MA 266 Lecture 15

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Sec 3.3-1 Homogeneous Eqs. Constant Coefficients

Solving nth-Order Equations

- A general solution of an nth-order homogeneous linear equation is a linear combination of n linearly independent particular solutions.
- Q: How to find a single solution?
- The solution of a linear differential equation with _________________________ coefficients ordinarily requires numerical methods or infinite series methods.
- In this lecture, we show how to find ______________________________________ of a given nth-order linear equation if it has *constant* coefficients.
- Consider the homogeneous equation:

  where the coefficients a_0, a_1, a_2, $\ldots$, a_n are real constants with $a_n \neq 0$.

Finding a single solution

- Consider the *ansatz*:
- and observe that any derivative is:
- Substituting ___________________ in ___________________ gives:

- Because e^{rx} is never zero, we see that $y = e^{rx}$ will be a solution of ________________ precisely when r is a root of the *algebraic equation*:

Defintion 1. *(The Characteristic Equation) The characteristic equation of*

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

is the algebraic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0.$$

- Fundamental theorem of algebra $\implies$ every nth-degree polynomial has n zeros, though not necessarily distinct and not necessarily real.
- Finding the exact values of these zeros may be difficult or even impossible.
- For equations of degree $n > 2$, we may need either to spot a fortuitous factorization or to apply a numerical technique such as Newton's method.

The case of *distinct* roots

- Assume (the simplest case) the characteristic equation has n distinct (no two equal) *real* roots:

- Then the functions

  are all solutions of ____.

- These n solutions are *linearly independent* on the entire real line.

Theorem (Distinct Real Roots) If the roots $r_1,\ r_2,\ \ldots,\ r_n$ of the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0$$

are real and distinct, then

is a general solution of the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0.$$

Example 1. *Find the general solution of*

$$2y'' - 7y' + 3y = 0.$$

- Characteristic equation:

- General solution:

Example 2. *Find the general solution of*

$$y'' + 5y' + 5y = 0.$$

- Characteristic equation:

- General solution:

Example 3. *Solve the initial value problem*

$$2y^{(3)} - 3y'' - 2y' = 0; \qquad y(0) = 1, y'(0) = -1, y''(0) = 3.$$

- Characteristic equation:

- General solution:

- Particular solution:

Polynomial Differential Operator

- If the roots of are *not* distinct $\implies$ there are repeated roots
- We cannot produce n linearly independent solutions of the differential equation

  $$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0.$$

  by the method of Theorem (Distinct Real Roots).
- The problem, then, is to produce the missing linearly independent solutions.
- For this purpose, it is convenient to adopt "operator notation" and write

  operates on the n-times differentiable function $y(x)$.
- The result is the linear combination

  of y and its first n derivatives.
- We also denote by __________________ the operation of differentiation with respect to x, so that

  and so on.
- In terms of D, the operator L may be written
- We will find it useful to think of the right-hand side of this equation as a (formal) nth-degree polynomial in the "variable" D.
- It is a *polynomial differential operator*.

Properties of Differential Operators

- A first-degree polynomial operator with leading coefficient 1 has the form $D - a$, where a is a real number.

- It operates on a function $y = y(x)$ to produce

- The important fact about such operators is that any two of them *commute*:

  for any twice differentiable function $y = y(x)$.

- The proof of this formula is:

- We see here also that

- Similarly, it can be shown by induction on the number of factors that an operator product of the form

  expands—by multiplying out and collecting coefficients—in the same way as does an ordinary product of linear factors, with x denoting a real variable.

The Operator Method and Repeated Real Roots

- Let us now consider the possibility that the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0 \qquad (3)$$

  has *repeated* roots.

- For example, suppose that this equation has only two distinct roots, r_0 of multiplicity 1 and r_1 of multiplicity $k = n - 1 > 1$.

Two distinct real roots

- Then (after dividing by a_n) the characteristic equation can be rewritten in the form

- Similarly, the corresponding operator L can be written as the order of the factors

  making no difference because of the commutativity discussed earlier.

- Two solutions of the differential equation $Ly = 0$ are ______________________________.

- This is, however, not sufficient.

- We need $k + 1$ linearly independent solutions in order to construct a general solution, because the equation is of order $k + 1$.

- To find the missing $k - 1$ solutions, we note that

- Consequently, *every* solution of the kth-order equation

  will also be a solution of the original equation $Ly = 0$.

- Hence our problem is reduced to that of finding a general solution of this differential equation.

- The fact that $y_1 = e^{r_1 x}$ is one solution of this equation suggests that we try the substitution

  where ____________________ is a function yet to be determined.

- Observe that

- Upon k applications of this fact, it follows that

  for any sufficiently differentiable function $u(x)$.

- Hence $y = ue^{r_1 x}$ will be a solution of

  if and only if

- But this is so if and only if

  a polynomial of degree at most $k - 1$.

- Hence our desired solution of

  is

- In particular, we see here the additional solutions ____________________________________ of the original differential equation $Ly = 0$.

- The preceding analysis can be carried out with the operator $D - r_1$ replaced with an arbitrary polynomial operator, resulting in the following theorem.

Theorem (Repeated Roots) If the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0$$

has a repeated root ______________ of multiplicity ______________, then the part of a general solution of the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

corresponding to r is of the form:

Root of Multiplicity k

- It is easy to verify that the k functions ____________________________________ in the expression

  $$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{rx}$$

  are linearly independent on the real line.

- Thus a root of multiplicity k corresponds to k linearly independent solutions of the differential equation.

Example 4. *Find the general solution of*

$$5y^{(4)} + 3y^{(3)} = 0.$$

- Characteristic equation:

- General solution:

Example 5. *Find the general solution of*

$$y^{(3)} + y'' - y' - y = 0.$$

- Characteristic equation:

- General solution:

Example 6. *Find a function* $y(x)$ *such that* $y^{(4)}(x) = y^{(3)}(x)$ *for all* x *and* $y(0) = 18$, $y'(0) = 12$, $y''(0) = 13$, *and* $y^{(3)} = 7$.

- Characteristic equation:

- General solution:

- Particular solution: