MA 266 Lecture 12

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Sec 2.4 Numerical Approximation: Euler's method

Explicit solutions

- It is the exception rather than the rule when a differential equation of the general form

  can be solved exactly and explicitly by elementary methods.

- For example, consider the simple equation

- A solution of this equation is just an antiderivative of ________________. However, every antiderivative of this function is known to be a *nonelementary* function—one that cannot be expressed as a finite combination of the familiar functions of elementary calculus.

Alternative approach

- Construct a solution curve that starts ___________ and follows the slope field of the given differential equation $y' = f(x, y)$.

Euler's Method

- To approximate the solution of the initial value problem:

- We first select a fixed (horizontal) *step size* ____________ to use in making each step from one point to the next.
- Suppose we've started at the initial point (x_0, y_0) and after n steps have reached the point (x_n, y_n). How do we compute the coordinates of the new point (x_{n+1}, y_{n+1})?

- Thus, the coordinates of the new point __________________ are given in terms of the old coordinates by

- Given the above initial value problem, *Euler's method* with step size h consists of starting with the initial point (x_0, y_0) and applying the above iterative formulas

$$\begin{aligned} x_1 &= x_0 + h \qquad & y_1 &= y_0 + h \cdot f(x_0, y_0) \\ x_2 &= x_1 + h \qquad & y_2 &= y_1 + h \cdot f(x_1, y_1) \\ x_3 &= x_2 + h \qquad & y_3 &= y_2 + h \cdot f(x_2, y_2) \\ &\vdots & &\vdots \\ &\vdots & &\vdots \end{aligned}$$

Algorithm: Euler's method

- Given the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0,$$

- *Inputs - Euler's method:* the step size h and the initial condition (x_0, y_0).

- Apply the iterative formula

$$y_{n+1} = y_n + h \cdot f(x_n, y_n) \quad (n \geq 0)$$

- Results successive approximations:

  to the [true] values:

  of the [exact] solution __________________ at the points:

Example 1. *Apply Euler's method to approximate the solution of the following IVP on the interval* $[0, 1/2]$*:*

$$\frac{dy}{dx} = y + 1, \qquad y(0) = 1,$$

a) first with $h = 0.25$*.*

b) then with $h = 0.1$*.*

Note that the particular solution of this IVP is: $y(x) = 2e^x - 1$*.*

Solution a) With $x_0 = 0$ and $y_0 = 1$, $f(x, y) = y + 1$, and $h = 0.25$ the Euler's iterative formula yields the approximate values at the points $x_1 = 0.25$ and $x_2 = 0.5$:

$$y_1 = y_0 + h \cdot [y_0 + 1] = (1) + (0.25)\,[1 + 1] = 1.5$$

$$y_2 = y_1 + h \cdot [y_1 + 1] = (1.5) + (0.25)\,[1.5 + 1] = 2.125$$

- Note how the result of each calculation feeds into the next one.
- The resulting table of approximate values is

x	0	0.25	0.5
Approx. y	1	1.5	2.125

Solution b) With $x_0 = 0$ and $y_0 = 1$, and $h = 0.1$ the Euler's iterative formula yields the approximate values at the points $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = 0.4$, and $x_5 = 0.5$:

$$y_1 = y_0 + h \cdot [y_0 + 1] = (1) + (0.1)\,[1 + 1] = 1.2$$

$$y_2 = y_1 + h \cdot [y_1 + 1] = (1.2) + (0.1)\,[1.2 + 1] = 1.42$$

$$y_3 = y_2 + h \cdot [y_2 + 1] = (1.42) + (0.1)\,[1.42 + 1] = 1.662$$

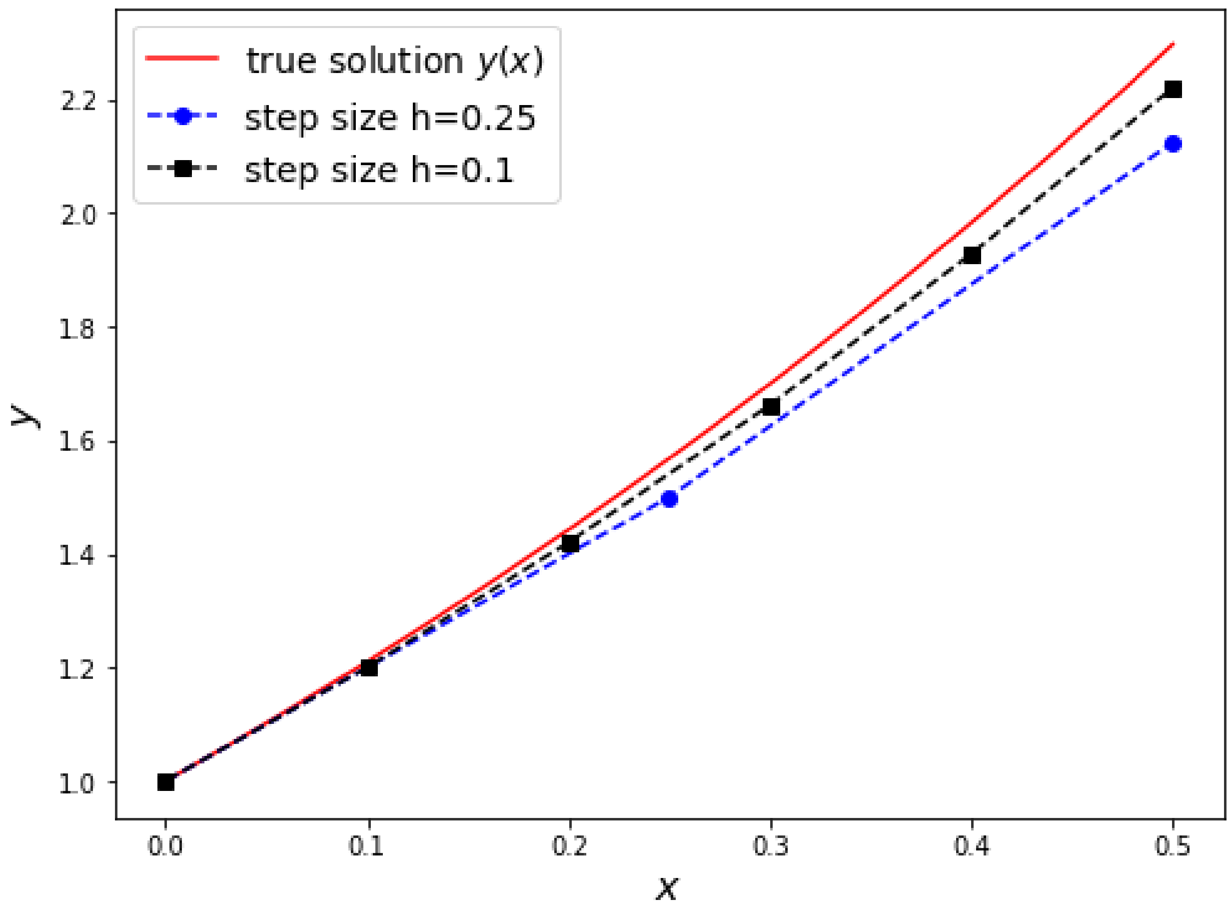
$$y_4 = y_3 + h \cdot [y_3 + 1] = (1.662) + (0.1)\,[1.662 + 1] = 1.9282$$

$$y_5 = y_4 + h \cdot [y_4 + 1] = (1.9282) + (0.1)\,[1.9282 + 1] = 2.221$$

- The resulting table of approximate values is

x	0	0.1	0.2	0.3	0.4	0.5
Approx. y	1	1.2	1.42	1.662	1.9282	2.221

- The next figure shows the graph of the true solution $y(x) = 2e^x - 1$, together with the graphs of the Euler approximations obtained with step sizes $h = 0.25$ and 0.1.



Remarks:

- __________________ the step size h __________________ the accuracy.

- __________________ the step size h __________________ the number of operations.

- Yet with any single approximation, the accuracy decreases with distance from the initial point.

Local and Cumulative Errors

- There are several sources of error in Euler's method that may make the approximation:

  unreliable for large values of n, those for which x_n is not sufficiently close to x_0.

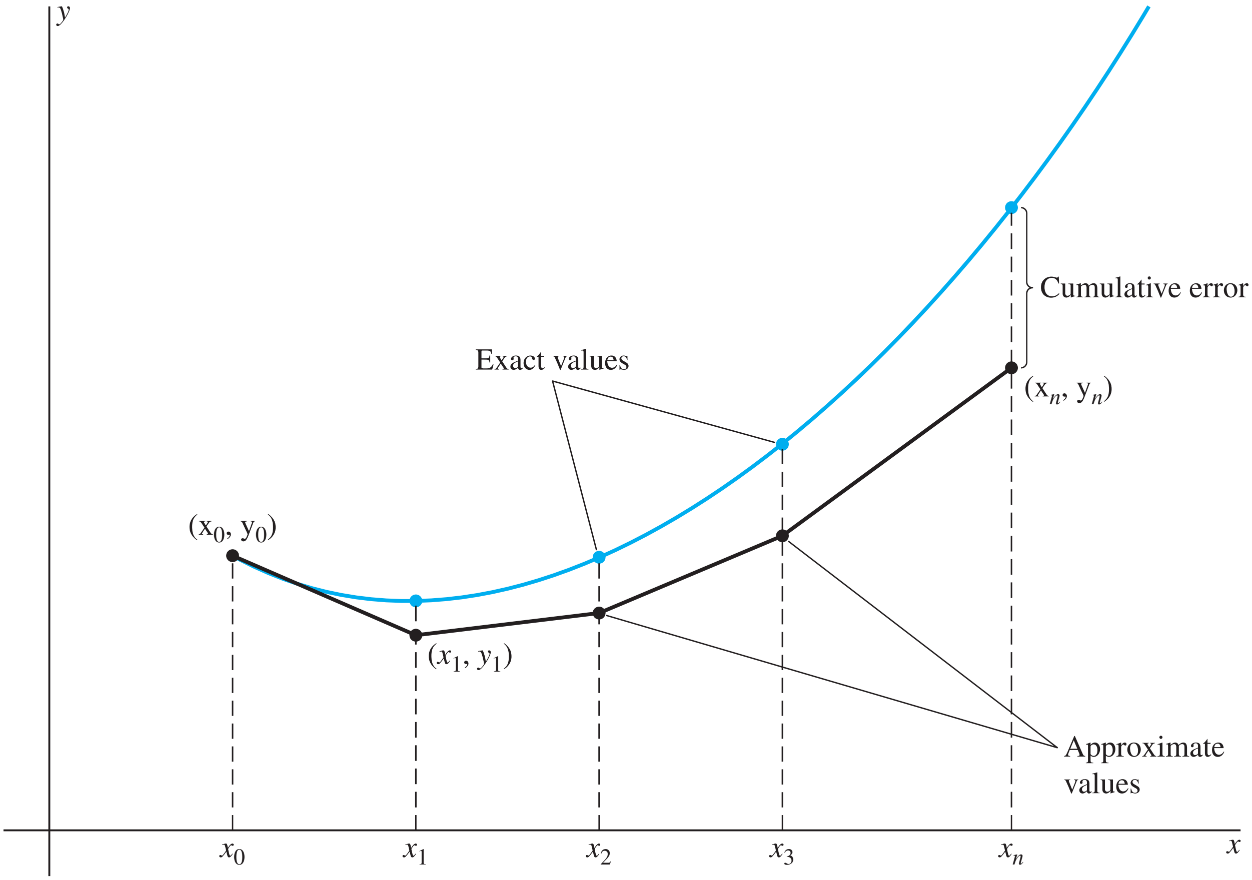
- The error in the linear approximation formula:

  is the amount by which the tangent line at (x_n, y_n) departs from the solution curve through (x_n, y_n).

Defintion 1. *The error introduced at each step in the process, is called the ________________ in Euler's method.*

- Note that y_n itself is merely an approximation to the actual value $y(x_n)$.

Defintion 2. *The* ______________________________ *at y_n is a measure of all the accumulated effects of all the local errors introduced at the previous steps*



Reducing Cumulative Error

- The usual way of attempting to reduce the cumulative error in Euler's method is to decrease the step size h.
- However, if h is too small, then (i) the number of operations may be too large, (ii) we may have to deal with computer precision/roundoff errors.

Example 2. *Consider the following logistic initial value problem:*

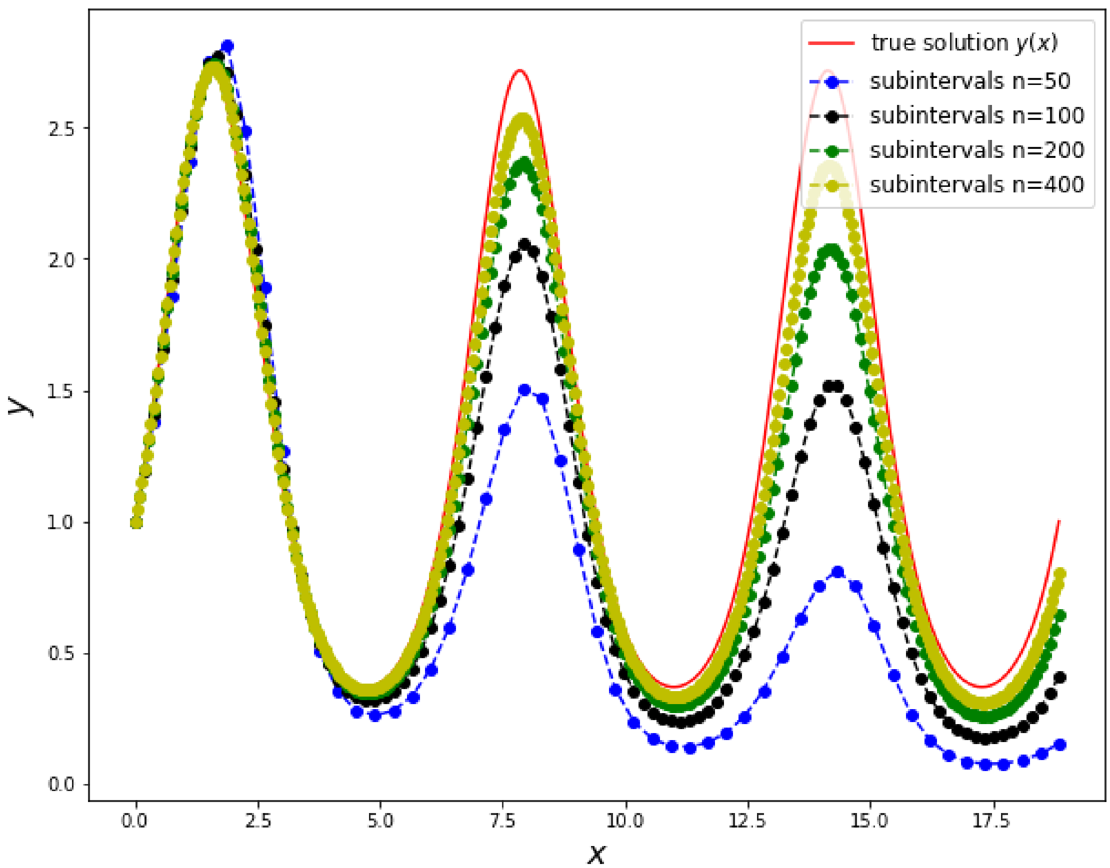
$$\frac{dy}{dx} = y\cos x, \quad y(0) = 1$$

The exact solution of the above equation is the ***periodic*** *function:* $y(x) = e^{\sin x}$. *Use Euler's method to approximate the solution in the interval* $0 \leqq x \leqq 6\pi$ *and using* $n \in \{50, 100, 200, 400\}$ *subintervals.*

- Euler's iterative formula for this examples is:

- Computing the step size h from n:

- Next figure shows the exact solution curve and approximate solution curves obtained by applying Euler's method.

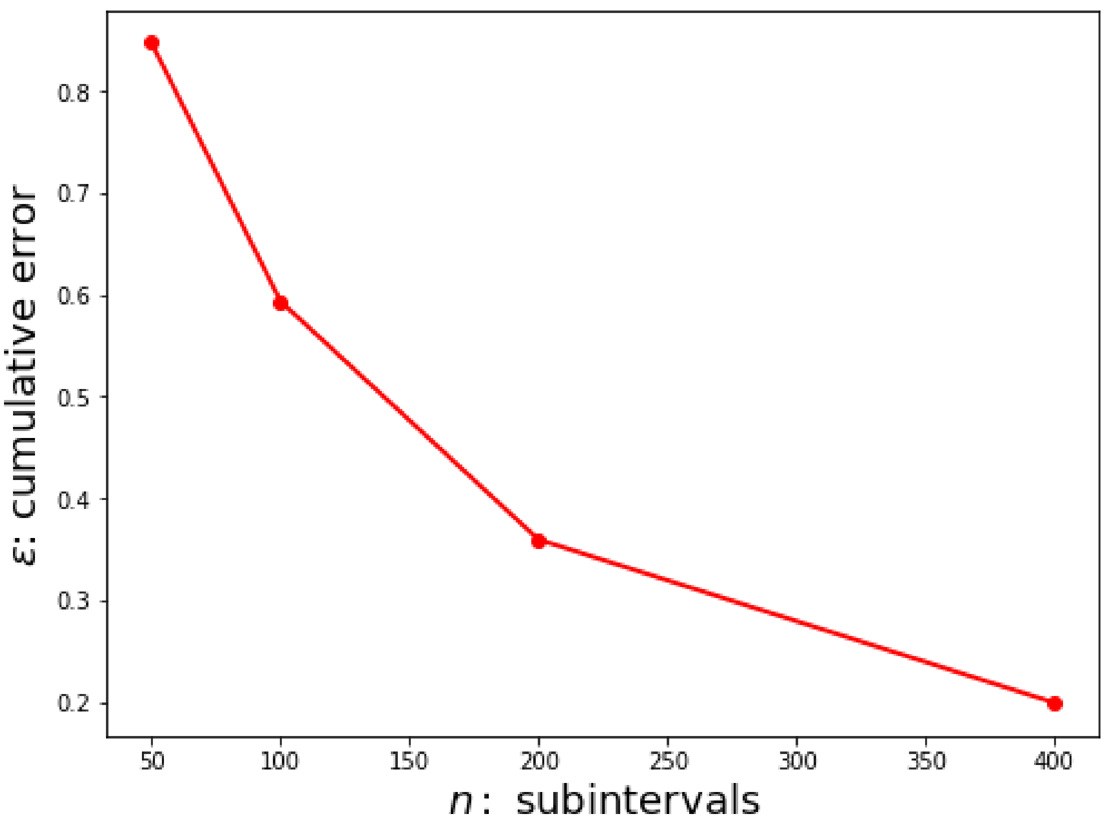


A Common Strategy

- The computations in the preceding example illustrate the common strategy of applying a numerical algorithm, such as Euler's method, several times in succession.
- We begin with a selected number n of subintervals for the first application, then double n for each succeeding application of the method.
- Visual comparison of successive results often can provide an "intuitive feel" for their accuracy.

Cumulative Error vs. Number of Intervals

- Next figure illustrates a graph comparing the cumulative error ϵ with the number of subintervals n.



A Word of Caution

Example 3. *Use Euler's method to approximate the solution of the initial value problem*

$$\frac{dy}{dx} = x^2 + y^2, \quad y(0) = 1$$

on the interval $[0, 1]$.

- The iterative formula of Euler's method:

- With step size $h = 0.1$ we obtain

$$y_1 = 1 + (0.1) \cdot [(0)^2 + (1)^2] = 1.1,$$

$$y_2 = 1.1 + (0.1) \cdot \left[(0.1)^2 + (1.1)^2\right] = 1.222,$$

$$y_3 = 1.222 + (0.1) \cdot \left[(0.2)^2 + (1.222)^2\right] \approx 1.3753,$$

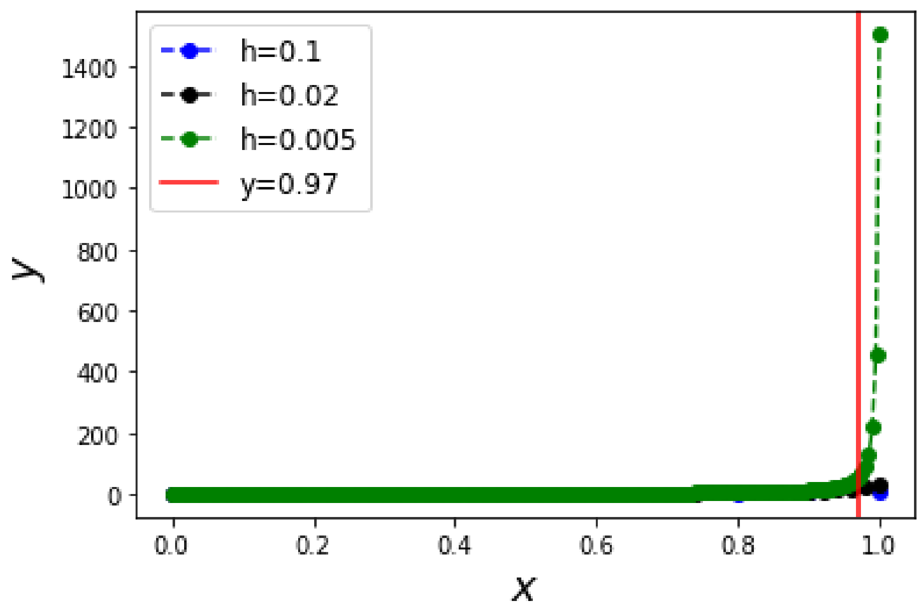
  and so forth.

- Rounded to four decimal places, the first ten values obtained in this manner are

$$\begin{aligned} y_1 &= 1.1000 & y_6 &= 2.1995 \\ y_2 &= 1.2220 & y_7 &= 2.7193 \\ y_3 &= 1.3753 & y_8 &= 3.5078 \\ y_4 &= 1.5735 & y_9 &= 4.8023 \\ y_5 &= 1.8371 & y_{10} &= 7.1895 \end{aligned}$$

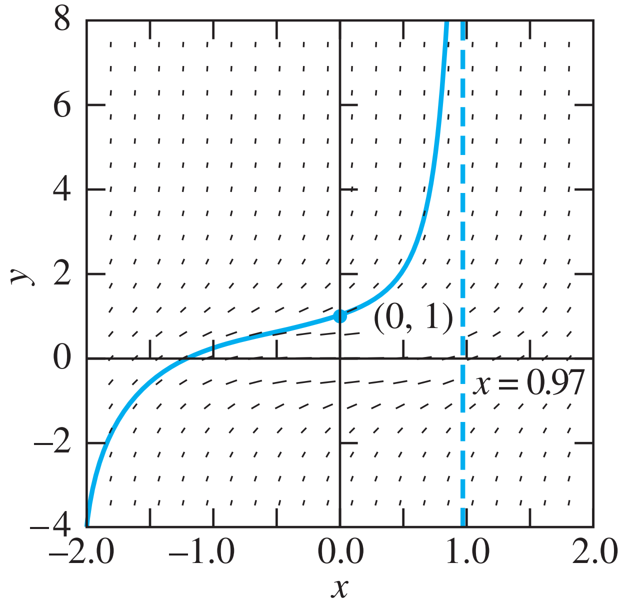
- We could naively accept these results as accurate approxmations.
- We instead can use a computer to repeat the computations with smaller values of h.
- The table on the next page shows the results obtained with step sizes $h = 0.1$, $h = 0.02$, and $h = 0.005$.

x	y with $h = 0.1$	y with $h = 0.02$	y with $h = 0.005$
0.1	1.1000	1.1088	1.1108
0.2	1.2220	1.2458	1.2512
0.3	1.3753	1.4243	1.4357
0.4	1.5735	1.6658	1.6882
0.5	1.8371	2.0074	2.0512
0.6	2.1995	2.5201	2.6104
0.7	2.7193	3.3612	3.5706
0.8	3.5078	4.9601	5.5763
0.9	4.8023	9.0000	12.2061
1.0	7.1895	30.9167	1502.2090



- Observe that now the “stability” of the numerical procedure is missing.
- Indeed, it seems obvious that something is going wrong near $x = 1$.

- Next figure provides a clue to the difficulty for approximating the solution.



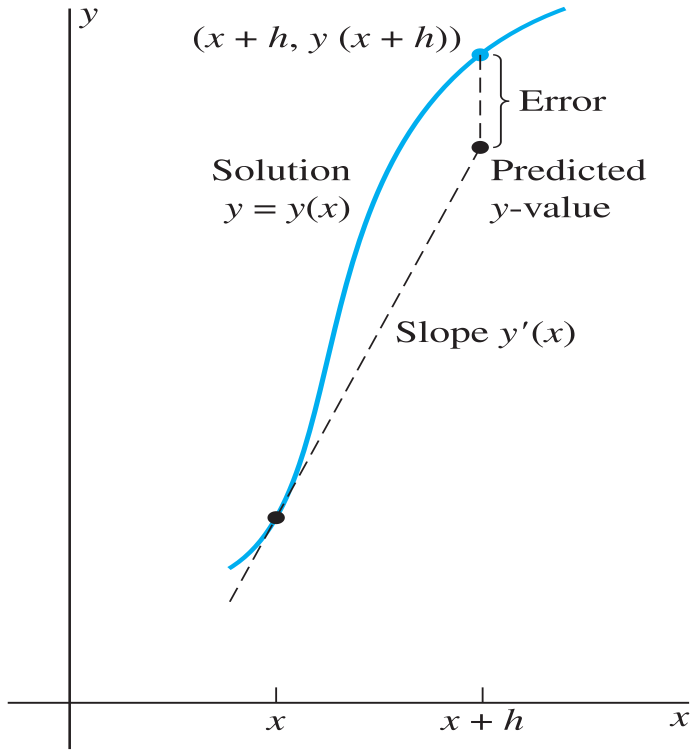
- It appears that this solution curve may have a vertical asymptote near $x = 0.97$.

- Indeed, an exact solution using Bessel functions can be used to show that $y(x) \to +\infty$ as $x \to 0.969811$ (approximately).

- Although Euler's method gives values (albeit spurious ones) at $x = 1$, the actual solution does not exist on the entire interval $[0, 1]$.

- Moreover, Euler's method is unable to "keep up" with the rapid changes in $y(x)$ that occur as x approaches the infinite discontinuity near 0.969811.

- As the figure shows, Euler's method is rather unsymmetrical.



- It uses the predicted slope $k = f(x_n, y_n)$ of the graph of the solution at the left-hand endpoint of the interval $[x_n, x_n + h]$ as if it were the actual slope of the solution over that entire interval.

- To increase the accuracy of our approximation, we can use the *improved Euler Method*.

Improved Euler Method

- Given the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0,$$

  suppose that after carrying out n steps with step size h we have computed the approximation y_n to the actual value $y(x_n)$ of the solution at $x_n = x_0 + nh$.

- We can use the Euler method to obtain a first estimate—which we now call u_{n+1} rather than y_{n+1}—of the value of the solution at $x_{n+1} = x_n + h$:

- Now that $u_{n+1} \approx y(x_{n+1})$ has been computed, we can take

  as a second estimate of the slope of the solution curve $y = y(x)$ at $x = x_{n+1}$.

- Note that, the approximate slope

  has already been calculated.

- Why not *average* these two slopes to obtain a more accurate estimate of the average slope of the solution curve over the entire subinterval $[x_n, x_{n+1}]$?

- This idea is the essence of the *improved* Euler method.

- The algorithm for this method is presented next.

Algorithm: The Improved Euler Method

- Given the initial value problem

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0,$$

  the *improved Euler method with step size h* consists in applying the iterative formulas:

$$\begin{aligned} k_1 &= f(x_n, y_n), \\ u_{n+1} &= y_n + h \cdot k_1, \\ k_2 &= f(x_{n+1}, u_{n+1}), \\ y_{n+1} &= y_n + h \cdot \frac{1}{2}(k_1 + k_2). \end{aligned}$$

- These formulas compute successive approximations $y_1,\ y_2,\ y_3, \ldots$ to the [true] values $y(x_1),\ y(x_2),\ y(x_3), \ldots$ of the [exact] solution $y = y(x)$ at the points $x_1, x_2, x_3, \ldots$, respectively.

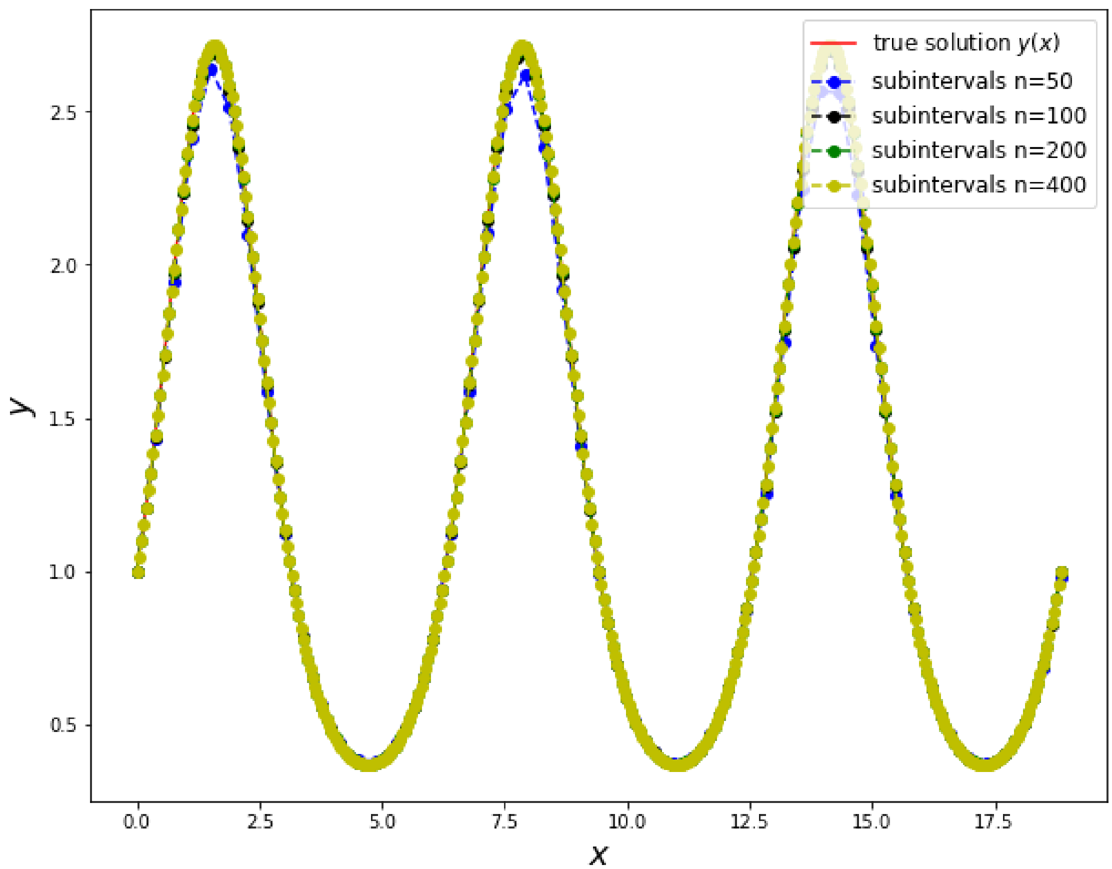
Example 4. *Consider the following logistic initial value problem:*

$$\frac{dy}{dx} = y\cos x, \quad y(0) = 1$$

The exact solution of the above equation is the ***periodic*** *function:* $y(x) = e^{\sin x}$. *Use the improved Euler's method to approximate the solution in the interval* $0 \leqq x \leqq 6\pi$ *and using* $n \in \{50, 100, 200, 400\}$ *subintervals.*

- Improved Euler's iterative formula for this example is:

- Next figure shows the exact solution curve and approximate solution curves obtained by applying the Improved Euler's method.



Cumulative Error vs. Number of Intervals

- Next figure illustrates a graph comparing the cumulative error ϵ with the number of subintervals n.