MA 266 Lecture 20

Christian Moya, Ph.D.

Sec 3.5 Nonhomogeneous Equations

Method - Variation of Parameters

- If the *nonhomogeneous* equation:

$$L[y] := y'' + P(x)y' + Q(x)y = f(x)$$

- has *complementary* function:

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x)$$

- The particular solution is :

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

- To find u_1 and u_2, we first solve the following system of equations for u_1' and u_2':

$$u_1' y_1 + u_2' y_2 = 0 \tag{1a}$$
$$u_1' y_1' + u_2' y_2' = f(x). \tag{1b}$$

- We the find u_1 and u_2 via integration:

$$u_1(x) = \int u_1'(x)dx$$
$$u_2(x) = \int u_2'(x)dx.$$

- The determinant of (1) is the *Wronksian* of the two linear independent solutions y_1 and y_2: $W(y_1, y_2) = W(x)$.

Theorem - Variation of Parameters

- If the *nonhomogeneous* equation:

$$L[y] := y'' + P(x)y' + Q(x)y = f(x)$$

- has *complementary* function:

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x)$$

- Then a particular solution is given by:

$$y_p(x) = -y_1(x) \int \frac{y_2(x) f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)} dx$$

- where $W(x) = W(y_1, y_2)$ is the Wronskian of the two independent solutions y_1 and y_2 of the associated homogeneous equation $L[y] = 0$.

Example 1. *Find the particular solution of*

$$y'' + 9y = \sin 3x.$$

- Complementary solution:

- The Wronksian $W(x)$ is

- The desired functions are then

- Particular solution:

Example 2. *Find the particular solution of*

$$y'' - 4y = xe^x.$$

- Complementary solution:

- The Wronksian $W(x)$ is

- The desired functions are then

- Particular solution:

Sec 3.6 Forced Oscillations

Forced Mass-Spring System

- In a previous lecture, we derived the differential equation

  that models the motion of a mass m that is attached to a spring (with constant k) and a dashpot (with constant c) and is also acted on by an external force $F(t)$.

- Machines with rotating components commonly involve mass-spring systems (or their equivalents) in which the external force is simple harmonic:

Undamped Forced Oscillations

Undamped Forced Oscillations

- Consider the external force $F(t) = F_0 \cos \omega t$ and let $c = 0$. Then, we have:

$$mx'' + kx = F_0 \cos \omega t,$$

- The *complementary* function is:

$$x_c(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

- The (circular) *natural frequency* of the mass–spring system is:

$$\omega_0 = \sqrt{\frac{k}{m}}$$

- Assuming $\omega \neq \omega_0$, the *particular* solution is:

$$x_p(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t.$$

- The general solution $x = x_c + x_p$ is given by:

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t,$$

  where the constants c_1 and c_2 are determined by the initial values $x(0)$ and $x'(0)$.

- As we saw earlier, this can be rewritten as

$$x(t) = C \cos(\omega_0 t - \alpha) + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t.$$

Example 3. *Use the method of undetermined coefficients to find the particular solution $x_p(t)$ of:*

$$mx'' + kx = F_0 \cos \omega t.$$

- The *trial* particular solution is:

- **Note:** No sine term is needed in x_p because there is no term involving x' on the L.H.S.

  of ____________.

- This gives

- So,

- Particular solution:

Example 4. *Find the solution* $x(t) = x_c(t) + x_p(t)$ *of the following initial value problem:*

$$L[x] := x'' + 4x = 5\sin 3t, \qquad x(0) = 0, x'(0) = 0.$$

- The *complementary* solution:

- The trial *particular* solution:

- $L[x_p] = 5\sin 3t$ gives:

- The *general* solution $x = x_c + x_p$ is:

- Using the ICs, we find c_1 and c_2:

Damped Forced Oscillations

Damped Forced Oscillations

- Consider the external force $F(t) = F_0 \cos \omega t$ and let $c \neq 0$. Then, we have:

$$mx'' + cx' + kx = F_0 \cos \omega t,$$

- The *complementary* function takes one of the three forms depending on:

$$c > c_{cr} := \sqrt{4km}, c = c_{cr}, \text{ or } c < c_{cr}.$$

Transient solution

- In our previous lectures, we demonstrated that:

$$x_c(t) \to 0 \text{ as } t \to +\infty.$$

- Thus, $x_c(t)$ is the *transient* solution of the *damped forced motion.*
- $\implies x_c(t)$ dies out with the passage of time.

Particular solution

- The *particular* function is:

$$x(t) = A \cos \omega t + B \sin \omega t$$

where

$$A = \frac{(k - m\omega^2)\, F_0}{(k - m\omega^2)^2 + (c\omega)^2}, \quad B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}.$$

- We can show that the resulting $x_p(t)$ corresponds to the steady periodic oscillation:

$$x_p(t) = C \cos(\omega t - \alpha)$$

has amplitude $C = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}$

- Phase angle α:

$$\alpha = \begin{cases} \tan^{-1} \frac{c\omega}{k-m\omega^2} & \text{if } k > m\omega^2, \\ \pi + \tan^{-1} \frac{c\omega}{k-m\omega^2} & \text{if } k < m\omega^2 \end{cases}$$

Example 5. *Find the particular solution $x_p(t)$ of:*

$$mx'' + cx' + kx = F_0 \cos \omega t.$$

- The method of undetermined coefficients indicates $\Longrightarrow$ the *trial* particular function:

- Replacing $L[x_p] = F_0 \cos \omega t$ gives:

- Two equations:

- The undetermined coefficients:

- If we write:

- Results in the *steady* periodic oscillation:

Example 6. *Find the steady state periodic solution of the differential equation:*

$$x'' + 3x' + 5x = -4\cos 5t.$$

- Trial particular solution:

- Replacing into the differential equation gives:

- The undetermined coefficients are:

- Write as *steady* periodic solution:

- Amplitude and angle:

Example 7. *Find the transient solution and the steady periodic solution of the initial value problem:*

$$x'' + 8x' + 25x = 200\cos t + 520\sin t, \qquad x(0) = -30, x'(0) = -10.$$

- Transient solution:

- Steady periodic solution: