MA 266 Midterm Review

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Midterm Review

Example 1. ***(Fall 2018) - Sec 3.3-b.*** *Find the solution of the initial value problem*

$$y'' + y' + \frac{5}{4}y = 0, \qquad y(0) = 3, y'(0) = 1.$$

i) ch. eq'n:

$$r^2 + r + \frac{5}{4} = 0.$$

$$r_{1,2} = \frac{-1 \pm \sqrt{1-5}}{2} = -\frac{1}{2} \pm i.$$

$$\{e^{-t/2}\cos t,\ e^{-t/2}\sin t\}.$$

gen sol'n: $\boxed{y(t) = e^{-t/2}(c_1 \cos t + c_2 \sin t)}$

ii) $y'(t) = -\frac{1}{2}e^{-t/2}(c_1\cos t + c_2 \sin t) + e^{-t/2}(-c_1 \sin t + c_2 \cos t)$

ICs:

$$3 = y(0) = c_1 \Rightarrow c_1 = 3.$$

$$1 = y'(0) = -\frac{3}{2} + c_2 \Rightarrow c_2 = \frac{5}{2}.$$

sol'n IVP: $\boxed{y(t) = 3e^{-t/2}\cos t + \frac{5}{2}e^{-t/2}\sin t.}$

Example 2. ***(Spring 2017) - Sec 3.4.*** *The charge in the capacitor of an electric circuit satisfies the differential equation*

$$\boxed{9Q'' + Q = 0,}$$

with the initial conditions $Q(0) = 1$, $Q'(0) = 1$. If we write $Q(t) = R\cos(wt - \delta)$, then find R and δ.

Free undamped - motion.

$$mx'' + kx = 0.$$

$$x'' + w_0^2 x = 0.$$

i) $9r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i/3.$

$\{\cos t/3, \sin t/3\}.$

$$\boxed{Q(t) = A\cos t/3 + B.\sin t/3.}$$

ii) $Q'(t) = -\frac{A}{3}\sin t/3 + \frac{B}{3}\cos t/3.$

ICs: $1 = Q(0) = A \Rightarrow \boxed{A=1}$

$1 = Q'(0) = \frac{B}{3} \Rightarrow \boxed{B=3.}$

sol'n IVP: $Q(t) = \cos\frac{t}{3} + 3\sin\frac{t}{3}. \Leftrightarrow R\left(\cos\frac{t}{3} - \delta\right)$

Amplitude: $\boxed{R = \sqrt{A^2+B^2} = \sqrt{10}.}$

phase angle: $\cos\delta = \frac{A}{C} > 0$, $\sin\delta = \frac{B}{C} > 0.$

$\delta \in [0, \pi/2]$ $\quad\boxed{\delta = \tan^{-1}(B/A) = \tan^{-1}(3)}$

Example 3. ***(Fall 2018) - Sec 3.5.*** *Find the correct form of the particular solution of the equation*

$$y^{(4)} - y = 3e^t + te^{-t} + 4\cos t.$$

i) ch. eqn:

$$r^4 - 1 = 0.$$

$$(r^2-1)(r^2+1) = 0$$

$r_{1,2} = \pm 1.$

$r_{3,4} = \pm i$

gen sol'n:

$$\boxed{y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t.}$$

ii)

$$y_p(t) = Ae^t + (Bt + C)e^{-t} + D\cos t + E\sin t.$$

$$\boxed{y_p(t) = Ate^t + (Bt^2 + Ct)e^{-t} + Dt\cos t + Et\sin t.}$$

Example 4. ***(Spring 2019) - Sec 3.5.*** *The homogeneous differential equation*

$$\boxed{x^2y'' - xy' + y = 0.}$$

has a set of fundamental solutions given by $y_1(x) = x$ and $y_2(x) = x\ln x$. Applying the method of Variation of Parameters, find a particular solution to the nonhomogeneous equation

$$\boxed{x^2y'' - xy' + y = x, \quad x > 0.}$$

$y'' + P(x)y' + Q(x) = f(x$

$y'' - \frac{1}{x}y' + \frac{y}{x^2} = \frac{1}{x}.$

(1) $u_1'y_1 + u_2'y_2 = 0.$

(2) $u_1'y_1' + u_2'y_2' = f(x)$

$u_1 = -\int \frac{u_2 f}{W}.$

$u_2 = \int \frac{y_1 f}{W}.$

i) $y_1(x) = x \qquad y_1'(x) = 1.$

$y_2(x) = x\ln x \qquad y_2'(x) = \ln x + 1.$

$f(x) = \frac{1}{x}.$

(1): $u_1'(\cancel{x}) + u_2'(\cancel{x}\ln x) = 0 \qquad x > 0$

$\Rightarrow \boxed{u_1' = -u_2'\ln(x).}$

$-u_2'(\cancel{\ln x}) + u_2'(\cancel{\ln x}+1) = \frac{1}{x}. \Rightarrow u_2' = \frac{1}{x}$

$\boxed{u_2 = \ln(x)}$

$u_1' = -\frac{\ln(x)}{x} \Rightarrow u_1 = -\int \frac{\ln(x)}{x}\,dx$

$\boxed{u_1 = -\frac{\ln(x)^2}{2}.}$

$y_p(x) = u_1y_1 + u_2y_2 = -\frac{\ln(x)^2}{2}x + \ln(x)^2 x$

$\boxed{y_p(x) = \frac{\ln(x)^2}{2}x}$

Example 5. ***(Spring 2017) - Sec 3.6.*** *A system with external forcing term is represented by the equation*

$$y'' + 2y' + 2y = 4\cos t + 2\sin t.$$

Then, the steady state solution of the system is

$y_p(t)$

i) $r^2 + 2r + 2 = 0.$

$r_{1,2} = -1 \pm i.$

$$\boxed{y_c(t) = e^{-t}(c_1 \cos t + c_2 \sin t)}$$

ii) $y_p(t) = A\sin t + B\cos t.$

$y_p'(t) = A\cos t - B\sin t.$

$y_p''(t) = -A\sin t - B\cos t.$

$(-A - 2B + 2A)\sin t + (-B + 2A + 2B)\cos t = 4\cos t + 2\sin t$

$A - 2B = 2.$
$2A + B = 4$ $\Rightarrow$ $A = 2$, $B = 0.$

$$\boxed{y_p(t) = 2\sin t.}$$

Example 6. ***(Fall 2019) - Sec 3.5.*** *Let $y_p(t)$ be a particular solution to the following second-order differential equation obtained from the method of variation of parameters*

$$\boxed{y'' - 2y' + y = \frac{e^t}{t}. \quad t > 0.}$$

$=: \tilde{f}(t)$

Then the general solution is

$y(t) = y_c + y_p.$

i)

$r^2 - 2r + 1 = 0.$

$(r-1)^2 = 0.$

$r_{1,2} = 1 \quad (k=2)$

$\{e^t, te^t\}.$

$$\boxed{y_c(t) = c_1 e^t + c_2 t e^t}.$$

ii) $y_1(t) = e^t \qquad y_1' = e^t \qquad f(t) = \frac{e^t}{t}$

$y_2(t) = te^t \qquad y_2' = e^t + te^t.$

iii) $u_1' y_1 + u_2' y_2 = 0.$

$u_1' y_1' + u_2' y_2' = \frac{e^t}{t}.$

(1) $u_1' \cancel{e^t} + u_2' t\cancel{e^t} = 0. \Rightarrow u_1' = -t u_2'$

$\cancel{-u_2'(te^t)} + u_2'(\cancel{e^t} + \cancel{te^t}) = \frac{\cancel{e^t}}{t}$

$y_p = u_1 y_1 + u_2 y_2$

$u_2' = \frac{1}{t} \Rightarrow \boxed{u_2 = \ln(t)}$

$u_1' = -t u_2' = -1 \Rightarrow \boxed{u_1 = -t}$

$\therefore \boxed{\begin{aligned} y_p(t) &= -te^t + \ln(t) te^t \\ &= te^t(\ln(t) - 1) \end{aligned}}$

$$\boxed{y(t) = y_c(t) + y_p(t).}$$

Example 7. ***(Spring 2019) Sec 3.5.*** *Find a particular solution of the second-order differential equation*

$$y'' - 2y' + 5y = 20\sin t.$$

i)

$\vdots$

$$\boxed{y_c(t) = c_1 e^t \cos 2t + c_2 e^t \sin 2t.}$$

ii) Trial sol'n:

$$\boxed{y_p(t) = A\sin t + B\cos t.}$$

$$y_p'(t) = A\cos t - B\sin t.$$

$$y_p''(t) = -A\sin t - B\cos t.$$

$\vdots$

$$\begin{aligned} 2B + 4A &= 20. \\ -2A + 4B &= 0 \end{aligned} \implies \begin{aligned} A &= 4 \\ B &= 2. \end{aligned}$$

So, $\boxed{y_p(t) = 4\sin t + 2\cos t.}$

Example 8. ***(Spring 2018) - Sec 5.3.*** *Find the type of equilibrium point the origin* $(0,0)$ *is for the linear system:*

$$\mathbf{x}' = \begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix} \mathbf{x}.$$

i) $\det(A - \lambda I) = \det\begin{pmatrix} -2-\lambda & 3 \\ 1 & -4-\lambda \end{pmatrix}$

$\vdots$

$$\boxed{= \lambda^2 + 6\lambda + 5 = 0.}$$

ii) eig's A: $\lambda_1 = -1$, $\lambda_2 = -5$. $\Rightarrow$ $(0,0)$ stable node

or improper node sink.

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Example 5. ***(Spring 2019) - Sec 5.3*** *Given*

$$\mathbf{x}' = \begin{pmatrix} 1 & \alpha \\ 3 & 1 \end{pmatrix} \mathbf{x}$$

For which values of α, *the origin* $(0,0)$ *is a saddle-point.*

1) $\boxed{\lambda_1 > 0 \;;\; \lambda_2 < 0 \Rightarrow \text{saddle point.}}$

2)

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & \alpha \\ 3 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)^2 - 3\alpha.$$

$$\boxed{= \lambda^2 - 2\lambda + 1 - 3\alpha = 0.}$$

3) quad. formula.

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4 - 4(1-3\alpha)}}{2}.$$

$$= \frac{\cancel{2} \pm \cancel{2}\sqrt{3\alpha}}{\cancel{2}} = 1 \pm \sqrt{3\alpha}.$$

i) $\underline{3\alpha > 0}$

ii) $\sqrt{3\alpha} > 1$

$\rightarrow \boxed{\alpha > \frac{1}{3}.}$

Example 6. ***(Fall 2019) - Sec 5.3*** *For the system*

$$\mathbf{x}' = \begin{pmatrix} 5 & 5 \\ -8 & -7 \end{pmatrix} \mathbf{x}$$

the origin $(0, 0)$ *is:*

a) a saddle point

b) a proper node source

c) a center point

d) $\boxed{\text{a spiral sink}}$

e) a spiral source

$x' = Ax$.

$Ax = 0$, $\Rightarrow \exists A^{-1}$ $\Rightarrow x = 0$

1) $\det(A - \lambda I) = \begin{pmatrix} 5-\lambda & 5 \\ -8 & -7-\lambda \end{pmatrix}$

$= (5-\lambda)(-7-\lambda) + 40.$

$\boxed{= \lambda^2 + 2\lambda + 5 = 0.}$

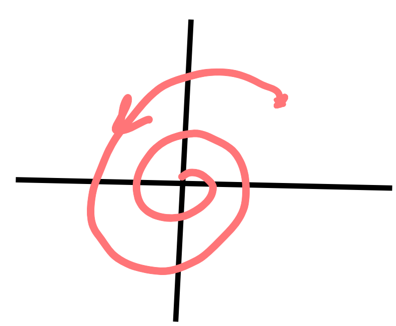
2) A:

$\lambda_{1,2} = \dfrac{-2 \pm \sqrt{4-20}}{2}.$

$= -1 \pm 2i.$

$= p \pm qi$

$p < 0.$



spiral sink.

Example 7. ***(Fall 2018) - Sec 5.2*** *The real* 2×2 *matrix* $\mathbf{A}$ *has an eigenvalue* $\lambda_1 = 3 + i2$ *with corresponding eigenvector* $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$*. Find the general solution of the linear system* $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Solution.

Formula for distinct complex eigenvalues

- *Distinct* complex conjugate eigenvalues $\lambda_{1,2} = p \pm iq$ with eigenvectors $\mathbf{v}_{1,2} = \mathbf{a} \pm i\mathbf{b}$ produce two linearly independent *real-valued* vector solutions:

$$\mathbf{x}_1(t) = e^{pt}(\mathbf{a}\cos qt - \mathbf{b}\sin qt)$$
$$\mathbf{x}_2(t) = e^{pt}(\mathbf{b}\cos qt + \mathbf{a}\sin qt)$$

$$\lambda_1 = 3 + 2i = p + iq.$$

$$v_1 = \begin{pmatrix} 1 \\ 2i \end{pmatrix} = a + ib.$$

$$= \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{=:a} + i \underbrace{\begin{pmatrix} 0 \\ 2 \end{pmatrix}}_{=:b}.$$

$$\boxed{\begin{aligned} x_1(t) &= e^{3t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ 2 \end{pmatrix}\sin 2t\right) \\ x_2(t) &= e^{3t}\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix}\cos 2t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin 2t\right). \end{aligned}}$$

$$\Rightarrow \boxed{x(t) = c_1 x_1(t) + c_2 x_2(t)}.$$

Example 8. ***(Spring 2018) - Sec 5.5*** *Find the general solution of the linear system*

$$\mathbf{x}' = \begin{pmatrix} -4 & 4 \\ -1 & -8 \end{pmatrix} \mathbf{x}$$

1) $\det(A-\lambda I) = \det\begin{pmatrix} -4-\lambda & 4 \\ -1 & -8-\lambda \end{pmatrix}$

$$\boxed{= \lambda^2 + 12\lambda + 36 = 0.}$$

2) A:

$\lambda_1 = -6 \quad (K=2)$

i) $(A-\lambda I)^2 v_2 = 0 \rightarrow v_2.$

ii) $(A-\lambda I) v_2 = v_1 \neq 0$

$$\boxed{\begin{aligned} x_1(t) &= v_1 e^{\lambda t} \\ x_2(t) &= (v_1 t + v_2) e^{\lambda t}. \end{aligned}}$$

i) $(A-\lambda I)^2 = (A+6I)^2 = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$0\, v_2 = 0$

$\boxed{v_2 = (1,0)^T}$

$v_1 = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \neq 0$

$\boxed{v_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}}$

$$\boxed{x(t) = v_1 e^{-6t} + (v_1 t + v_2) e^{-6t}.}$$