MA 266 Lecture 10

Christian Moya, Ph.D.

Sec 2.2 Equilibrium Solutions and Stability

Qualitative analysis

- Often we use explicit solutions of differential eq's to answer specific numerical questions.
- However, when the DE impossible to solve explicitly, it is often possible to extract

  ______________________________ about general properties of solutions.
- For example,

Example 1. *(Newton's Law of Cooling:) Let:*

- temperature of body: __________
- initial temperature: __________
- body immersed in a medium with temperature = __________

Newton's law of cooling:

Separating variables:

Q: $\lim_{t\to\infty} x(t) =$**?**

Is _______________ a solution?

Autonomous equations

Defintion 1. *An __________________ first-order differential equation takes the form:*

That is, the R.H.S. is __________________ of ____________.

Critical points and equilibrium solutions

Defintion 2. *The critical points of __________________ are the solutions of the algebraic equation:*

If __________________ is a critical point of this equation. Then

- Such a solution is called an ________________________ of the differential equation.
- Qualitative information (behavior) can be described in terms of *critical points*.

Example 2. *Find the critical points of:*

a) $\frac{dx}{dt} = x^2 - 4$

b) $\frac{dx}{dt} = (2 - x)^3$

Stability of critical points (Stable vs. Unstable)

Defintion 3. *A critical point* ____________ *of* ____________ *is* ***stable*** *provided that:*

- If __________________ is sufficiently close to c, then __________________ remains close to c for all $t > 0$.
- (**Formally**) the critical point is *stable* if, for every __________________, there exists

  a __________________ such that

- Otherwise, the critical point is *unstable*.

Example 3. *Consider the logistic initial value problem:*

Critical points:

The particular solution:

Equilibrium solutions:

We observed (in previous lecture):

But if __________________________:

Example 4. *Consider:*

$$\frac{dx}{dt} = kx - x^3$$

a) *Let $k \leq 0$. Show that the only critical point is stable:*

b) Let $k > 0$. Analyze the stability of critical point(s).

Bifurcation points

- As we gradually increase the value of the parameter __________________.

- We have seen that the differential equation has

- The value ____________, for which the qualitative nature of the solutions changes

  as the parameter increases, is called a _________________________ for the differential equation containing the parameter.

Bifurcation diagram

- A common way to visualize the corresponding "bifurcation" in the solutions is to plot the *bifurcation diagram* for the equation.

- This diagram consists of all points _________________________, where c is a critical point of the equation

Example 5. *Construct the bifurcation diagram of the following logistic equation with harvesting:*

$$\frac{dx}{dt} = x(4 - x) - h$$

Critical points:

Bifurcation diagram:

Example 6. *Given the differential equation:*

$$\frac{dx}{dt} = -(3 - x)^2$$

Analyze the stability of the critical point(s).

Critical points:

Separating variables:

Particular solution: