MA 266 Lecture 6

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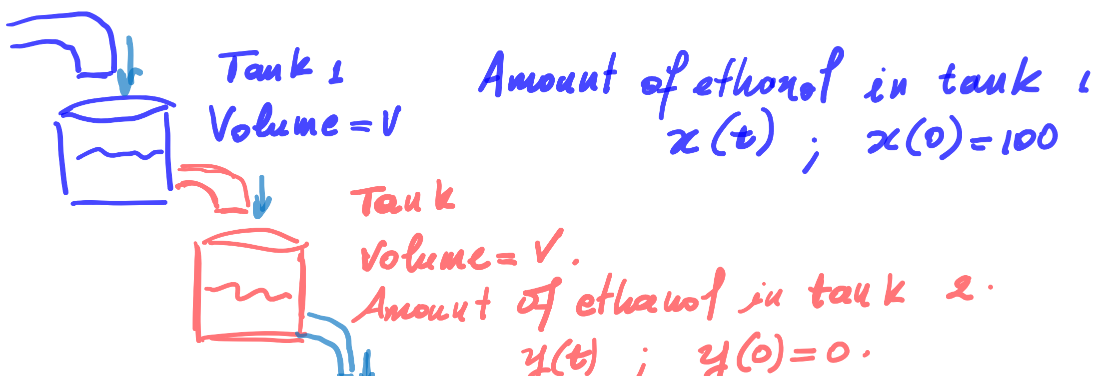
Cascade of Tanks

Example 1. *Suppose we have a cascade of tanks. Tank 1 initially contains 100 gal of pure ethanol and tank 2 initially contains 100 gal of water. Pure water flows into tank 1 at 10 gal/min and the two other flow rates are 10 gal/min. a) Find the amount $x(t)$ and $y(t)$ of ethanol in the two tanks at time $t \geq 0$.*



Let $x(t)$ denote the amount of ethanol at time t for tank 1 and let $y(t)$ denote the amount of ethanol at time t for tank 2. Let's find the differential equation for $x(t)$. The data for tank 1 is:

- inflow rate: $r_i^1 = 10$ gal/min
- inflow concentration: $c_i^1 = 0$ (water)
- outflow rate: $r_o^1 = 10$ gal/min.
- volume: $V^1 = V = 100$ gals (remains constant).
- initial amount of ethanol: $x(0) = 100$ gal

The differential equation for $x(t)$ is then:

$$\frac{dx}{dt} = -\frac{r_o^1}{V}x(t) = -\frac{1}{10}x(t).$$

The above is a linear equation with $P(t) = 1/10$ and $Q(t) = 0$. So, the integrating factor is:

$$\rho(t) = e^{\int .1dt} = e^{t/10}.$$

The *general* solution is then

$$x(t) = e^{-t/10}\left(\int 0 \cdot \rho(t)dt + C\right) = Ce^{-t/10}.$$

Using the initial condition $x(0) = 100$, we find that **the amount of ethanol in tank 1** for $t \geq 0$ is:

$$x(t) = 100e^{-t/10}.$$

The data for tank 2 is:

- inflow rate: $r_i^2 = r_o^1 = 10$ gal/min
- inflow concentration: $c_i^2 = \frac{x(t)}{V}$
- outflow rate: $r_o^2 = 10$ gal/min.
- volume: $V = 100$ gals (remains constant).
- initial amount of ethanol: $y(0) = 0$ gal (only water)

The differential equation for $y(t)$ is

$$\begin{aligned}
\frac{dy}{dt} &= r_i^2 c_i^2 - \frac{r_o^2}{V} y(t) \\
&= 10 \cdot \frac{x(t)}{V} - \frac{1}{10} y(t) \\
&= \frac{10}{100} \cdot 100e^{-t/10} - \frac{1}{10} y(t) \\
&= 10e^{-t/10} - \frac{1}{10} y(t).
\end{aligned}$$

The above is a linear equation with $P(t) = \frac{1}{10}$ and $Q(t) = 10e^{-t/10}$. To solve the linear equation, we use the integrating factor:

$$\rho(t) = e^{t/10}$$

Thus, the general solution is:

$$\begin{aligned}
y(t) &= e^{-t/10}\left(\int 10e^{-t/10} e^{t/10} dt + C\right) \\
&= e^{-t/10} \cdot (10t + C).
\end{aligned}$$

Using the initial condition $y(0) = 0$, we find that $C = 0$. Thus, **the amount of ethanol in tank 2** for $t \geq 0$ is:

$$y(t) = 10te^{-t/10}.$$

Example 2. *Find the maximum amount of ethanol ever in tank 2.*

We know that for $t \geq 0$, the amount of ethanol in tank 2 is given by:

$$y(t) = 10te^{-t/10}.$$

To find the maximum value, we first need to solve

$$y'(t^*) = 0$$

for t^*. Using the product rule, we have

$$\begin{aligned} y'(t) &= 10(e^{-t/10} - \frac{t}{10}e^{-t/10}) \\ &= e^{-t/10}(10 - t). \end{aligned}$$

The above equation is zero when $t^* = 10$. We now check if $y(10)$ is a maximum.

- for $t \in [0, 10)$, $y'(t) = e^{-t/10}(10 - t) > 0$. So, $y(t)$ increases for $t \in [0, 10)$.
- for $t > 10$, $y'(t) = e^{-t/10}(10 - t) < 0$. So, $y(t)$ decreases for $t > 0$.

The above implies that $y(t)$ reaches its maximum at $t = 10$ min. Thus, **the maximum amount of ethanol in tank 2** is:

$$y(10) = 100e^{-1} \approx 36.79 \text{ gal}.$$