MA 266 Lecture 14

Christian Moya, Ph.D.

Sec 3.2 General Solutions of Linear Equations

Review from last class:

- Consider the *homogeneous* ODE with constant coefficients ($a, b, c \in \mathbb{R}$):

$$ay'' + by' + cy = 0$$

- Look for a solution of the form: $y(x) = e^{rx}$. Then, we find that $(ar^2 + br + c)e^{rx} = 0$ results:

$$ar^2 + br + c = 0$$

- The above equation is called *characteristic equation* of the differential equation.
- By solving the characteristic equation, we find r (three possibilities):
  - roots $r_1 \neq r_2$ are real.
  - roots $r_1 = r_2$ is real.
  - roots r_1, r_2 are complex.
- (First case) Consider *distinct* real roots $r_1 \neq r_2$. (Theorem 3) The general solution of the homogeneous ODE is:

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

- Consider the *homogeneous* ODE with constant coefficients ($a, b, c \in \mathbb{R}$):

$$ay'' + by' + cy = 0$$

- (Second case) Consider *repeated* or equal real root $r_1 = r_2$. Here, we only one have solution

$$y_1(x) = e^{r_1 x}$$

- The problem is to produce the "missing" second solution.

- Note that the equal root $r = r_1$ occurs when the *characteristic equation* is a constant multiple of:

$$(r - r_1)^2 = r^2 - 2r_1 r + r_1^2$$

- Any differential equation with the above characteristic equation is equivalent to:

$$y'' - 2r_1 y' + r_1^2 = 0 \tag{1}$$

- However, it is easy to verify that $y(x) = xe^{r_1 x}$ is a second (*linearly independent*) solution of (1).

- Thus, by Theorem 3, the general solution of (1) is:

Example 1. *Find the general solution of the differential equation:*

$$9y'' - 12y' + 4y = 0$$

- Characteristic equation:

- Solution:

Example 2. *Let* $y(x) = c_1 + c_2 e^{-10x}$ *be a general solution of a homogeneous second-order differential equation of the form*

$$ay'' + by' + c = 0,$$

with constant coefficients. Find such coefficients.

- Roots:

- Characteristic equation:

- Homogeneous equation:

General Linear Equations

- Consider the *nth-order linear* differential equation:

- We assume $P_i(x)$ and $F(x)$ are continuous on some open interval I.

- If ____________________, we obtain:

- The *homogeneous linear equation* associated with this differential equation is:

Theorem (Principle of Superposition for Homogeneous Equations) If $y_1, y_2, \ldots, y_n$ are n solutions of the linear equation on the interval I. If $c_1, c_2, \ldots, c_n$ are constants, then the linear combination

is also a solution on I.

Theorem (Existence and Uniqueness of Linear equations) Suppose that the functions p_1, p_2, ..., p_n, and f are continuous on the open interval I containing the point a. Then, given n numbers b_0, b_1, ..., b_{n-1}, the nth-order linear equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)$$

has a unique (that is, one and only one) solution on the entire interval I that satisfies the n initial conditions

$$y(a) = b_0, \quad y'(a) = b_1, \quad \ldots, \quad y^{(n-1)}(a) = b_{n-1}.$$

Example 3. *Without solving the ODE, find the existence and uniqueness interval I of the solution of the IVP:*

$$x(x-3)y'' + 2xy' - (x+1) = 0, \qquad y(1) = 1, \quad y'(1) = 2.$$

- Rewrite it in standard form:

- Use Theorem:

Linear Independent Solutions

- Based on our knowledge of general solutions of second-order linear equations, we would expect that a general solution of the *homogeneous* nth-order linear equation

  $$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$$

  will be a linear combination

  $$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n,$$

  where y_1, y_2, ..., y_n are particular solutions of

  $$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0.$$

- However these n particular solutions must be "sufficiently independent" that we can always choose the coefficients c_1, c_2, ..., c_n to satisfy arbitrary initial conditions of the form $y(a) = b_0$, $y'(a) = b_1$, ..., $y^{(n-1)}(a) = b_{n-1}$.

Linear Dependence of Two Functions

- Recall that *two* functions f_1 and f_2 are linearly *dependent* if one is a constant multiple of the other. That is, if either $f_1 = kf_2$ or $f_2 = kf_1$ for some constant k.

- If we write these equations as

  we see that the linear dependence of f_1 and f_2 implies that there exist two constants c_1 and c_2 *not both zero* such that

- By analogy, we say that n functions $f_1,\ f_2,\ \ldots,\ f_n$ are ________________________ provided that some *nontrivial* linear combination of them vanishes identically.

- *Nontrivial* means that *not all* of the coefficients $c_1, c_2, \ldots, c_n$ are zero (although some of them may be zero).

Defintion 1. *(Linear Dependence of Functions) The n functions $f_1,\ f_2,\ \ldots,\ f_n$ are said to be **linearly dependent** on the interval I provided that there exist constants $c_1,\ c_2,\ \ldots,\ c_n$ not all zero such that*

$$c_1f_1 + c_2f_2 + \cdots + c_nf_n = 0$$

on I, that is,

$$c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x) = 0$$

for all x in I.

Remarks:

- If not all the coefficients in

  $$c_1f_1 + c_2f_2 + \cdots + c_nf_n = 0$$

  are zero, then clearly we can solve for at least one of the functions as a linear combination of the others, and conversely.

- Thus the functions $f_1,\ f_2,\ \ldots,\ f_n$ are linearly dependent if and only if at least one of them is a linear combination of the others.

Example 4. *Show that the functions* $f(x) = 0$*,* $g(x) = \sin(x)$ *and* $h(x) = e^x$ *are linearly dependent on* $\mathbb{R}$*.*

Defintion 2. *(Linear Independent Functions) The* n *functions* $f_1, f_2, \ldots, f_n$ *are called* linearly independent *on the interval* I *if they are not linearly dependent there. Equivalently, they are linearly independent on* I *provided that the identity*

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

holds on I *only in the trivial case*

that is, no *nontrivial linear combination of these functions vanishes on* I*.*

- To show that n given functions are linearly independent, we use the *Wronksian Determinant.*

The Wronskian Determinant

- Suppose that the n functions $f_1, f_2, \ldots, f_n$ are each __________________ times differentiable.
- Then their *Wronskian* is the __________________ determinant
- The Wronskian of n ________________________ $f_1, f_2, \ldots, f_n$ *is identically zero.*

Example 5. *Use the Wronskian to show that the functions* $y_1(x) = e^x$*,* $y_2(x) = \cos(x)$*, and* $y_3(x) = \sin(x)$ *are linearly independent on* $\mathbb{R}$*.*

Wronksians of Solutions

- Provided that ______________________, it turns out (Theorem General Solutions of Homogeneous Equations) that we can always find values of the coefficients in the linear combination

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

  that satisfy any given initial conditions of the form

Theorem (Wronksians of Solutions) Suppose that y_1, y_2, $\ldots$, y_n are n solutions of the homogeneous nth-order linear equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$$

on an open interval I, where each p_i is continuous. Let

(a) If y_1, y_2, $\ldots$, y_n are linearly dependent, then __________________ on I.

(b) If y_1, y_2, $\ldots$, y_n are linearly independent, then __________________ on I.

Capturing All Solutions of a Homogeneous Equation

- Given any fixed set of n linearly independent solutions of a *homogeneous* nth-order equation, *every* (other) solution of the equation can be expressed as a linear combination of those n particular solutions.

Theorem (General Solutions of Homogeneous Equations)

- Let y_1, y_2, …, y_n be n linearly independent solutions of the homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$$

  on an open interval I where the p_i are continuous.

- If Y is any solution whatsoever of this equation, then there exist numbers c_1, c_2, …, c_n such that

  for all x in I.

Nonhomogeneous Equations

Example 6. *Solutions of nonhomogeneous equations.*

- Consider the *nonhomogeneous* nth-order linear differential equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)$$

  with associated homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0.$$

- Suppose that a single fixed particular solution ____________ of the above nonhomogeneous equation is known

- Let Y is any other solution of this equation.

- Show that if __________________, then ____________ is the solution of the associated homogeneous Equation

- We call ____________ a *complementary function* of the nonhomogeneous equation.

Theorem (Solutions Homogeneous Equations)

- Let y_p be a particular solution of the nonhomogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)$$

on an open interval I where the functions p_i and f are continuous.

- Let $y_1, y_2, \ldots, y_n$ be linearly independent solutions of the associated homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0.$$

- If Y is any solution whatsoever of the equation nonhomogeneous equation on I, then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$Y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x)$$

for all x in I.

Example 7. *We are given (i) the homogeneous IVP:*

$$y'' + y = 3x, \qquad y(0) = 2, \quad y'(0) = -2$$

(ii) the complementary solution: $y_c = C_1\cos(x) + C_2\sin(x)$, *and (iii) the particular solution:* $y_p = 3x$. *Find a solution for the IVP.*