MA 266 Lecture 11

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Sec 2.2-b Bifurcation Points

Sec 2.3 Acceleration-Velocity Models

Review example from Lecture 10

Example 1. *Consider:*

$$\frac{dx}{dt} = kx - x^3$$

a) *Let $k \leq 0$. Show that the only critical point is stable.*

Solution: Note that if $k \leq 0$, then we can let $k := -a^2$. The **first** step is to find *critical points*. To this end, we solve the following algebraic equation:

$$f(x) = 0 \iff (k - x^2)x = -\underbrace{(a^2 + x^2)}_{>0} x = 0.$$

So, the only *critical point* is:

$$c = 0.$$

The **second** step is to compute the solution of the above differential equation. Observe that this equation is *separable*. Let $a > 0$. Then by separating variables, we obtain:

$$\int \frac{dx}{(a^2 + x^2)x} = \int -dt + C$$

$$\iff \frac{x^2}{(a^2 + x^2)} = Ce^{-2a^2 t}$$

By solving the above for x^2, we obtain the general solution of the differential equation:

$$x^2 = \frac{a^2 Ce^{-2a^2 t}}{1 - Ce^{-2a^2 t}}$$

In the **third** step, we use the above general solution to determine the stability of the critical point $c = 0$. To this end, we check what happens with the solution as $t \to \infty$. Clearly, if we let $t \to \infty$, $Ce^{-2a^2 t} \to 0$. As a result:

$$x(t) \to 0 \equiv c \text{ as } t \to \infty.$$

Thus, we conclude that the critical point $c = 0$ is *stable*.

b) Let $k > 0$. Analyze the stability of critical point(s).

Solution: Note that if $k > 0$, then we can let $k = a^2$ with $a \neq 0$. As before, the **first** step is to find the critical points by solving the following algebraic equation:

$$f(x) = 0 \iff (k - x^2)x = \underbrace{(a^2 - x^2)}_{\geq 0} x = 0.$$

So, when $k > 0$, we have *three* critical point:

$$\begin{cases} c_1 = 0 \\ c_2 = +a = +\sqrt{k} \\ c_3 = -a = -\sqrt{k}. \end{cases}$$

The **second** step is to compute the solution of the differential equation. Note that we can write the differential equation as follows:

$$\frac{dx}{dt} = -x(x-a)(x+a).$$

This is a separable equation. Thus, by separating variables, we obtain:

$$\int \frac{2a^2}{x(x-a)(x+a)} dx = -\int 2a^2 dt + C.$$

The partial fractions method yields:

$$\begin{aligned} &\int -\frac{2}{x} + \frac{1}{x-a} + \frac{1}{x+a} dx = -\int 2a^2 dt + C \\ \iff\ & -2\ln(x) + \ln(x-a) + \ln(x+a) = -2a^2 t + C \\ \iff\ & \frac{x^2 - a^2}{x^2} = Ce^{-2a^2 t}. \end{aligned}$$

By solving for x, we obtain the general solution:

$$x(t) = \frac{\pm\sqrt{k}}{\sqrt{1 - Ce^{-2a^2 t}}}$$

For this example, it is convenient to also compute the particular solution for the initial condition $x(0) = x_o$. Using this initial condition, we find that is $C = 1 - k/x_o^2$. As a result, the particular solution is:

$$x(t) = \frac{\pm\sqrt{k}}{\sqrt{1 - (1 - k/x_o^2)e^{-2a^2 t}}}.$$

In the **third step**, we use the above particular solution to determine the stability of the three critical points. To this end, we analyze four cases.

(Case 1) Let $x_o \in (0, \sqrt{k})$ and use the particular solution:

$$x(t) = \frac{+\sqrt{k}}{\sqrt{1-(1-k/x_o^2)e^{-2a^2t}}}.$$

- Note that for this case $(1 - k/x_o^2) < 0$.
- Hence the denominator satisfies: $\sqrt{1-(1-k/x_o^2)e^{-2a^2t}} \searrow 1$.
- As a result, $x(t)$ increases towards $+\sqrt{k}$, *i.e.*, $x(t) \nearrow +\sqrt{k}$ as $t \to \infty$.

(Case 2) Let $x_o > \sqrt{k}$ and use the particular solution:

$$x(t) = \frac{+\sqrt{k}}{\sqrt{1-(1-k/x_o^2)e^{-2a^2t}}}.$$

- Note that for this case $(1 - k/x_o^2) > 0$.
- Hence the denominator satisfies: $\sqrt{1-(1-k/x_o^2)e^{-2a^2t}} \nearrow 1$.
- As a result, $x(t)$ decreases towards $+\sqrt{k}$, *i.e.*, $x(t) \searrow +\sqrt{k}$ as $t \to \infty$.

Case 1) and 2) show that the critical point $c_2 = +\sqrt{k}$ is *stable*.

(Case 3) Let $x_o \in (-\sqrt{k}, 0)$ and use the particular solution:

$$x(t) = \frac{-\sqrt{k}}{\sqrt{1 - (1 - k/x_o^2)e^{-2a^2 t}}}.$$

- Note that for this case $(1 - k/x_o^2) < 0$.
- Hence the denominator satisfies: $\sqrt{1 - (1 - k/x_o^2)e^{-2a^2 t}} \searrow 1$.
- As a result, $x(t)$ becomes more negative ; $x(t)$ decreases towards $-\sqrt{k}$, *i.e.*, $x(t) \searrow -\sqrt{k}$ as $t \to \infty$.

(Case 4) Let $x_o < -\sqrt{k}$ and use the particular solution:

$$x(t) = \frac{-\sqrt{k}}{\sqrt{1 - (1 - k/x_o^2)e^{-2a^2 t}}}.$$

- Note that for this case $(1 - k/x_o^2) > 0$.
- Hence the denominator satisfies: $\sqrt{1 - (1 - k/x_o^2)e^{-2a^2 t}} \nearrow 1$.
- As a result, $x(t)$ becomes less negative; $x(t)$ increases towards $-\sqrt{k}$, *i.e.*, $x(t) \nearrow -\sqrt{k}$ as $t \to \infty$.

Case 3) and 4) show that the critical point $c_3 = -\sqrt{k}$ is *stable.*

Finally, **Case 1)** and **Case 3)** show that the critical point $c_1 = 0$ is *unstable.*

Bifurcation points

- As we gradually increase the value of the parameter ________________.

- We have seen that the differential equation has

- The value ___________, for which the qualitative nature of the solutions changes

  as the parameter increases, is called a ______________________ for the differential equation containing the parameter.

Bifurcation diagram

- A common way to visualize the corresponding "bifurcation" in the solutions is to plot the *bifurcation diagram* for the equation.

- This diagram consists of all points ______________________, where c is a critical point of the equation

Example 2. *Construct the bifurcation diagram of the following logistic equation with harvesting:*

$$\frac{dx}{dt} = x(4 - x) - h$$

Critical points:

Bifurcation diagram:

Acceleration-velocity models

- In chapter 1, we studied vertical motion of a mass m without considering air resistance.
- Newton's second law:

- __________________ is the (downward-direction) force of gravity.

In section 2.3, we want to take into account air resistance.

- __________________: force exerted by air resistance on the moving mass m.
- Newton's second law:

- For many problems, it suffices to model the force as:

  where ____________ and ____________ depends on the size and shape of the body, as well as the density and viscosity of the air.

Generally speaking, we have:

- ____________ for relatively low speeds.

- ____________ for high speeds.

- ____________ for intermediate speeds.

Example 3. *Consider the vertical motion of an object near the surface of the earth.*

- ____________: *mass.*

Subject to two forces:

- ____________: *downward gravitational force.*

- ____________: *air resistance force, where* ____________.

Find the particular solutions $v(t)$ *and* $y(t)$ *for the initial conditions* $v(0) = v_o$ *and* $y(0) = y_o$*.*

- Newton's law of motion:

- Separating variables:

- Velocity equation:

- Q: $\lim_{t\to\infty} v(t)$?

- Position equation:

Example 4. *Consider a body that moves horizontally with resistance* $-kv^2$ *such that:*

$$\frac{dv}{dx} = -kv^2.$$

Show that the velocity and position equations are:

$$v(t) = \frac{v_o}{1 + v_o kt} \qquad \textit{and} \qquad x(t) = x_o + \frac{1}{k}\ln(1 + v_o kt).$$

where $x(0) = x_o$ *and* $v(0) = v_o$.

- Separating variables:

- Velocity equation:

- Position equation:

- Q: $\lim_{t\to\infty} v(t), x(t)$?

Variable Gravitational Acceleration

- Consider a body (m) in vertical motion.
- Unless the body remains in the immediate vicinity of the earth's surface, the gravitational acceleration acting on the body is *not constant*.
- According to Newton's law of gravitation,
  - the gravitational force between two point masses ____________________
  - located at a distance ____________________ is:

Example 5. ***Escape velocity.*** *Consider a body with mass m and let*

- ____________________*: body's distance from earth's center at time t.*
- ____________________*: earth's radius.*
- ____________________*: earth's mass.*

Find ____________________ *such that $v(t) > 0$ for all t.*

- Velocity equation:

- Escape velocity $v_o = v(R)$: