MA 266 Lecture 18

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Sec 3.5-1 Nonhomogeneous Equations

- Consider the nonhomogeneous nth-order linear equation with constant coefficients:

- Recall the following theorem:

Theorem (Solutions Nonhomogeneous Equations)

- Let y_p be a particular solution of the nonhomogeneous equation on an open interval I where the functions p_i and f are continuous.
- Let $y_1, y_2, \ldots, y_n$ be linearly independent solutions of the associated homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0.$$

- If Y is any solution whatsoever of the equation nonhomogeneous equation on I, then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$Y(x) = \underbrace{c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)}_{=:y_c(x)} + y_p(x)$$

for all x in I.

- *Question:* How to find ____________ ?

Undetermined Coefficients

- Suppose that ____________ is a polynomial of degree ____________.
- Note that the derivatives of a polynomial are themselves polynomials of lower degree.
- Thus it is reasonable to *guess* a particular solution has the form

Example 1. *Find a particular solution of*

$$y'' + 3y' + 4y = 3x + 2.$$

- Here __________________ is a polynomial of degree ____________, so our *guess* is:
- Then
- Solve for the *undetermined coefficients* ____________ and ____________.

- Similarly, suppose that

- Then it is reasonable to expect a particular solution of the same form:

  a linear combination with undetermined coefficients ____________ and ____________.

- Any derivative of the linear combination of $\cos kx$ and $\sin kx$ has the same form.

Example 2. *Find a particular solution of*

$$3y'' + y' - 2y = 2\cos x.$$

- We try the *guess*:

- Then

- Solve for the *undetermined coefficients* ____________ and ____________.

Eligible Functions

- The method of *undetermined coefficients* applies whenever the function $f(x)$ in

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x).$$

  is a linear combination of (finite) products of functions of the following three types:

Example 3. *Find a particular solution of*

$$y'' + y' + y = \sin^2 x.$$

- We try the *guess*:

- Then

- Solve for the *undetermined coefficients* ____________ and ____________.

Caution!!!

Example 4. *Find a particular solution of*

$$y'' - 4y = 2e^{2x}.$$

- We try the *guess*:

- Then

- Solve for the *undetermined coefficient:*

Rule 1 Method of Undetermined Coefficients

- Suppose that $Ly = f(x)$ is a nonhomogeneous linear equation with constant coefficients and that $f(x)$ is a linear combination of finite products of *eligible* functions.
- Also suppose that no term appearing either in $f(x)$ or in any of its derivatives satisfies the associated homogeneous equation $Ly = 0$.
- Then take as a *guess/trial* solution for y_p a linear combination of all linearly independent such terms and their derivatives.
- Then determine the coefficients by substitution of this trial solution into the nonhomogeneous equation $Ly = f(x)$.

Example 5. *Find a particular solution of*

$$y'' + 9y = 2x^2e^{3x} + 5.$$

- We try the *guess*:

- Then

- Solve for the *undetermined coefficients:*

Remarks on Rule 1

- In practice we check the supposition made in Rule 1 by first using the characteristic equation to find the complementary function y_c.
- Then we make a list of all the terms appearing in $f(x)$ and its successive derivatives.
- If none of the terms in the list duplicates a term in y_c, then we use Rule 1.

The Case of Duplication

- Consider the case in which *Rule* 1 does not apply.
- That is, some of the terms involved in $f(x)$ and its derivatives satisfy the associated homogeneous equation.
- For instance, suppose that we want to find a particular solution for:

- Proceeding as in Rule 1, our first guess would be

- This form of $y_p(x)$ will not be adequate because the complementary function is

  so substitution would yield zero rather than $(2x - 3)e^{rx}$.

Amending Our Initial Guess

- To amend our first guess, we observe that

  by our earlier discussion of differential operators.

- If $y(x)$ is *any* solution of our differential equation and we apply the operator $(D - r)^2$ to both sides, we see that $y(x)$:

- The general solution of this *homogeneous* equation is:

Form of the General Solution

- Thus *every* solution of our original equation is:

- Note that the RHS can be obtained by multiplying each term of our first guess

  by

  that is, the least positive power of x (in this case, x^3) that eliminates duplication between the terms of the resulting trial solution $y_p(x)$ and the complementary function $y_c(x)$.

The General Case

- To simplify the general statement of *Rule 2*, we observe that to find a particular solution of the nonhomogeneous linear differential equation

  it suffices to find *separately* particular solutions $Y_1(x)$ and $Y_2(x)$ of the two equations

  respectively.

- Linearity then gives

  and therefore ________________________ is a particular solution of

- (This is a type of "superposition principle" for nonhomogeneous linear equations.)
- Now our problem is to find a particular solution of the equation $Ly = f(x)$, where $f(x)$ is a linear combination of products of the elementary functions listed earlier.
- *Notation:* $f(x)$ can be written as a sum of terms each of the form

  where $P_m(x)$ is a polynomial in x of degree m.

Rule 2 Method of Undetermined Coefficients

- If the function $f(x)$ is of the form

  take as the *guess/trial* solution

  where s is the smallest nonnegative integer such that no term in y_p duplicates a term in the complementary function y_c.
- Then determine the coefficients in y_p by substituting y_p into the nonhomogeneous equation.

The next table lists the form of y_p in various common cases:

$f(x)$	$\boldsymbol{y_p}$
$P_m(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$	$x^s(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)$
$a\cos kx + b\sin kx$	$x^s(A\cos kx + B\sin kx)$
$e^{rx}(a\cos kx + b\sin kx)$	$x^se^{rx}(A\cos kx + B\sin kx)$
$P_m(x)e^{rx}$	$x^s(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)e^{rx}$
$P_m(x)(a\cos kx + b\sin kx)$	$x^s[(A_0 + A_1x + \cdots + A_mx^m)\cos kx +$ $(B_0 + B_1x + \cdots + B_mx^m)\sin kx]$

Example 6. *Determine the appropriate form for a particular solution of*

$$y'' - 6y' + 13y = xe^{3x}\sin 2x.$$

- Characteristic equation:

- The complementary solution is:

- We examine:

- Eliminate duplication terms:

- Particular solution:

Example 7. *Determine the appropriate form for a particular solution of*

$$y^{(4)} + 5y'' + 4y = \sin x + \cos 2x.$$

- Characteristic equation:

- The complementary solution is:

- We examine:

- Eliminate duplication terms:

- Particular solution: