# MA 266 Lecture 6

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## Sec 1.5-b  Linear First-Order Equations - part 2

- To solve the linear first order equation:

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{1}$$

- We use the integrating factor ____________.

- Obtain the *explicit* general solution:

**Theorem 1.** *If the functions $P(x)$ and $Q(x)$ are continuous on the open interval $I$ containing the point $x_0$, then the initial value problem*

$$y' + P(x)y = Q(x), \qquad y(x_0) = y_0$$

*has a unique solution $y(x)$ on $I$ , given by the formula in* ____________ *with an appropriate value of $C$.*

- To find the particular solution, i.e., to find ____________,

**Example 1.** *Solve the initial value problem*

$$x^2 \frac{dy}{dx} + xy = \sin(x), \qquad y(1) = y_0$$

- Linear first order form:

- $P(x) =$ __________________ and $Q(x) =$ __________________.

- With __________________, the integrating factor:

- The desired ________________________ solution is:

**Example 2.**

a) Show that

$$y_c(x) = Ce^{-\int P(x)dx}$$

is a *general* solution of $\frac{dy}{dx} + P(x)y = 0$.

b) Show that

$$y_p(x) = e^{-\int P(x)dx}\left[\int \left(Q(x)e^{\int P(x)dx}\right) dx\right]$$

is a solution of $\frac{dy}{dx} + P(x)y = Q(x)$.

c) Show that $y(x) = y_c(x) + y_p(x)$ is a general solution of $\frac{dy}{dx} + P(x)y = Q(x)$.

## Mixture problems

Consider a mixture of a solute and a solvent (e.g., salt dissolved in water). There is both inflow and outflow. Our goal is to compute the amount $x(t)$ of the solute in the tank at time $t$, given the amount $x(0)$ at time $t = 0$. We assume that a solution with a concentration of $c_i$ (g/L) of solute flows into the tank at constant rate $r_i$ (L/s), and that the solution in the tank flows out at constant rate $r_o$ (L/s).

- The amount of solute $x(t)$ in the tank satisfies the differential equation:

$$\frac{dx}{dt} = r_i c_i - \frac{r_o}{V} x.$$

**Example 3.** *Assume that a lake A has a volume of 480 $km^3$ and that its rate of inflow from Lake B and outflow to Lake C are both 350 $km^3$ per year. Suppose that $t = 0$ (years), the pollutant concentration is five times that of Lake B. If the outflow henceforth is perfectly mixed lake water, how long it will take to reduce the pollution concentration in Lake A to twice that of Lake B?*

- We have:

- The differential equation:

- The particular solution:

- To find when ___________________, we solve:

**Example 4.** *Rework the previous example for the case of Lake C, which empties to a river X and receives inflow from Lake A. The volume of Lake C is 1640 km$^3$ and an inflow-outflow rate of 410 km$^3$ per year.*

- We have:

- The differential equation:

- The particular solution:

- To find when ____________________, we solve:

**Example 5.** *Suppose we have a cascade of tanks. Tank 1 initially contains 100 gal of pure ethanol and tank 2 initially contains 100 gal of water. Pure water flows into tank 1 at 10 gal/min and the two other flow rates are 10 gal/min. a) Find the amount* $x(t)$ *and* $y(t)$ *in the two tanks at time* $t \geq 0$.

**Example 6.** *Find the maximum amount of ethanol ever in tank 2.*