MA 266 Lecture 16

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Sec 3.3-2 Homogeneous Eqs. Constant Coefficients

Theorem (Repeated Roots) If the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0$$

has a repeated root r of multiplicity k, then the part of a general solution of the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

corresponding to r is of the form:

$$(c_1 + c_2 x + \ldots + c_k x^{k-1}) e^{rx}$$

Example 1. *Find a function $y(x)$ such that $y^{(4)}(x) = y^{(3)}(x)$ for all x and $y(0) = 18$, $y'(0) = 12$, $y''(0) = 13$, and $y^{(3)} = 7$.*

- Characteristic equation:

- General solution:

- Particular solution:

Complex-Valued Functions and Euler's Formula

Complex roots

- Any complex (nonreal) roots will occur in complex conjugate pairs:

- This raises the question of what might be meant by an exponential such as

- Recall from elementary calculus the Taylor series for the exponential function

- If we substitute ____________ in this series and recall that ________________________, and so on, we get

Euler's formula

- Because the two real series in the last line are the Taylor series for $\cos\theta$ and $\sin\theta$, respectively, this implies that

- This result is known as *Euler's formula*.

- Because of it, we *define* the exponential function e^z, for $z = x+iy$ an arbitrary complex number, to be

Complex-Valued functions

- Thus it appears that complex roots of the characteristic equation will lead to complex-valued solutions of the differential equation.

- A *complex-valued function* F of the real variable x associates with each real number x (in its domain of definition) the complex number

- The real-valued functions f and g are called the *real* and *imaginary* parts, respectively, of F.

Complex Exponentials

- The particular complex-valued functions of interest here are of the form:

- We note from Euler's formula that

  and

- The most important property of e^{rx} is that

  if r is a complex number.

- The proof is straightforward:

Complex Characteristic Roots

- As a result, when r is complex (just as when r is real), e^{rx} will be a solution of the differential equation
$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$
  if and only if r is a root of its characteristic equation.

- If the complex conjugate pair of roots $r_1 = a + bi$ and $r_2 = a - bi$ are nonrepeated, then the corresponding part of a general solution of this differential equation is

  where the arbitrary constants C_1 and C_2 can be complex.

- For instance, the choice $C_1 = C_2 = \dfrac{1}{2}$ gives the real-valued solution

while the choice $C_1 = -\dfrac{1}{2}i$, $C_2 = \dfrac{1}{2}i$ gives the independent real-valued solution

- This yields the following result.

Theorem (Complex Roots) If the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0$$

has an unrepeated pair of complex conjugate roots $a \pm bi$ (with $b \neq 0$), then the corresponding part of a general solution of the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

has the form

Polar Form

- We can employ the *polar form*

  of the complex number z.

- This form follows from Euler's formula upon writing

- Here r is the *modulus*

  of the number z.

- The *argument* of z is the angle θ.

- For instance, the imaginary number has modulus and argument __________________.

- Similarly, __________________.

- Another consequence is the fact that the nonzero complex number $z = re^{i\theta}$ has the two square roots

  where $\sqrt{r}$ denotes (as usual for a positive real number) the positive square root of the modulus of z.

Repeated Complex Roots

- Theorem can be extended for repeated complex roots.
- If the conjugate pair ____________________ has multiplicity ______________, then the corresponding part of the general solution has the form
- It can be shown that the $2k$ functions

  appearing above are linearly independent.

Example 2. *Find the general solution of the differential equation:*

$$y'' - 6y' + 13y = 0.$$

- Characteristic equation:

- General solution:

Example 3. *Find the general solution of the differential equation:*

$$y^{(4)} + 18y'' + 81y = 0.$$

- Characteristic equation:

- General solution:

Example 4. *Solve the following initial value problem:*

$$9y'' + 6y' + 4y = 0, \qquad y(0) = 3, y'(0) = 4.$$

- Characteristic equation:

- General solution:

- Particular solution:

Example 5. *Find a linear homogeneous constant-coefficient equation with the given general solution:*

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + c_3 \cosh 2x + c_4 \sinh 2x.$$