MA 266 Midterm Review

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Sec 5.3 Gallery of Solns for Linear Systems

Example 1. *Solve the linear system:*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}}_{=\mathbf{A}} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Graph the corresponding phase portrait/diagram.

Solution

- The characteristic equation is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \lambda^2 + \lambda - 6 = 0.$$

- Hence, the eigenvalues of $\mathbf{A}$ are

$$\lambda_1 = 2 > 0 \text{ and } \lambda_2 = -3 < 0 \implies (0,0) \text{ saddle point}$$

- The eigenvector equation is:

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- For $\lambda = 2$, this yields

$$(\mathbf{A} - 2\mathbf{I})\mathbf{v} = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- The corresponding non-trivial eigenvector is

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- Similarly, for $\lambda = -3$, this yields

$$(\mathbf{A} + 3\mathbf{I})\mathbf{v} = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- The corresponding non-trivial eigenvector is

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

- The general solution is

$$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$$

.

- The *phase portrait is*

Example 2. *Determine the nature of the eigenvalues and eigenvectors associated to the following phase portrait.*

Example 3. *Determine what happens when the eigenvalues of the* 2×2 *linear system are complex numbers.*

Solution

- Let's write the eigenvalues as

- **Case** if $p = 0$, the general solution is

$$\mathbf{x}(t) = c_1(\mathbf{a}\cos qt - \mathbf{b}\sin qt) + c_2(\mathbf{b}\cos qt + \mathbf{a}\sin qt).$$

- **Case** if $p \neq 0$, the general solution is

$$\mathbf{x}(t) = e^{pt}(c_1(\mathbf{a}\cos qt - \mathbf{b}\sin qt) + c_2(\mathbf{b}\cos qt + \mathbf{a}\sin qt)).$$

Example 4. *Determine what happens when the eigenvalues of the 2×2 linear system are equal.*

Solution

- **Complete eigenvalue.** If there are two independent eigenvectors, then they span the plane and so *every vector is an eigenvector with this same eigenvalue.* To see this, let

- **Defective eigenvalue.** The eigenspace corresponding to λ is one-dimensional.

Example 5. ***(Spring 2019) - Sec 5.3*** *Given*

$$\mathbf{x}' = \begin{pmatrix} 1 & \alpha \\ 3 & 1 \end{pmatrix} \mathbf{x}$$

For which values of α*, the origin* $(0,0)$ *is a saddle-point.*

Example 6. ***(Fall 2019) - Sec 5.3*** *For the system*

$$\mathbf{x}' = \begin{pmatrix} 5 & 5 \\ -8 & -7 \end{pmatrix} \mathbf{x}$$

the origin $(0, 0)$ *is:*

a) a saddle point

b) a proper node source

c) a center point

d) a spiral sink

e) a spiral source

Example 7. ***(Fall 2018) - Sec 5.2*** *The real* 2×2 *matrix* $\mathbf{A}$ *has an eigenvalue* $\lambda_1 = 3 + i2$ *with corresponding eigenvector* $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$. *Find the general solution of the linear system* $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Solution.

Formula for distinct complex eigenvalues

- *Distinct* complex conjugate eigenvalues $\lambda_{1,2} = p \pm iq$ with eigenvectors $\mathbf{v}_{1,2} = \mathbf{a} \pm i\mathbf{b}$ produce two linearly independent *real-valued* vector solutions:

$$\mathbf{x}_1(t) = e^{pt}(\mathbf{a}\cos qt - \mathbf{b}\sin qt)$$
$$\mathbf{x}_2(t) = e^{pt}(\mathbf{b}\cos qt + \mathbf{a}\sin qt)$$

Example 8. ***(Spring 2018) - Sec 5.5*** *Find the general solution of the linear system*

$$\mathbf{x}' = \begin{pmatrix} -4 & 4 \\ -1 & -8 \end{pmatrix} \mathbf{x}$$