these n solutions.

Distinct Real Eigenvalues

- If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real and distinct, then we substitute each of them in turn in the equation

  $$\boxed{(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}}$$

  and solve for the associated eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

- Then it can be proved that the particular solution vectors

  $$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}, \quad \mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}, \quad \ldots, \quad \mathbf{x}_n(t) = \mathbf{v}_n e^{\lambda_n t}.$$

  are always linearly independent.

Example 4. *Use the eigenvalue method to find the general solution of:*

$$x_1' = \underline{2x_1 + 3x_2},\ x_2' = 2x_1 + x_2.$$

- Linear system in matrix form: $x' = Ax.$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

- Characteristic equation:

$$\det(A - \lambda I) = \det\left(\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$$

$$= \det \begin{pmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix}$$

$$= (2-\lambda)(1-\lambda) - 6.$$

ch eq'n: $\boxed{\lambda^2 - 3\lambda - 4 = 0.}$

- Eigenvector equation:

$$(A - \lambda I)v = 0$$

$$\boxed{\begin{cases} \lambda_1 = -1 \\ \lambda_2 = 4. \end{cases}}$$

Case 1: $\lambda = -1.$

$$(A + I)v = 0. \iff \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

singular.

$$3a + 3b = 0.$$
$$2a + 2b = 0.$$

$a = 1 \Rightarrow b = -1.$

$$\boxed{v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}} \quad v_1 = c\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Case 2: $\lambda = 4 \Rightarrow \boxed{v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}}$

- General solution:

$$\boxed{x(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-1t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}.}$$

Example 5. *Use the eigenvalue method to find the general solution of:*

$$x_1' = 4x_1 + x_2,\ x_2' = 6x_1 - x_2.$$

- Linear system in matrix form: $x' = Ax.$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 4 & 1 \\ 6 & -1 \end{pmatrix}}_{=:A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- Characteristic equation:

$$\det(A - \lambda I) = \det \begin{pmatrix} 4-\lambda & 1 \\ 6 & -1-\lambda \end{pmatrix}$$

$$= (4-\lambda)(-1-\lambda) - 6.$$

ch. eq'n: $\boxed{\lambda^2 - 3\lambda - 10 = 0.}$ $\boxed{\begin{cases} \lambda_1 = 5 \\ \lambda_2 = -2. \end{cases}}$

- Eigenvector equation:

$$(A - \lambda I)v = 0$$

Case 1: $\lambda = 5$

$$\begin{pmatrix} -1 & 1 \\ 6 & -6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $-a + b = 0.$
$6a - 6b = 0.$

$$\boxed{V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.}$$

Case 2: $\lambda = -2.$

$$\begin{pmatrix} 6 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boxed{V_2 = \begin{pmatrix} 1 \\ -6 \end{pmatrix}}$$

- General solution:

$$\boxed{x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 1 \\ -6 \end{pmatrix} e^{-2t}.}$$

Example 6. *Find the solution of the IVP:*

$$x_1' = 9x_1 + 5x_2,\ x_2' = -6x_1 - 2x_2,\ x_1(0) = 1, x_2(0) = 0.$$

- Linear system in matrix form:

$$x' = \begin{pmatrix} 9 & 5 \\ -6 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

eigenvalues

- Characteristic equation:

$$\det(A - \lambda I) = \det \begin{pmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{pmatrix}$$

$$= (9-\lambda)(-2-\lambda) + 30$$

$$\Rightarrow \text{Ch. eq'n: } \boxed{\lambda^2 - 7\lambda + 12 = 0.}$$

$$\boxed{\begin{cases} \lambda_1 = 4 \\ \lambda_2 = 3. \end{cases}}$$

- Eigenvector equation:

$$(A - \lambda I)v = 0$$

Case 1 : $\lambda = 4$. $\begin{pmatrix} 5 & 5 \\ -6 & -6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\boxed{v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

Case 2: $\lambda = 3$ $\boxed{v_2 = \begin{pmatrix} 5 \\ -6 \end{pmatrix}}$.

$X(t)c.$

- General solution:

$$x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}.$$

$$x(t) = \begin{pmatrix} c_1 e^{4t} + 5c_2 e^{3t} \\ -c_1 e^{4t} - 6c_2 e^{3t} \end{pmatrix} = \begin{pmatrix} e^{4t} & 5e^{3t} \\ -e^{4t} & -6e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

- Particular solution:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = x(0) = X(0)c$$

$$= \begin{pmatrix} 1 & 5 \\ -1 & -6 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Rightarrow \begin{cases} c_1 = 6. \\ c_2 = -1. \end{cases}$$

$$\boxed{x(t) = 6\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} - \begin{pmatrix} 5 \\ -6 \end{pmatrix} e^{3t}.}$$

MA 266 Lecture 26

Christian Moya, Ph.D.

Sec 5.2 Eigenvalue Method for Homogeneous Systems

The Eigenvalue Method

- To solve the $n \times n$ homogeneous constant-coefficient linear system:

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

1. Solve the characteristic equation

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0.$$

for the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix $\mathbf{A}$.

2. Attempt to find n *linearly independent* eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ associated with these eigenvalues using

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}.$$

3. Step 2 is not always possible, but when it is, we get n linearly independent solutions:

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}, \mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}, \ldots, \mathbf{x}_n(t) = \mathbf{v}_n e^{\lambda_n t}.$$

- In this case, the *general solution* of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is a linear combination

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) + \ldots + c_1 \mathbf{x}_n(t).$$

of these n solutions.

Distinct Real Eigenvalues

- If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are **real and distinct**, then we substitute each of them in turn in the equation

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}$$

and solve for the associated eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$.

- Then, the particular solution vectors

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}, \quad \mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}, \quad \ldots, \quad \mathbf{x}_n(t) = \mathbf{v}_n e^{\lambda_n t}.$$

are always **linearly independent**.

Example 1. *Find the solution of the IVP:*

$$x_1' = 9x_1 + 5x_2, \; x_2' = -6x_1 - 2x_2, \; x_1(0) = 1, x_2(0) = 0.$$

Solution

- The matrix form of the system is

$$\mathbf{x}' = \underbrace{\begin{pmatrix} 9 & 5 \\ -6 & -2 \end{pmatrix}}_{=:\mathbf{A}} \mathbf{x}.$$

- The characteristic equation of the coefficient matrix is

$$\begin{aligned} \det \begin{pmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{pmatrix} &= (9-\lambda)(-2-\lambda) + 30 \\ &= \lambda^2 - 7\lambda + 12 = (\lambda - 4)(\lambda - 3) = 0, \end{aligned}$$

so we have the *distinct real* eigenvalues $\lambda_1 = 4$ and $\lambda = 3$.

- For the coefficient matrix $\mathbf{A}$, the eigenvector equation $(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}$ is

$$\begin{pmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- **Case 1:** $\lambda_1 = 4$. Substitution of the first eigenvalue $\lambda_1 = 4$ in $(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}$ yields

$$\begin{pmatrix} 5 & 5 \\ -6 & -6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- The choice $a = 1$ yields $b = -1$, and thus $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

- **Case 2:** $\lambda_1 = 3$. Exercise - Answer: $\mathbf{v}_2 = \begin{pmatrix} 5 \\ -6 \end{pmatrix}$

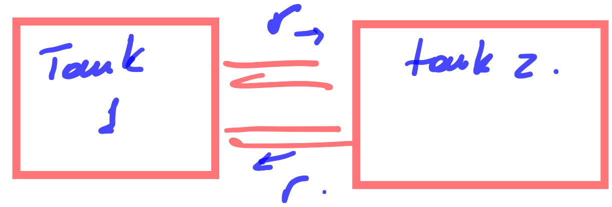
- The *general solution*:
$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}$$
therefore takes the form:
$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 5 \\ -6 \end{pmatrix} e^{3t}$$

- The resulting scalar equations are
$$\begin{aligned} x_1(t) &= c_1 e^{4t} + 5c_2 e^{3t} \\ x_2(t) &= -c_1 e^{4t} - 6c_2 e^{3t}. \end{aligned}$$
$\Longleftrightarrow x(t) = X(t)c.$

- When we impose the initial conditions $x_1(0) = 1$ and $x_2(0) = 0$, we get

$\Longleftrightarrow x(0) = X(0)c.$
$$\boxed{\begin{pmatrix} 1 & 5 \\ -1 & -6 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.}$$
that is readily solved (in turn) for $c_1 = 6$ and $c_2 = -1$. Thus, finally, the solution of the initial value problem is:
$$\mathbf{x}(t) = 6 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} - \begin{pmatrix} 5 \\ -6 \end{pmatrix} e^{3t}$$
or equivalently
$$\begin{aligned} x_1(t) &= 6e^{4t} - 5e^{3t} \\ x_2(t) &= -6e^{4t} + 6e^{3t}. \end{aligned}$$

$x_1(t)$
$\sim$
V_1



Example 2. *The amounts $x_1(t)$ and $x_2(t)$ of salt in two brine tanks satisfy the differential equations*

$x_2(t)$
V_2.

$$\frac{dx_1}{dt} = -k_1x_1 + k_2x_2,$$
$$\frac{dx_2}{dt} = k_1x_1 - k_2x_2,$$

where

$$k_i = \frac{r}{V_i}, \qquad i = 1, 2.$$

Find the general solution assuming that $r = 10$ (gal/min), $V_1 = 25$ (gal), and $V_2 = 40$ (gal).

Solution

- If $r = 10$ (gal/min), $V_1 = 25$ (gal), and $V_2 = 40$ (gal), then

$$k_1 = \frac{r}{V_1} = \frac{10}{25} = \frac{2}{5} \; ; \quad k_2 = \frac{r}{V_2} = \frac{10}{40} = \frac{1}{4}$$

- The matrix form of the system is $x' = Ax$.

$$A = \begin{pmatrix} -2/5 & 1/4 \\ 2/5 & -1/4 \end{pmatrix}$$

- The characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} -2/5 - \lambda & 1/4 \\ 2/5 & -1/4 - \lambda \end{vmatrix} = \left(-\frac{2}{5} - \lambda\right)\left(-\frac{1}{4} - \lambda\right) - \frac{1}{10}$$

$$\boxed{= \lambda^2 + \frac{13}{20}\lambda = 0}$$

$$= \lambda\left(\lambda + \frac{13}{20}\right) = 0.$$

- Thus, the coefficient matrix $\mathbf{A}$ has

$$\boxed{\lambda_1 = 0 \quad ; \quad \lambda_2 = -\frac{13}{20}}$$

- **Case 1.** $\lambda =$ 0. Substituting $\lambda_1 = 0 \to (A - 0I)v = 0$

$$Av = 0 \iff \begin{pmatrix} -2/5 & 1/4 \\ 2/5 & -1/4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-\frac{2}{5}a + \frac{1}{4}b = 0.$$

$$a = 5 \implies b = 8 \implies \boxed{v_1 = \begin{pmatrix} 5 \\ 8 \end{pmatrix}.}$$

- **Case 2.** $\lambda =$ $-13/20.$ Exercise - Answer

$$\boxed{v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

- The *general solution*:

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}$$

therefore takes the form:

$$\boxed{x(t) = c_1 \begin{pmatrix} 5 \\ 8 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\frac{13}{20}t}.}$$

Distinct Complex Eigenvalues

- Even if some eigenvalues are *complex*, so as long as they are *distinct* the eigenvalue method yields *n linearly independent* solutions.

- **Problem:** The eigenvectors associated to *complex eigenvalues* are complex valued.

- Thus, we will have *complex-valued* solutions. $x(t) = v e^{\lambda t} \in \mathbb{C}^{2\times 1}$ complex numbers.

- Suppose by solving characteristic equation

$$\det(A - \lambda I) = 0.$$

- we get the pair of complex-conjugate eigenvalues

$$\boxed{\lambda = p + iq} \quad ; \quad \boxed{\bar{\lambda} = p - iq.}$$

Eigenvectors

- $\mathbf{v}$ is the eigenvector associated to λ, so that

$$(A - \lambda I)v = 0$$

- Similarly, $\bar{\mathbf{v}}$ is the eigenvector associated to the complex conjugate $\bar{\lambda}$, so that

$$(A - \bar{\lambda} I)\bar{v} = 0$$

- $\mathbf{v}$ defined componentwise:

$$v = \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ \vdots \\ a_n + ib_n \end{pmatrix} = \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}}_{=:a} + i \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{=:b} = a + ib.$$

$$\boxed{\bar{v} = a - ib.}$$

$e^{i\theta} = \cos\theta + i\sin\theta.$

Complex-Valued Solution

- The *complex-valued* solution associated with λ and $\mathbf{v}$ is then

$$x(t) = \mathbf{v}e^{\lambda t} = \mathbf{v}e^{(p+iq)t} = \mathbf{v}e^{pt}\cdot e^{iqt} = (a+ib)\cdot e^{pt}(\cos qt + i\sin qt)$$

that is,

(1)
$$\boxed{x(t) = e^{pt}(a\cos qt - b\sin qt) + i\, e^{pt}(b\cos qt + a\sin qt).}$$

Real-Valued Solution

of (1)

- Because the *real* and *imaginary* parts of a *complex-valued* solution are also solutions, we thus get two *real-valued* solutions

$x' = Ax.$

(1)
$$x_1(t) = \mathrm{Re}\{x(t)\} = e^{pt}(a\cos qt - b\sin qt)$$
$$x_2(t) = \mathrm{Im}\{x(t)\} = e^{pt}(b\cos qt + a\sin qt).$$

- It is easy to check that the same two real-valued solutions result from taking real and imaginary parts of $\bar{\mathbf{v}}e^{\bar{\lambda}t}$

Procedure for Finding Real-Valued Solutions

1. Find explicitly a single *complex-valued* solution $\mathbf{x}(t)$ associated with the complex eigenvalue λ;

2. Then, find the *real* and *imaginary* parts $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ to get two *independent* real-valued solutions corresponding to the complex conjugate eigenvalues λ and $\bar{\lambda}$.

Example 3. *Find the solution of the IVP:*

$$x_1' = 2x_1 - 5x_2,\ x_2' = 4x_1 - 2x_2,\ x_1(0) = 2, x_2(0) = 3.$$

Solution

- The matrix form of the system is $x' = Ax.$

$$A = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$$

- The characteristic equation of the coefficient matrix is

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & -5 \\ 4 & -2-\lambda \end{vmatrix} = \boxed{\lambda^2+16=0.}$$

charpoly(A).
(1, 0, 16)

so we have the complex eigenvalues:

eig(A) $\quad \boxed{\lambda_{1,2} = \pm 4i.}$

- Substituting $\lambda_1 = 4i \rightarrow (A - \lambda I)v = 0$

$$\Leftrightarrow \begin{pmatrix} 2-4i & -5 \\ 4 & -2-4i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2a - 4ai - 5b = 0.$$

$$\Leftrightarrow \frac{2}{5}a - \frac{4}{5}aj - b = 0.$$

if $a = 5 \Rightarrow b = 2-4i.$

$$\boxed{v_1 = \begin{pmatrix} 5 \\ 2-4i \end{pmatrix}}$$

$$v = a + ib.$$

$$= \underbrace{\begin{pmatrix} 5 \\ 2 \end{pmatrix}}_{a} + i\underbrace{\begin{pmatrix} 0 \\ -4 \end{pmatrix}}_{b.}$$

- The corresponding complex-valued solution

$$x(t) = \begin{pmatrix} 5 \\ 2-4i \end{pmatrix} e^{4it} = \begin{pmatrix} 5 \\ 2-4i \end{pmatrix} (\cos 4t + i \sin 4t) = \begin{pmatrix} 5\cos 4t + i 5 \sin 4t \\ 2(\cos 4t + i \sin 4t) - 4i(\cos 4t + i \sin 4t) \end{pmatrix}$$

- The real and imaginary parts of $\mathbf{x}(t)$ are the *real-valued* solutions

$$\boxed{x_1(t) = \operatorname{Re}\{x(t)\} = \begin{pmatrix} 5\cos 4t \\ 2\cos 4t + 4\sin 4t \end{pmatrix}} \qquad \boxed{x_2(t) = \operatorname{Im}\{x(t)\} = \begin{pmatrix} 5\sin 4t \\ 2\sin 4t - 4\cos 4t \end{pmatrix}}$$

- The real-valued *general solution* is

$$c_1 x_1 \begin{pmatrix} 0 \\ - \end{pmatrix} +$$

$$\boxed{x(t) = c_1 x_1(t) + c_2 x_2(t)}.$$

- The resulting scalar equations are

$$(*) \quad \begin{aligned} x_1(t) &= c_1 5\cos 4t + c_2 5 \sin 4t \\ x_2(t) &= c_1(2\cos 4t + 4\sin 4t) + c_2 \cdot (2\sin 4t - 4\cos 4t) \end{aligned}$$

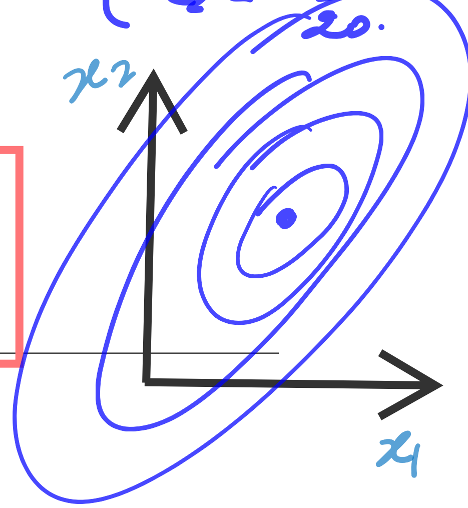
- When we impose the initial conditions $x_1(0) = 2 \quad x_2(0) = 3.$

Exercise

$$\begin{aligned} &i) \ (*) \equiv x(t) = X(t)c. \\ &ii) \ x(0) = X(0)c. \end{aligned} \Rightarrow \begin{cases} c_1 = \frac{2}{5} \\ c_2 = -\frac{11}{20}. \end{cases}$$

- Thus, finally, the solution of the IVP is:

$$\boxed{\begin{aligned} x_1(t) &= 2\cos 4t - \frac{11}{4} \sin 4t \\ x_2(t) &= 3\cos 4t + \frac{1}{2} \sin 4t. \end{aligned}}$$



MA 266 Lecture 27

Christian Moya, Ph.D.

Sec 5.2 Eigenvalue Method for Homogeneous Systems

Distinct Complex Eigenvalues

Formula for distinct complex eigenvalues Method 1

- *Distinct* complex conjugate eigenvalues $\lambda_{1,2} = p \pm iq$ with eigenvectors $\mathbf{v}_{1,2} = \mathbf{a} \pm i\mathbf{b}$ produce two linearly independent *real-valued* vector solutions:

$$\mathbf{x}_1(t) = e^{pt}(\mathbf{a}\cos qt - \mathbf{b}\sin qt)$$
$$\mathbf{x}_2(t) = e^{pt}(\mathbf{b}\cos qt + \mathbf{a}\sin qt)$$

Procedure for Finding Real-Valued Solutions Method 2

1. Find explicitly a single *complex-valued* solution $\mathbf{x}(t) = \mathbf{v}_1 e^{\lambda_1 t}$ associated with the complex eigenvalue λ_1 and eigenvector $\mathbf{v}_1$;

2. Then, find the *real* and *imaginary* parts $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ to get two *independent* real-valued solutions corresponding to the complex conjugate eigenvalues $\lambda_{1,2}$.

Example 1. *Find the the general solution of:*

$$\mathbf{x}' = \underbrace{\begin{pmatrix} 1 & -5 \\ 1 & 3 \end{pmatrix}}_{=\mathbf{A}} \mathbf{x}$$

Solution

- The corresponding characteristic equation

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \det \begin{pmatrix} 1-\lambda & -5 \\ 1 & 3-\lambda \end{pmatrix} \\ &= (1-\lambda)(3-\lambda) + 5 \\ &\boxed{= \lambda^2 - 4\lambda + 8 = 0} \end{aligned}$$

- Using the quadratic formula, we obtain the pair of complex eigenvalues:

$$\lambda_{1,2} = p \pm iq = 2 \pm 2i.$$

- Substituting $\lambda_1 = 2 + 2i$ into the *eigenvector* equation $(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v}_1 = \mathbf{0}$ gives

$$\begin{pmatrix} 1-2i & -5 \\ 1 & -1-2i \end{pmatrix} \underbrace{\begin{pmatrix} y \\ z \end{pmatrix}}_{=: v_1.} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Note that y and z satisfy

$$\boxed{-\frac{y}{5} - \frac{2y}{5} - z = 0}$$

.

- Choose $y = -5$, then $z = 1 + 2i$. Hence, the eigenvector is

$$\mathbf{v}_1 = \mathbf{a} + i\mathbf{b} = \underbrace{\begin{pmatrix} -5 \\ 1 \end{pmatrix}}_{=: a} + i \underbrace{\begin{pmatrix} 0 \\ 2 \end{pmatrix}}_{=: b.} = \begin{pmatrix} -5 \\ 1+2i \end{pmatrix}$$

$\lambda = 2+2i = p+iq.$

$v = \underbrace{\begin{pmatrix}-5\\1\end{pmatrix}}_{a} + i \underbrace{\begin{pmatrix}0\\2\end{pmatrix}}_{b}.$

Method 1.

- Use the formulae:

$x(t) = x_1(t) + i\, x_2(t)$

$$\mathrm{Re}\{x(t)\} = \mathbf{x}_1(t) = e^{pt}(\mathbf{a}\cos qt - \mathbf{b}\sin qt)$$
$$\mathrm{Im}\{x(t)\} = \mathbf{x}_2(t) = e^{pt}(\mathbf{b}\cos qt + \mathbf{a}\sin qt)$$

$\boxed{x_1(t) = e^{2t}\begin{pmatrix}-5\cos 2t\\ \cos 2t - 2\sin 2t\end{pmatrix}}$ ✓

$\boxed{x_2(t) = e^{2t}\begin{pmatrix}-5\sin 2t\\ 2\cos 2t + \sin 2t\end{pmatrix}}$. ✓

Method 2.

- Compute the corresponding complex-valued solution

$$\mathbf{x}(t) = \mathbf{v}_1 e^{\lambda_1 t} = \begin{pmatrix}-5\\1+2i\end{pmatrix} e^{(2+2i)t} = \begin{pmatrix}-5\\1+2i\end{pmatrix} e^{2t}(\cos 2t + i\sin 2t)$$
$$= e^{2t}\begin{pmatrix}-5\cos 2t - 5i\sin 2t\\ (\cos 2t + i\sin 2t) + 2i\cos 2t - 2\sin 2t\end{pmatrix}$$

$= x_1(t) + i\, x_2(t).$

- The real and imaginary parts of $\mathbf{x}(t)$ are the real-valued solutions:

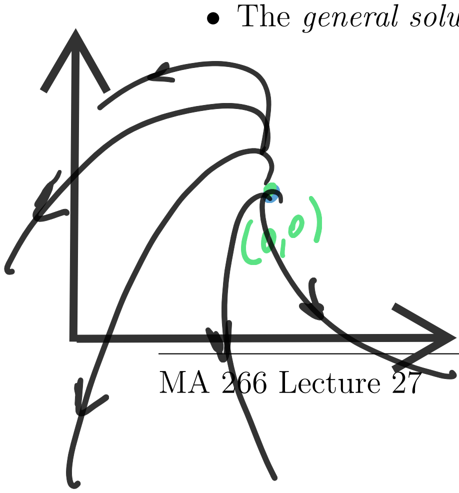
✓ $\boxed{\underset{\mathrm{Re}\{x(t)\}}{\underset{\parallel}{x_1(t)}} = e^{2t}\begin{pmatrix}-5\cos 2t\\ \cos 2t - 2\sin 2t\end{pmatrix}}$

$\boxed{\underset{\mathrm{Im}\{x(t)\}}{\underset{\parallel}{x_2(t)}} = e^{2t}\begin{pmatrix}-5\sin 2t\\ \sin 2t + 2\cos 2t\end{pmatrix}}$

- The *general solution* is

$$\boxed{\begin{aligned}\mathbf{x}(t) &= c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)\\ &= e^{2t}\begin{pmatrix}-5c_1\cos 2t - 5c_2\sin 2t\\ (c_1+2c_2)\cos 2t + (-2c_1+c_2)\sin 2t\end{pmatrix}.\end{aligned}}$$

$\lim_{t\to\infty} |x(t)| = +\infty.$



Sec 5.5 Multiple Eigenvalue Solutions

Repeated Roots

- Suppose the characteristic equation

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

does *not* have n distinct roots, and thus has *at least one repeated root.*

Definition. An eigenvalue is of *multiplicity* k if it is a k-fold root of the characteristic equation.

- An eigenvalue of multiplicity $k > 1$ may have *fewer* than k linearly independent associated eigenvectors.
- In this case we are unable to find a "complete set" of n linearly independent eigenvectors of $\mathbf{A}$, as needed to form the general solution of the system.

Complete Eigenvalues

An eigenvalue of multiplicity k is said to be complete. if it has k linearly independent associated eigenvectors.

- If every eigenvalue of the matrix $\mathbf{A}$ is complete, then—because eigenvectors associated with different eigenvalues are linearly independent—it follows that $\mathbf{A}$ does have a complete set of n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ associated with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (each repeated with its multiplicity).
- In this case a general solution of

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

is still given by the usual combination

$$\boxed{\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n \mathbf{v}_n e^{\lambda_n t}.}$$

Example 2. *Find a general solution of the system*

$$\mathbf{x}' = \underbrace{\begin{pmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{pmatrix}}_{=\mathbf{A}} \mathbf{x}.$$

Solution

- The characteristic equation of the coefficient matrix $\mathbf{A}$ is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{pmatrix} 9-\lambda & 4 & 0 \\ -6 & -1-\lambda & 0 \\ 6 & 4 & 3-\lambda \end{pmatrix}$$
$$= (0)\cdot\det\begin{pmatrix} -6 & -1-\lambda \\ 6 & 4 \end{pmatrix} - (0)\cdot\det\begin{pmatrix} 9-\lambda & 4 \\ 6 & 4 \end{pmatrix} + (3-\lambda)\cdot\det\begin{pmatrix} 9-\lambda & 4 \\ -6 & -1-\lambda \end{pmatrix}$$
$$\boxed{= (5-\lambda)(3-\lambda)^2 = 0.}$$

- Thus $\mathbf{A}$ has

$\lambda_1 = 5 \quad (k=1) \quad ; \quad \lambda_2 = 3 \quad (k=2)$

- **Case 1.** $\lambda_1 = 5$. The eigenvector equation $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_1 = \mathbf{0}$ is:

$$(\mathbf{A} - 5\mathbf{I})\mathbf{v}_1 = \begin{pmatrix} 4 & 4 & 0 \\ -6 & -6 & 0 \\ 6 & 4 & -2 \end{pmatrix} \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{=:\mathbf{v}_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

- Each of the first two eq'ns imply $\boxed{b = -a.}$ Then, one can reduce the third equation to

$6a - 4a - 2c = 0. \iff 2a - 2c = 0.$

$\boxed{a = c.}$

- The choice of $a = 1$ yields the eigenvector:

$b = -1, c = 1.$

$$\boxed{v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}$$

- **Case 2.** $\lambda_2 = 3$. Here the eigenvector equation is:

$$(\mathbf{A} - 3\mathbf{I})\mathbf{v}_2 = \begin{pmatrix} 6 & 4 & 0 \\ -6 & -4 & 0 \\ 6 & 4 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{=: v_2.} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

- Here $\mathbf{v}_2$ is an eigenvector if and only if

$$\boxed{b = -\frac{3}{2} a.}$$

- The above does not involve c. Thus c is arbitrary.

1) if $c = 0$ $\Rightarrow$ $a \neq 0$. $a = 2$. $\Rightarrow$ $b = -3$.

$$\boxed{v_2 = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}}$$

2) if $c = 1$ $\Rightarrow$ $a = b = 0$.

$$\boxed{v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.}$$

- Thus, we have found the complete set $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ of linearly independent eigenvectors associated to the eigenvalues $\lambda_1 = 5, \lambda_2 = 3$, and $\lambda_3 = 3$. Thus, the corresponding *general solution* is

$$\boxed{x(t) = c_1 v_1 e^{5t} + c_2 v_2 e^{3t} + c_3 v_3 e^{3t}.}$$

Defective Eigenvalues

We start with an illustrative example.

Example 3. *Find the eigenvalues and the eigenvectors of the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & -4 \\ 4 & 9 \end{pmatrix}$$

Solution

- The coefficient matrix has characteristic equation

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \det \begin{pmatrix} 1-\lambda & -4 \\ 4 & 9-\lambda \end{pmatrix} \\ &= (1-\lambda)(9-\lambda) + 16 \\ &\boxed{= \lambda^2 - 10\lambda + 25 = 0} \quad \Longleftrightarrow \quad (\lambda - 5)^2 = 0. \end{aligned}$$

- Thus $\mathbf{A}$ has $\boxed{\lambda_1 = 5 \quad (k = 2)}$

- The corresponding eigenvector equation is

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{=: v_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-4a - 4b = 0$ and $4a + 4b = 0.$

- Hence $\Rightarrow$ $b = -a.$

$$\boxed{v_1 = c \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

- Thus the multiplicity 2 eigenvalue $\lambda = 5$ has only one. independent eigenvector. Hence $\lambda = 5$ is i) incomplete. ii) defective (the defect is 1)

$d = 1$
$= k - p = 2 - 1 = 1.$ ✓

Definition. An eigenvalue λ of multiplicity $k > 1$ is called defective. if it is not complete.

- If λ has only $p < k$ linearly independent eigenvectors, then the number

$$\boxed{d = k - p.}$$

of "missing" eigenvectors is called the *defect* of the defective eigenvalue λ.

- If the eigenvalues of the $n \times n$ matrix $\mathbf{A}$ are not all complete, then the eigenvalue method as yet described will produce *fewer* than the needed n linearly independent solutions of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

- We therefore need to discover how to find the "missing solutions" corresponding to a defective eigenvalue λ of multiplicity $k > 1$.

The Case $k = 2$

- Suppose there is a single eigenvector $\mathbf{v}_1$ associated with the defective eigenvalue λ.

- Then at this point we have found only the single solution

$$\boxed{\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t}}$$

of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

The Second Solution

- We explore a second solution of the form

$$\boxed{x_2(t) = (v_1 t + v_2) e^{\lambda t} = v_1 t e^{\lambda t} + v_2 e^{\lambda t}}$$

- When we substitute $x = v_1 t e^{\lambda t} + v_2 e^{\lambda t}$ in $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we get

$$\underbrace{v_1 e^{\lambda t} + \lambda v_1 t e^{\lambda t} + \lambda v_2 e^{\lambda t}}_{x'} = A v_1 t e^{\lambda t} + A v_2 e^{\lambda t}$$

$$= Ax.$$

$$\lambda v_1 t\cancel{e^{\lambda t}} = A v_1 t \cancel{e^{\lambda t}}.$$

$$v_1 \cancel{e^{\lambda t}} + \lambda v_2 \cancel{e^{\lambda t}} = A v_2 \cancel{e^{\lambda t}}$$

- We obtain the two equations

$$v_1 + \lambda v_2 = A v_2.$$

$$\lambda v_1 = A v_1 \iff \boxed{(A - \lambda I) v_1 = 0} \iff \boxed{(A - \lambda I) v_2 = v_1.}$$

that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ must satisfy in order for

$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t} = \mathbf{v}_1 t e^{\lambda t} + \mathbf{v}_2 e^{\lambda t}$$

to give a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

- The *first* eq'n confirms that $\mathbf{v}_1$ is an eigenvector of $\mathbf{A}$ associated with eigenvalue λ.
- Then the *second* equation says that the vector $\mathbf{v}_2$ satisfies

$$(A - \lambda I)^2 v_2 = (A - \lambda I)[\underbrace{(A - \lambda I) v_2}_{=:\, v_1}] = (A - \lambda I) v_1 = 0$$

$$\boxed{(A - \lambda I)^2 v_2 = 0}$$

- To solve the two equations simultaneously, it suffices to find a solution $\mathbf{v}_2$ of the single equation $(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_2 = \mathbf{0}$ such that the resulting vector $\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2$ is *nonzero*.

Algorithm Defective Multiplicity **2** Eigenvalues

1. First find a nonzero solution $\mathbf{v}_2$ of the equation

$$\boxed{(A - \lambda I)^2 v_2 = 0}$$

such that

$$\boxed{(A - \lambda I) v_2 = v_1 \neq 0}$$

is nonzero, and therefore is an eigenvector $\mathbf{v}_1$ associated with λ.

2. Then form the two independent solutions

$$\boxed{x_1(t) = v_1 e^{\lambda t}}$$

and

$$\boxed{x_2(t) = (v_1 t + v_2) e^{\lambda t}.}$$

of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ corresponding to λ.

Example 4. *Find a general solution of the system*

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & 9 \end{pmatrix} \mathbf{x}.$$

Solution

- In the previous example, we showed that $\mathbf{A}$ has a *defective* eigenvalue:

$$\lambda = 5 \quad (k=2) \quad d=1.$$

- Following the Algorithm, we start by calculating

$$\underbrace{(A-5I)}^2 = \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$(A-5I)^2 V_2 = 0 \Leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} V_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad V_2 \neq V_1$$

- If we try $V_2 = (1, 0)^T$

$$(A-5I)V_2 = \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} = V_1. \checkmark$$

$$V_1 = c\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- Therefore the two solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are:

$$\boxed{\begin{aligned} x_1(t) &= V_1 e^{\lambda t} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} e^{5t} \\ x_2(t) &= (V_1 t + V_2) e^{\lambda t} = \begin{pmatrix} -4t+1 \\ 4t \end{pmatrix} e^{5t}. \end{aligned}}$$

- The general solution is:

$$\boxed{x(t) = c_1 x_1(t) + c_2 x_2(t).}$$

MA 266 Lecture 28

Christian Moya, Ph.D.

Sec 5.5 Multiple Eigenvalue Solutions

Generalized Eigenvectors

- The vector $\mathbf{v}_2$ in the equation

$$(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_2 = \mathbf{0}$$

is an example of a generalized eigenvector.

Rank of a Generalized Eigenvector

- If λ is an eigenvalue of the matrix $\mathbf{A}$, then a *rank r generalized eigenvector* associated with λ is a vector $\mathbf{v}$ such that

$$\boxed{(A-\lambda I)^r v = 0} \quad \text{but.}$$

$$\boxed{(A-\lambda I)^{r-1} v \neq 0}$$

- A rank 1 generalized eigenvector is an ordinary eigenvector because

$$(A-\lambda I)^1 v = 0 \qquad (A-\lambda I)^0 v = Iv = v \neq 0$$

- The vector $\mathbf{v}_2$ in the equation

$$(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_2 = \mathbf{0}$$

is a rank 2 generalized eigenvector (and not an ordinary eigenvector).

change // new page

Chains of Generalized Eigenvectors

i) $(A-\lambda I)^2 v_2 = 0.$

ii) $(A-\lambda I) v_2 = v_1 \neq 0$

eq) $0 v_2 = 0$

i) $0 v_1 = 0.$

ii) $(A-\lambda I) v_1 = 0$

- The multiplicity 2 method described earlier boils down to finding a pair $\{v_1, v_2\}$ of generalized eigenvectors, one of rank 1 and one of rank 2, such that

$$(A-\lambda I) v_2 = v_1 \neq 0$$

- Higher multiplicity methods involve longer "chains" of generalized eigenvectors.

Length k Chain

- A *length k chain of generalized eigenvectors based on the eigenvector* $\mathbf{v}_1$ is a set

$$\{v_1, v_2, \dots, v_{k-1}, v_k\}.$$

of k generalized eigenvectors such that

$$(A-\lambda I) v_k = v_{k-1}$$
$$(A-\lambda I) v_{k-1} = v_{k-2}.$$
$$\vdots$$
$$(A-\lambda I) v_2 = v_1.$$

- Because $\mathbf{v}_1$ is an ordinary eigenvector, $(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_1 = \mathbf{0}$. Therefore,

$$\boxed{(A-\lambda I)^k v_k = 0}$$

Length 3 Chain

- Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a length 3 chain of generalized eigenvectors associated with the multiple eigenvalue λ of the matrix $\mathbf{A}$.

$(k=3)$

- It is easy to verify that three linearly independent solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are given by

$$\boxed{\begin{aligned} x_1(t) &= v_1 e^{\lambda t} \\ x_2(t) &= (v_1 t + v_2) e^{\lambda t}. \\ x_3(t) &= \left(\tfrac{1}{2} v_1 t^2 + v_2 t + v_3\right) e^{\lambda t}. \end{aligned}}$$

Example 1. *In this example, the eigenvalues are given. Find the general solution of*

$$\mathbf{x}' = \underbrace{\begin{pmatrix} 5 & -1 & 1 \\ 1 & 3 & 0 \\ -3 & 2 & 1 \end{pmatrix}}_{=\mathbf{A}} \mathbf{x}; \quad \lambda = 3, 3, 3.$$

Solution

- The eigenvector equation is

$$(\mathbf{A} - 3\mathbf{I})\mathbf{v} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The second row implies that $a = 0$. Then, $c = b$. Thus, to within a constant multiple, the eigenvalue $\lambda = 3$ has only one single eigenvector (with $b \neq 0$)

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ b \\ b \end{pmatrix}.$$

So the defect of $\lambda = 3$ is 2.

- To apply the method described for triple eigenvalues, we first calculate

$$(\mathbf{A} - 3\mathbf{I})^2 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & -1 & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

and

$$(\mathbf{A} - 3\mathbf{I})^3 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 2 & -1 & 1 \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Beginning with $V_3 = (1, 0, 0)^T$, for instance, we calculate

$$\mathbf{v}_2 = (\mathbf{A} - 3\mathbf{I})\mathbf{v}_3 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$$\mathbf{v}_1 = (\mathbf{A} - 3\mathbf{I})\mathbf{v}_2 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \checkmark \qquad \begin{pmatrix} 0 \\ b \\ b \end{pmatrix} = V_1.$$

- The linearly independent solutions are:

$$x_1(t) = v_1 e^{3t} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} e^{3t}.$$

$$x_2(t) = (v_1 t + v_2) e^{3t} = \begin{pmatrix} 2 \\ 2t + 1 \\ 2t - 3 \end{pmatrix} e^{3t}.$$

$$x_3(t) = \left(\frac{1}{2} v_1 t^2 + v_2 t + v_3\right) e^{3t} = \begin{pmatrix} 2t + 1 \\ t^2 + t \\ t^2 - 3t \end{pmatrix} e^{3t}.$$

the gen. soln:

$$\boxed{x(t) = c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t)}$$

Sec 5.3 Gallery of Solns for Linear Systems (2x2 linear systems)

Example 2. *Solve the linear system:*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

$\lambda_1 = a \quad \lambda_2 = -1$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Graph the phase portrait/diagram as a varies from $-\infty$ to ∞, showing the qualitatively different cases.

Solution

- Matrix multiplication yields

$$x' = ax$$
$$y' = -y.$$

uncoupled.

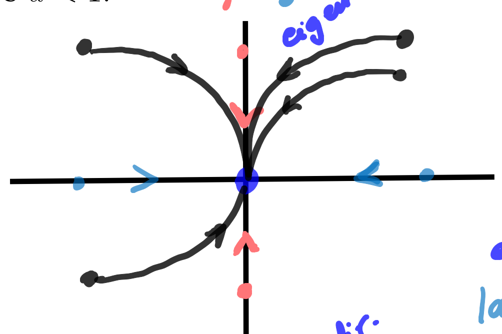
- The solution is

$$\boxed{\begin{aligned} x(t) &= x_0 e^{at} \\ y(t) &= y_0 e^{-t} \end{aligned}}$$

for IC: (x_0, y_0).

- The phase portrait for different values of a are shown next. In each case, $y(t)$ decays exponentially fast.

- **Case** $a < -1$:



i) $(0, y_0)$: $y_0 > 0$; $y_0 < 0$.

ii) $(x_0, 0)$: $x_0 > 0$; $x_0 < 0$.

iii) (x_0, y_0) : $x_0, y_0 > 0$.

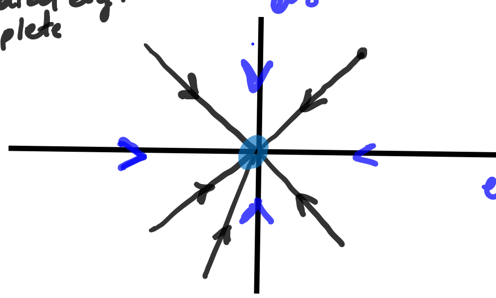
$\frac{dy}{dx} = \frac{y'}{x'}$

(0,0). stable node

Since $|a| > 1$. $x(t)$ decays faster than $y(t)$

- **Case** $a = -1$

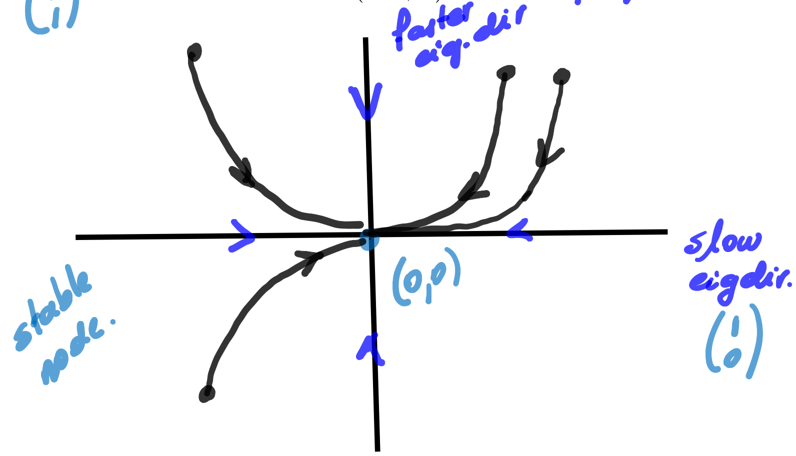
repeated eig. complete



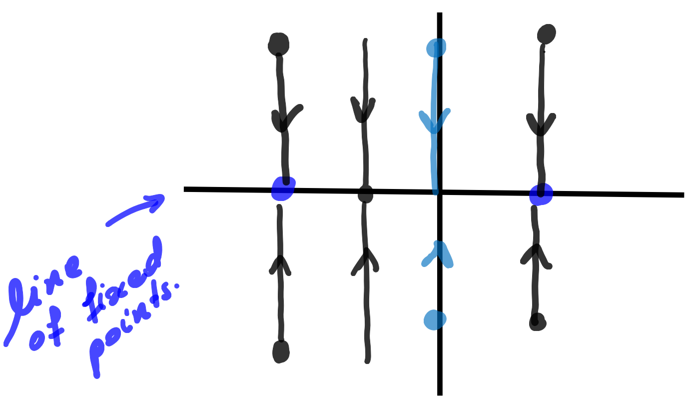
(0,0) symmetrical stable node or star

- **Case** $a \in (-1, 0)$

$|a| < 1.$



- **Case** $a = 0$



$\vec{x}(t) = c_1 v_1 + c_2 v_2 e^{-t}.$

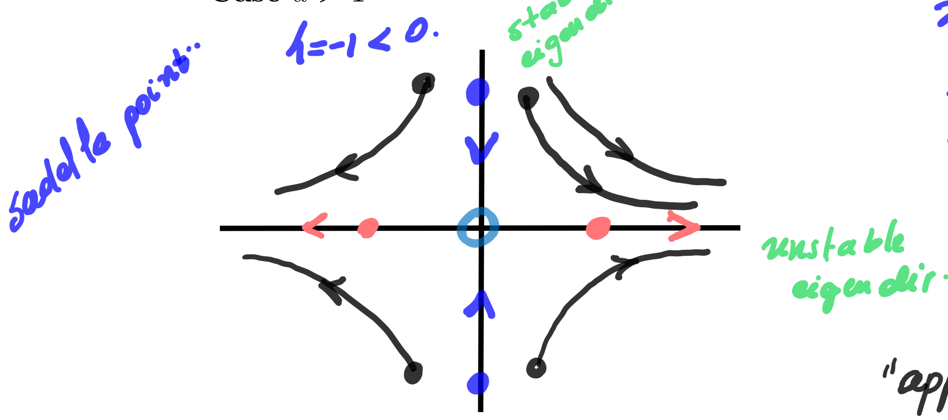
$x(t) = x_0.$

$y(t) = y_0 e^{-t}$

i) $(x_0, 0)$

ii) (x_0, y_0)

- **Case** $a > 1$



$x(t) = x_0 e^{|a|t}.$

$y(t) = y_0 e^{-t}.$

as $t \to \infty$. $x(t) \searrow 0$

as $t \to \infty$ $y(t) \nearrow +\infty$

"approach the unst. manifold as $t \to \infty$"

Example 3. *Solve the linear system:*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}}_{=\mathbf{A}} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Graph the corresponding phase portrait/diagram.

Solution

- The characteristic equation is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \lambda^2 + \lambda - 6 = 0.$$

- Hence, the eigenvalues of $\mathbf{A}$ are

$$\lambda_1 = 2 > 0 \quad \text{and} \quad \lambda_2 = -3 < 0$$

saddle point.

- The eigenvector equation is:

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- For $\lambda = 2$, this yields

$$(\mathbf{A} - 2\mathbf{I})\mathbf{v} = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- The corresponding non-trivial eigenvector is

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- Similarly, for $\lambda = -3$, this yields

$$(\mathbf{A} + 3\mathbf{I})\mathbf{v} = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

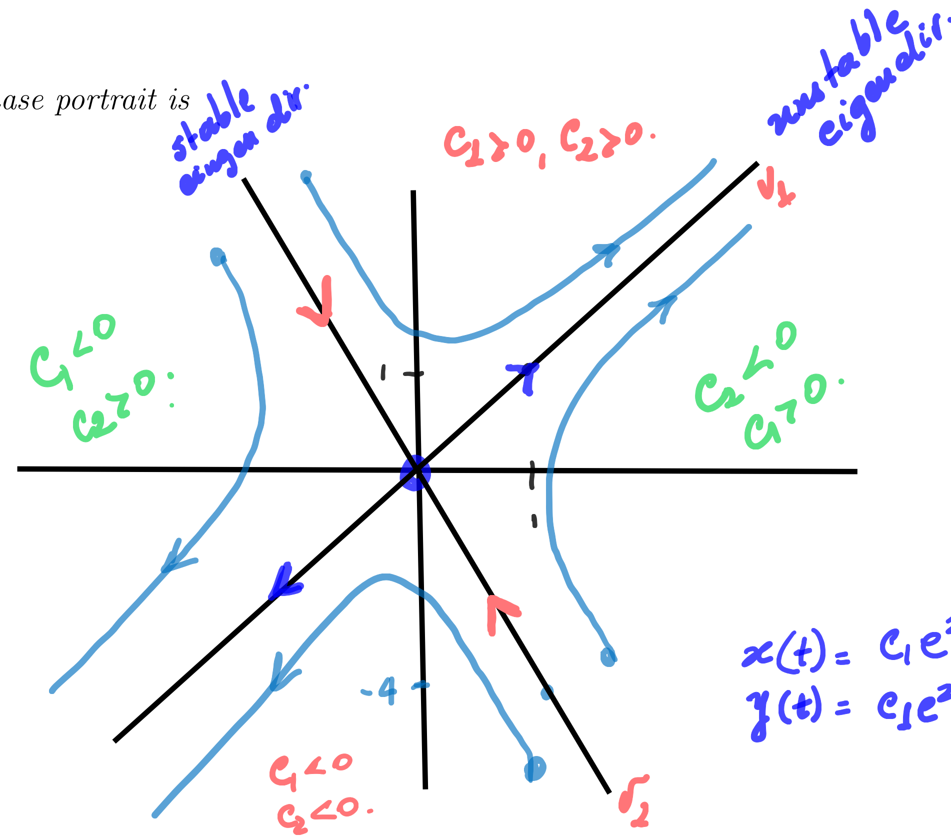
- The corresponding non-trivial eigenvector is

$$v_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

- The general solution is

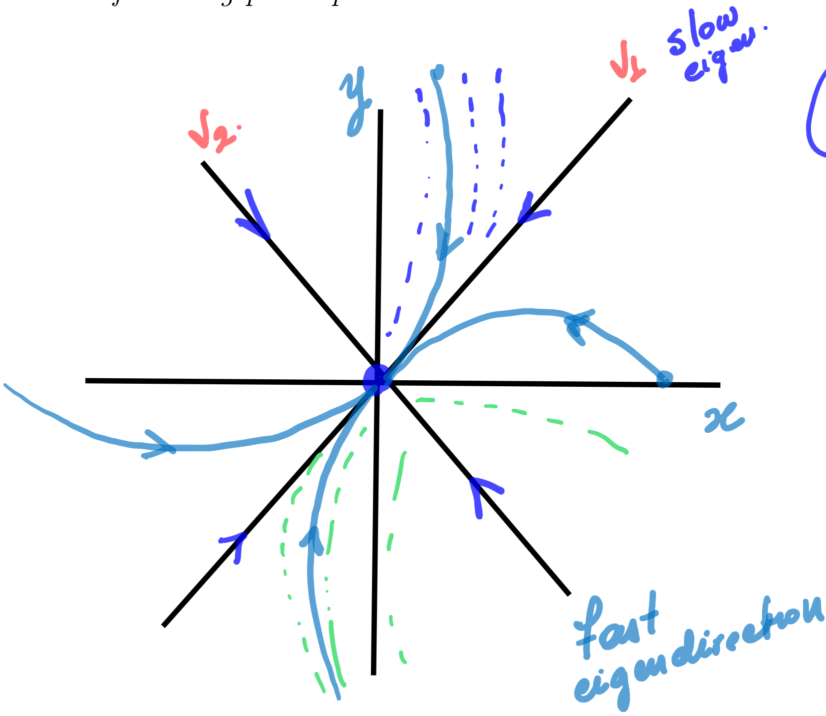
$$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}.$$

- The *phase portrait is*



$$x(t) = c_1 e^{2t} + c_2 e^{-3t}.$$
$$y(t) = c_1 e^{2t} - 4c_2 e^{-3t}$$

Example 4. *Determine the nature of the eigenvalues and eigenvectors associated to the following phase portrait.*



$(0,0)$: stable-node.

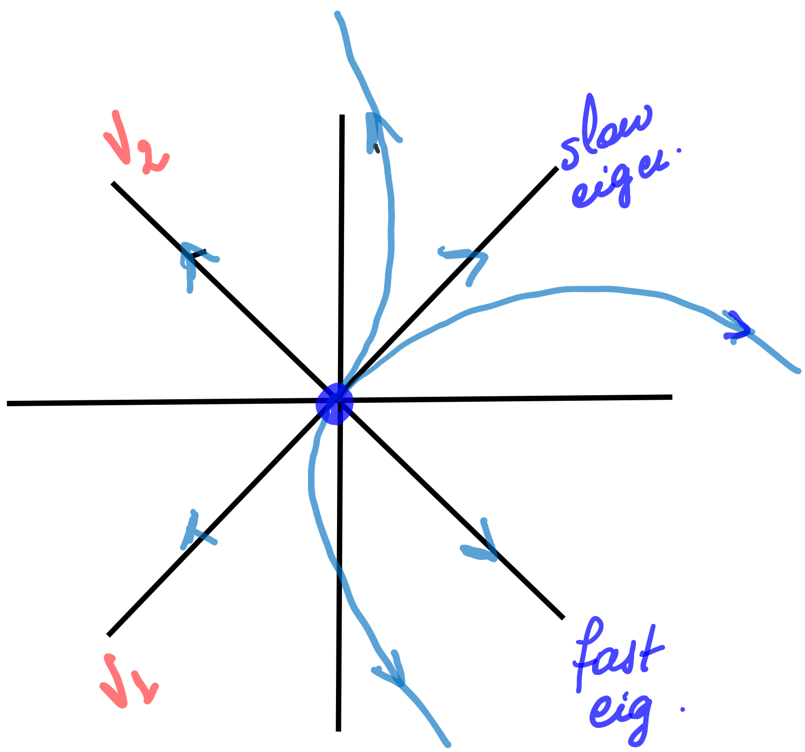
$\lambda_2 < \lambda_1 < 0.$

V_1: slow eigendirection.

V_2: fast eigendirection

$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



$(0,0)$: unstable node.

$\underline{\lambda_2 > \lambda_1 > 0}.$

Example 5. *Determine what happens when the eigenvalues of the* 2×2 *linear system are complex numbers.*

Solution

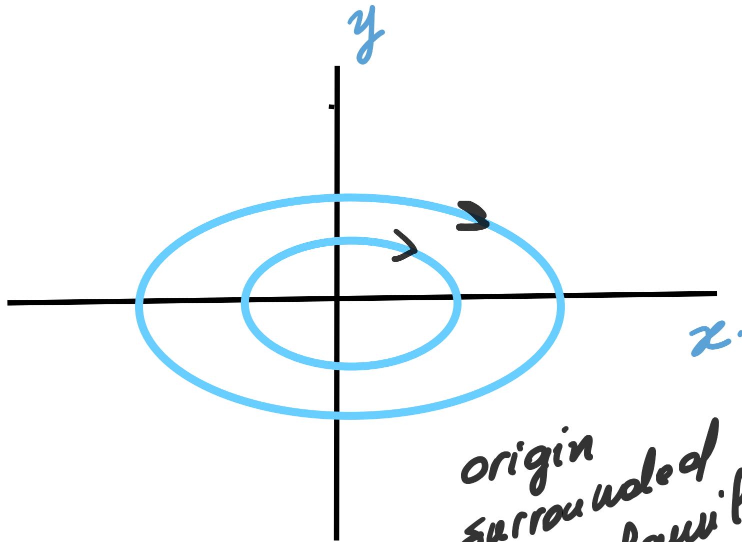
- Let's write the eigenvalues as

$\lambda_1 = p + iq \qquad \lambda_2 = p - iq. \qquad \lambda_{1,2} = a + i\,b.$

- **Case** if $p = 0$, the general solution is

$$\mathbf{x}(t) = c_1(\mathbf{a}\cos qt - \mathbf{b}\sin qt) + c_2(\mathbf{b}\cos qt + \mathbf{a}\sin qt).$$

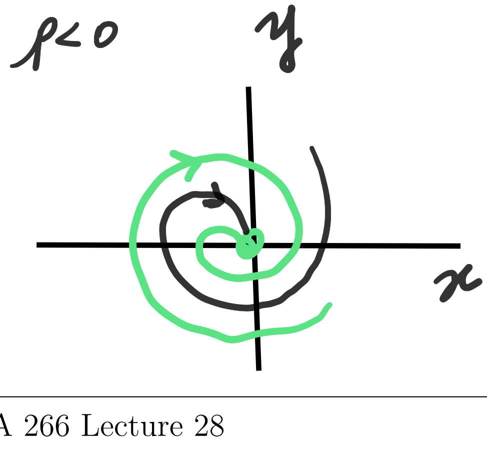
center.

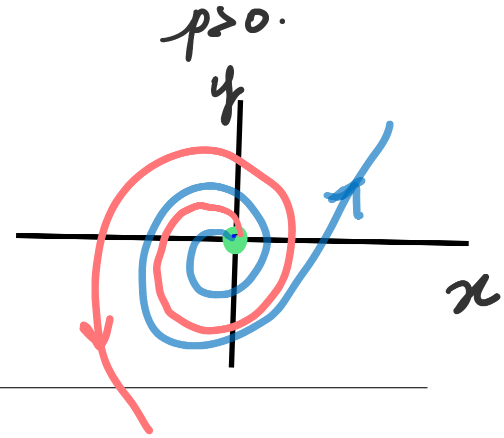


origin surrounded by a family of closed orbits.

- **Case** if $p \neq 0$, the general solution is

$$\mathbf{x}(t) = e^{pt}(c_1(\mathbf{a}\cos qt - \mathbf{b}\sin qt) + c_2(\mathbf{b}\cos qt + \mathbf{a}\sin qt)).$$





spirals.

Example 6. *Determine what happens when the eigenvalues of the* 2×2 *linear system are equal.*

Solution

v_1, v_2.

- **Complete eigenvalue.** If there are two independent eigenvectors, then they span the plane and so *every vector is an eigenvector with this same eigenvalue.* To see this, let

v_1: $Av_1 = \lambda v_1$

v_2: $Av_2 = \lambda v_2$.

$v = c_1 v_1 + c_2 v_2$.

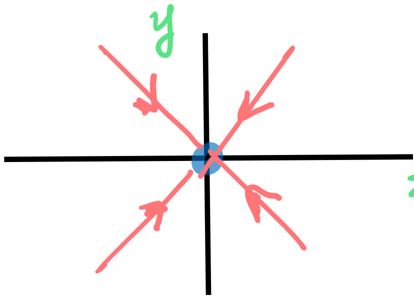
$Av = c_1 Av_1 + c_2 Av_2$.

$= c_1 \lambda v_1 + c_2 \lambda v_2$.

$= \lambda (c_1 v_1 + c_2 v_2) = \lambda v$.

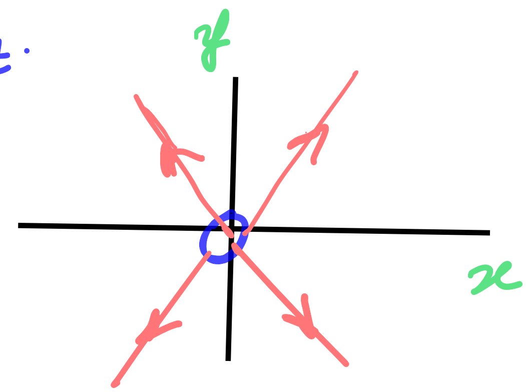
$\Rightarrow A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \Rightarrow$ $x(t) = x_0 e^{\lambda t}$, $y(t) = y_0 e^{\lambda t}$.

$\lambda < 0$



star

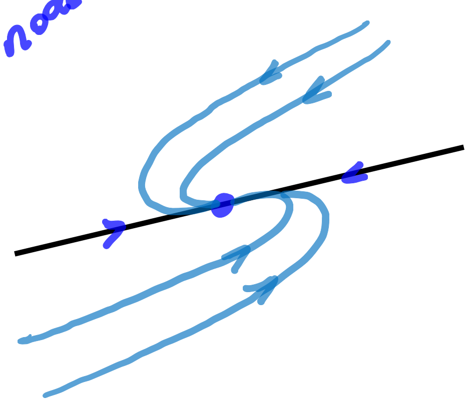
if $\lambda > 0$



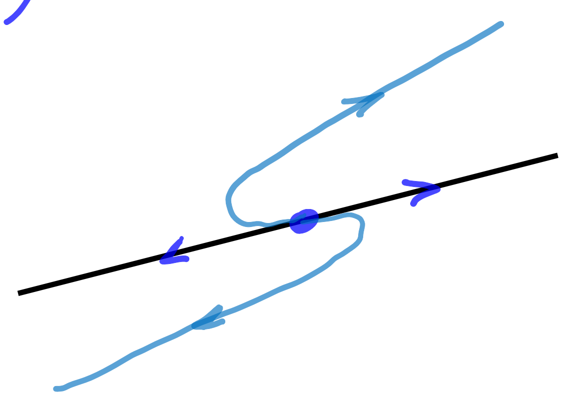
- **Defective eigenvalue.** The eigenspace corresponding to λ is one-dimensional.

degenerate node.

$\Rightarrow$ $A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$ $b \neq 0$.



eigendir.



MA 266 Lecture 29

Christian Moya, Ph.D.

Sec 5.6 Matrix Exponentials and Linear Systems

- Consider the *single* equation linear system IVP:

$$x' = ax \quad ; \quad x(0) = x_0.$$

- The corresponding *solution*:

$$\boxed{x(t) = e^{at} x_0.}$$

- Now consider the n equations linear system IVP:

$$x' = Ax \quad ; \quad x(0) = x_0.$$

$A \ (n \times n) \wedge x \ (n \times 1)$

- We expect the *solution* to be of the form:

$$\boxed{x(t) = e^{At} x_0.}$$

- We can verify $e^{At}x_0$ is the solution of $x' = Ax$.

Series definition of $e^{\mathbf{A}t}$

$$e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{3!}(At)^3 + \dots$$

$$x' = \frac{d}{dt}\left(e^{At}\right)x_0$$

$$\frac{d}{dt}\left(e^{At}\right) = A + A^2 t + \frac{1}{2!}A^3 t^2 + \dots$$

$$= A\underbrace{\left(I + At + \frac{1}{2}(At)^2 + \dots\right)}_{e^{At}}$$

$$x' = \frac{d}{dt}\left(e^{At}\right)x_0 = A\underbrace{e^{At}x_0}_{=:x} = Ax \checkmark$$

How to compute $e^{\mathbf{A}t}$?

1. Series definition*.
2. Fundamental matrix solution $\mathbf{\Phi}(t)$.
3. n linearly independent eigenvectors.

Properties of $e^{\mathbf{A}t}$

1. If $\mathbf{D} = \mathrm{diag}(d_1, \ldots, d_n)$, then

$$e^{D} = \begin{pmatrix} e^{d_1} & & & \\ & e^{d_2} & & \\ & & \ddots & \\ & & & e^{d_n} \end{pmatrix}$$

2. If $\mathbf{0}$ is the $n \times n$ zero matrix, then

$$e^{0} = I.$$

3. If $\mathbf{A}$ and $\mathbf{B}$ are two $n \times n$ matrices that commute, i.e.,

$$\boxed{AB = BA}$$

then

$$\boxed{e^{A+B} = e^{A} e^{B}.}$$

4. The inverse of $e^{\mathbf{A}}$

$$\left(e^{A}\right)^{-1} = e^{-A}.$$

Computing $e^{\mathbf{A}t}$ using the *series* definition

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \ldots + \mathbf{A}^n\frac{t^n}{n!} + \ldots$$

Q: When can we use this series?

$\boxed{A^n = 0}$ for some $n > 0$ (integer)

$\Rightarrow$ series terminates after finite # of terms.

Definition -**Nilpotent matrix**

- A $n \times n$ matrix is said to be *nilpotent* if

$A^n = 0$ for some $n > 0$ (int)

Example 1. *Show that the following matrix* $\mathbf{A}$ *is nilpotent and then use this fact to find the matrix exponential* $e^{\mathbf{A}t}$.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution.

$$A^2 = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \checkmark$$

- so $\mathbf{A}^n = \mathbf{0}$ for $n \geq 3$. It therefore follows from the series definition of the matrix exponential that

$$e^{At} = I + (At) + \frac{1}{2}(At^2)$$

$$= \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} + t\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1+t & -t & -t - t^2 \\ t & 1-t & t - t^2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 2. *Solve the following IVP*

$$\mathbf{x}' = \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}}_{=\mathbf{A}} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

Solution.

- Show that $\mathbf{A} = \mathbf{I} + \mathbf{B}$.

$$A = \underbrace{\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}}_{=:I} + \underbrace{\begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}}_{=:B}.$$

- Show $\mathbf{B}$ is nilpotent.

$$B^2 = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B^3 = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow B^n = 0$ for $n \geq 3$.

- Compute $e^{\mathbf{A}t}$.

i) $IB = BI$ exercise

$$e^{At} = e^{(I+B)t} = e^{It} + e^{Bt}$$

$$= e^t I \cdot \left(I + t\begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

$$= \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix} \begin{pmatrix} 1 & 2t & 3t+2t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t & e^t 2t & e^t(3t+2t^2) \\ 0 & e^t & e^t 2t \\ 0 & 0 & 1 \end{pmatrix}$$

$$e^{At} = e^t \begin{pmatrix} 1 & 2t & 3t+2t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix} \quad \blacksquare$$

- Solve the IVP:

$$x(t) = e^{At} x_0. \qquad x_0 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

$$= e^t \begin{pmatrix} 4 + 10t + 18t + 12t^2 \\ 5 + 12t \\ 6 \end{pmatrix}$$

$$= e^t \begin{pmatrix} 4 + 28t + 12t^2 \\ 5 + 12t \\ 6 \end{pmatrix} \quad \blacksquare$$

Fundamental Matrix Solutions

- The *general* solution of the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$:

$$x(t) = c_1 x_1(t) + \dots + c_n x_n(t)$$

can be written in the form

$$\boxed{x(t) = \Phi(t)\, c.}$$

where

$$\underset{(n\times n)}{\Phi(t) = \begin{pmatrix} | & | & & | \\ x_1(t) & x_2(t) & \dots & x_n(t) \\ | & | & & | \end{pmatrix}}; \quad \underset{(n\times 1)}{c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}$$

- To solve the IVP for a given initial condition

$$x(0) = x_0.$$

it suffices to find c

$$x(0) = \underbrace{\Phi(0)}_{\text{nonsingular}}\, c \implies \boxed{c = \Phi^{-1}(0)\, x_0.}$$

$$\boxed{x(t) = \Phi(t)\, \Phi^{-1}(0)\, x_0.}$$

- Conclusion:

$$x(t) = e^{At} x_0. \implies \boxed{e^{At} = \Phi(t)\, \Phi^{-1}(0)}$$

Example 3. *Compute the matrix exponential* $e^{\mathbf{A}t}$ *for the system*

$$x_1' = 5x_1 - 3x_2, x_2' = 2x_1.$$

$A = \begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix}$

Solution.

- Characteristic equation:

$$\det(A - \lambda I) = \det\begin{pmatrix} 5-\lambda & -3 \\ 2 & -\lambda \end{pmatrix}$$
$$= (5-\lambda)(-\lambda) + 6.$$
$$= \lambda^2 - 5\lambda + 6 \Rightarrow \lambda_1 = 2,\ \lambda_2 = 3$$
$$(\lambda-3)(\lambda-2)$$

- **Case 1:** $\lambda_1 = 2.$

$$(A - 2I)v = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow 3a - 3b = 0.$ if $a = 1 \Rightarrow b = 1$

$\Leftrightarrow a - b = 0$

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$

- **Case 2:** $\lambda_2 = 3.$

$$(A - 3I)v_2 = \begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow 2a_2 - 3b_2 = 0.$ $\therefore$ $v_2 = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix}$

if $a_2 = \frac{1}{2} \Rightarrow b = \frac{1}{3}$

$x_2(t) = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix} e^{3t}$

- The fundamental matrix is then

$$\Phi(t) = \begin{pmatrix} e^{2t} & \frac{1}{2}e^{3t} \\ e^{2t} & \frac{1}{3}e^{3t} \end{pmatrix}$$

$$\Phi(0) = \begin{pmatrix} 1 & 1/2 \\ 1 & 1/3 \end{pmatrix}$$

- The matrix exponential $e^{\mathbf{A}t}$:

$$\Phi^{-1}(0) = \frac{1}{\det \Phi} \begin{pmatrix} 1/3 & -1/2 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{\frac{1}{3} - \frac{1}{2}} \begin{pmatrix} 1/3 & -1/2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & +3 \\ 6 & -6 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^{2t} & \frac{1}{2}e^{3t} \\ e^{2t} & \frac{1}{3}e^{3t} \end{pmatrix} \begin{pmatrix} -2 & +3 \\ 6 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} -2e^{2t} + 3e^{3t} & 3e^{2t} - 3e^{3t} \\ -2e^{2t} + 2e^{3t} & 3e^{2t} - 2e^{3t} \end{pmatrix} \quad \blacksquare$$

Suppose

n **linearly independent eigenvectors**

$$A = V \Lambda V^{-1} \qquad A^2 = V \Lambda V^{-1} \cdot V \Lambda V^{-1} = V \Lambda^2 V^{-1}$$

$$e^{At} = \underset{VV^{-1}}{I} + V \Lambda V^{-1} t + \frac{1}{2} V \Lambda^2 V^{-1} t^2 + \dots$$

$$e^{At} = V \big(\underbrace{I + \Lambda t + \tfrac{1}{2} \Lambda^2 t^2 + \dots}_{e^{\Lambda t}} \big) V^{-1}$$

$$\boxed{e^{At} = V e^{\Lambda} V^{-1} = V \underbrace{\begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix}}_{} V^{-1}}$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

Example 4. *Compute the matrix exponential $e^{\mathbf{A}t}$ for the system*

$$x_1' = 5x_1 - 3x_2, x_2' = 2x_1.$$

Solution with the alternative method.

$$e^{At} = V e^{\Lambda} V^{-1}$$

$$V = \begin{pmatrix} 1 & 1/2 \\ 1 & 1/3 \end{pmatrix}; \quad V^{-1} = \begin{pmatrix} -2 & -3 \\ 6 & -6 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 1 & 1/2 \\ 1 & 1/3 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 6 & -6 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} e^{2t} & \frac{1}{2} e^{3t} \\ e^{2t} & \frac{1}{3} e^{3t} \end{pmatrix}}_{=: \Phi(t)} \underbrace{\begin{pmatrix} -2 & 3 \\ 6 & -6 \end{pmatrix}}_{=: \Phi^{-1}(0)} \quad \checkmark$$

MA 266 Lecture 30

Christian Moya, Ph.D.

Sec 5.6 Matrix Exponentials and Linear Systems

Example 1. *Compute the matrix exponential $e^{\mathbf{A}t}$ for the system*

$$x_1' = 5x_1 - 3x_2, x_2' = 2x_1.$$

Solution.

- The characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 5\lambda + 6 = 0.$$

  So, the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 2$ and $\lambda_2 = 3$.

- **Case 1:** $\lambda_1 = 2$. The associated eigenvector equation is

$$(\mathbf{A} - 2\mathbf{I})\mathbf{v}_1 = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \mathbf{v}_1 = \mathbf{0}.$$

  The eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The associated solution is:

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{2t} = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}.$$

- **Case 2:** $\lambda_2 = 3$. The associated eigenvector equation is

$$(\mathbf{A} - 3\mathbf{I})\mathbf{v}_2 = \begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix} \mathbf{v}_2 = \mathbf{0}.$$

  The eigenvector is $\mathbf{v}_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}$. The associated solution is:

$$\mathbf{x}_2(t) = \mathbf{v}_2 e^{3t} = \begin{pmatrix} \frac{e^{3t}}{2} \\ \frac{e^{3t}}{3} \end{pmatrix}.$$

- The fundamental matrix is then

$$\mathbf{\Phi}(t) = \begin{pmatrix} e^{2t} & e^{3t}/2 \\ e^{2t} & e^{3t}/3 \end{pmatrix}. = \begin{pmatrix} x_1 & x_2 \end{pmatrix}$$

- So,

$$\underline{\mathbf{\Phi}(0)} = \begin{pmatrix} 1 & 1/2 \\ 1 & 1/3 \end{pmatrix} \text{ and } \underline{\mathbf{\Phi}^{-1}(0)} = \begin{pmatrix} -2 & 3 \\ 6 & -6 \end{pmatrix}.$$

- The matrix exponential $e^{\mathbf{A}t}$ is

$$e^{\mathbf{A}t} = \underset{\sim}{\mathbf{\Phi}(t)}\underset{\sim}{\mathbf{\Phi}^{-1}(0)} = \begin{pmatrix} -2e^{2t} + 3e^{3t} & 3e^{2t} - 3e^{3t} \\ -2e^{2t} + 2e^{3t} & 3e^{2t} - 2e^{3t} \end{pmatrix}.$$

Suppose we have n linearly independent eigenvectors.

$$\boxed{\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}.}$$

Then, using the series definition of $e^{\mathbf{A}t}$, we have

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}t + \frac{1}{2!}\mathbf{V}\mathbf{\Lambda}^2\mathbf{V}^{-1}t^2 + \dots \\ &= \mathbf{V}\left(\mathbf{I} + \mathbf{\Lambda}t + \frac{1}{2!}(\mathbf{\Lambda}t)^2 + \dots\right)\mathbf{V}^{-1} \\ &= \mathbf{V}\begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix}\mathbf{V}^{-1} \end{aligned}$$

Example 2. *Compute the matrix exponential $e^{\mathbf{A}t}$ for the system*

$$x_1' = 5x_1 - 3x_2, x_2' = 2x_1.$$

Solution with the alternative method.

$$\underset{\sim}{\mathbf{V}} = \begin{pmatrix} 1 & 1/2 \\ 1 & 1/3 \end{pmatrix} \text{ and } \underset{\sim}{\mathbf{V}}^{-1} = \begin{pmatrix} -2 & 3 \\ 6 & -6 \end{pmatrix}. = \underline{\Phi}^{-1}(0)$$

$\uparrow v_1$ $\uparrow v_2$

So,

$$\begin{aligned} e^{\mathbf{A}t} = \mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1} &= \begin{pmatrix} 1 & 1/2 \\ 1 & 1/3 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 6 & -6 \end{pmatrix} \\ \underline{\Phi}(t) \leftarrow \quad &= \begin{pmatrix} e^{2t} & e^{3t}/2 \\ e^{2t} & e^{3t}/3 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 6 & -6 \end{pmatrix} = \underline{\Phi}(t)\,\underline{\Phi}^{-1}(0) \\ &= \begin{pmatrix} -2e^{2t} + 3e^{3t} & 3e^{2t} - 3e^{3t} \\ -2e^{2t} + 2e^{3t} & 3e^{2t} - 2e^{3t} \end{pmatrix}. \end{aligned}$$

Sec 5.7 Nonhomogeneous Linear Systems

- Given the *nonhomogeneous* linear system

$$x' = Ax + f(t)$$

$A\ (n \times n)$

$A = A(t)$

- The general solution of the *nonhomogeneous* system is

$$\underbrace{x(t)}_{(n\times 1)} = \underbrace{x_c(t)}_{(n\times 1)} + \underbrace{x_p(t)}_{(n\times 1)}$$

where

1. $x_c(t) = c_1 x_1(t) + \dots + c_n x_n(t)$ is the general solution of the associated *homogeneous* system $\mathbf{x}' = \mathbf{A}x$.

2. $x_p(t)$ is a particular solution of the nonhomogeneous system.

Undetermined Coefficients

Undetermined Coefficients

- Suppose $\mathbf{f}(t)$ is a *linear combination of products* of
  1. Polynomials
  2. Exponential functions
  3. Sines and cosines
- Make a guess of the *particular* solution $\mathbf{x}_p$.
- Then, we determine the undetermined *vector* coefficients by substitution in the original *nonhomogeneous* equation.

Example 3. *Apply the method of undetermined coefficients to find a particular solution of the system*

$$x' = 2x + 3y + 5, \;\; y' = 2x + y - 2t.$$

Solution

- The matrix form:

$$z = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \quad z' = \underbrace{\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}}_{=:A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{=:z} + \underbrace{\begin{pmatrix} 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} t}_{=:f(t)}.$$

- The nonhomogeneous term **f** is $(5 \;\; -2t)^T$ linear. So it is reasonable to select the particular solution of the form:

$$x_p(t) = a + bt = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} t$$

$$x_p'(t) = b.$$

- Upon substitution of $\mathbf{x} = \mathbf{x}_p$ in the nonhomogeneous system, we get

$$x_p' = Ax_p + f \Leftrightarrow \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix} + \begin{pmatrix} 5 \\ -2t \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 2a_1 + 2b_1 t + 3a_2 + 3b_2 t + 5 \\ 2a_1 + 2b_1 t + a_2 + b_2 t - 2t \end{pmatrix} = \begin{pmatrix} 2b_1 + 3b_2 \\ 2b_1 + b_2 - 2 \end{pmatrix} t + \begin{pmatrix} 2a_1 + 3a_2 + 5 \\ 2a_1 + a_2 \end{pmatrix}$$

$$\left.\begin{array}{ll} i) & 2b_1 + 3b_2 = 0 \\ ii) & 2b_1 + b_2 = 2. \end{array}\right\} \quad \begin{array}{l} b_1 = -\frac{3}{2} b_2 \\ -2b_2 = 2 \end{array} \Rightarrow \left\{ \begin{array}{l} b_1 = 3/2. \\ b_2 = -1 \end{array}\right.$$

$$\left.\begin{array}{ll} iii) & 2a_1 + 3a_2 + 5 = b_1 \\ iv) & 2a_1 + a_2 = b_2. \end{array}\right\} \quad \cdots \quad \left\{ \begin{array}{l} a_1 = 1/8 \\ a_2 = -5/4 \end{array}\right.$$

- The *particular* solution is then

$$\boxed{x_p(t) = \underbrace{\begin{pmatrix} 1/8 \\ -5/4 \end{pmatrix}}_{=:a} + \underbrace{\begin{pmatrix} 3/2 \\ -1 \end{pmatrix}}_{=:b} t}$$

Example 4. *Consider the system*

$$\mathbf{x}' = \underbrace{\begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}}_{=:A} \mathbf{x} - \underbrace{\begin{pmatrix} 15 \\ 4 \end{pmatrix} te^{-2t}}_{=:f(t)}.$$

$\{e^{\lambda t}, te^{\lambda t}\}$

$\{v_1 e^{\lambda t}, (v_1 t + v_2)e^{\lambda t}\}$

Solution.

- The complementary solution is:

$$\mathbf{x}_c(t) = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}.$$

- The trial particular solution is:

$$x_p(t) = \left(\underbrace{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{=:a} + \underbrace{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}_{=:b} t \right) e^{-2t}.$$

duplication

$$\boxed{x_p(t) = \left(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} t^2 + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right) e^{-2t}.}$$

→ six scalar coefficients.

→ it is easier to use the method of variation of parameters.

Variation of Parameters

- **Problem:** Find a *particular* solution $\mathbf{x}_p$ of the nonhomogeneous linear system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$$

given that we have already found the general solution of the homogeneous system

$$\mathbf{x}_c(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_n\mathbf{x}_n(t).$$

- Using the *Fundamental matrix of solutions*, the general solution can be written as

$$\mathbf{x}_c(t) = \mathbf{\Phi}(t)c = \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \ldots & x_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

- **Idea:** we seek a particular solution of the form

$$\boxed{x_p(t) = \Phi(t)\, u(t).}$$

- The derivative of the particular solution is

$$x_p'(t) = \Phi'(t)\, u(t) + \Phi(t)\, u'(t)$$

- Substitution of $\mathbf{x}_p$ and $\mathbf{x}_p'$ into the nonhomogenoeus equation yields

$$x_p'(t) = A x_p + f \iff \cancel{\Phi'(t)\, u(t)} + \Phi(t)\, u'(t) = \cancel{\underbrace{A(t)\, \Phi(t)}_{\Phi'(t)}\, u(t)} + f(t)$$

$$x_i' = A x_i$$

- Observe

$$\underbrace{\begin{pmatrix} | & | & & | \\ x_1' & x_2' & \ldots & x_n' \\ | & | & & | \end{pmatrix}}_{\Phi'(t)} = A(t) \underbrace{\begin{pmatrix} | & | & | & & | \\ x_1 & x_2 & x_3 & \ldots & x_n \\ | & | & | & & | \end{pmatrix}}_{=:\, \Phi(t)}$$

- Thus

$$\Phi(t)\, u'(t) = f(t).$$

$$u'(t) = \Phi^{-1}(t)\, f(t) \Rightarrow \boxed{u(t) = \int \Phi^{-1}(t)\, f(t)\, dt.}$$

$$\boxed{x_p(t) = \Phi(t) \int \Phi^{-1}(t)\, f(t)\, dt.}$$

Theorem - Variation of Parameters.

If $\mathbf{\Phi}(t)$ is a fundamental matrix for the homogeneous system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$, then a *particular solution* of the nonhomogeneous system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$$

is given by

$$\boxed{\mathbf{x}_p(t) = \mathbf{\Phi}(t)\int \mathbf{\Phi}(t)^{-1}\mathbf{f}(t)dt.}$$

- Consider the constant-coefficient case $\mathbf{A}(t) = \mathbf{A}$ of the nonhomogeneous IVP

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t), \quad \mathbf{x}(0) = \mathbf{x}_0.$$

- Here, we can use as a fundamental matrix

$$\boxed{\Phi(t) = e^{At}.} \Rightarrow \Phi(0) = I$$

$$\Rightarrow \Phi(t)\Phi^{-1}(0) = e^{At}I = e^{At}.$$

$$e^{At} = \Phi(t)\Phi^{-1}(0)$$

$$(e^{As})^{-1} = e^{-As}.$$

- Then, the above theorem yields

$$\boxed{x_p(t) = e^{At}\int_0^t \underbrace{(e^{As})^{-1}}_{e^{-As}.} f(s)\, ds.}$$

- Thus, the general solution of the homogeneous system is

$$x(t) = x_c(t) + x_p(t) = e^{At}x_0 + e^{At}\int_0^t e^{-As}f(s)\, ds.$$

$$\boxed{x(t) = e^{At}x_0 + \int_0^t e^{-A(s-t)}f(s)\, ds.}$$

Example 5. *Use the method of variations of parameters to solve the IVP*

$$\mathbf{x}' = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

Solution.

$$(e^{As})^{-1} = \begin{pmatrix} 1-3t & t \\ -9t & 1+3t \end{pmatrix}$$

- The corresponding *matrix exponential* is

$$e^{\mathbf{A}t} = \begin{pmatrix} 1+3t & -t \\ 9t & 1-3t \end{pmatrix}.$$

$$x(t) = e^{At}x_0 + e^{At}\int_0^t (e^{As})^{-1} f(s)\, ds.$$

$$e^{-At}x(t) = x_0 + \int_0^t \begin{pmatrix} 1-3s & s \\ -9s & 1+3s \end{pmatrix}\begin{pmatrix} 7 \\ 5 \end{pmatrix} ds.$$

$$= \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \int_0^t \begin{pmatrix} 7-16s \\ 5-48s \end{pmatrix} ds.$$

$$= \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \begin{pmatrix} \int_0^t 7-16s\, ds \\ \int_0^t 5-48s\, ds \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 7t-8t^2 \\ 5t-24t^2 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} 1+3t & -t \\ 9t & 1-3t \end{pmatrix}\begin{pmatrix} 7t-8t^2 \\ 5t-24t^2 \end{pmatrix}$$

$$= \quad \vdots$$

$$= \begin{pmatrix} 3+11t+8t^2 \\ 5+17t+24t^2 \end{pmatrix}$$

Example 6. *Use the method of variations of parameters to solve the IVP*

$$\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \sec t \\ 0 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Solution.

$$(e^{As})^{-1} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

- The corresponding *matrix exponential* is

$$e^{\mathbf{A}t} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

$$x(t) = \cancel{e^{At} x_0}^{\,0} + e^{At} \int_0^t (e^{As})^{-1} f(s)\, ds.$$

$$e^{-At} x(t) = \int_0^t (e^{As})^{-1} f(s)\, ds.$$

$$= \int_0^t \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix} \begin{pmatrix} \frac{1}{\cos s} \\ 0 \end{pmatrix} ds.$$

$$= \int_0^t \begin{pmatrix} 1 \\ -\tan s \end{pmatrix} ds = \begin{pmatrix} \int_0^t 1\, ds. \\ -\int_0^t \tan s\, ds \end{pmatrix}$$

$$= \begin{pmatrix} t \\ -\ln(\cos t) \end{pmatrix}$$

$$x(t) = e^{At} \begin{pmatrix} t \\ -\ln(\cos t) \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} t \\ -\ln(\cos t) \end{pmatrix}$$

$$\boxed{x(t) = \begin{pmatrix} t\cos t + \ln \cos t \sin t \\ t\sin t - \ln \cos t \cos t \end{pmatrix}}$$