MA 266 Lecture 21

Christian Moya, Ph.D.

Sec 3.6 Forced Oscillations

Damped Forced Oscillations

- Consider the external force $F(t) = F_0 \cos \omega t$ and let $c \neq 0$. Then, we have:

$$mx'' + cx' + kx = F_0 \cos \omega t,$$

Transient solution

- In our previous lectures, we demonstrated that:

$$x_c(t) \to 0 \text{ as } t \to +\infty.$$

- Thus, $x_c(t)$ is the *transient* solution of the *damped forced motion*.
- $\implies$ $x_c(t)$ dies out with the passage of time.

Particular solution

- The *particular* function is:

$$x(t) = A \cos \omega t + B \sin \omega t$$

where

$$A = \frac{(k - m\omega^2)\, F_0}{(k - m\omega^2)^2 + (c\omega)^2}, \quad B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}.$$

- We can show that the resulting $x_p(t)$ corresponds to the steady periodic oscillation:

$$x_p(t) = C \cos(\omega t - \alpha)$$

has amplitude $C = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$

- Phase angle α:

$$\alpha = \begin{cases} \tan^{-1} \frac{c\omega}{k - m\omega^2} & \text{if } k > m\omega^2, \\ \pi + \tan^{-1} \frac{c\omega}{k - m\omega^2} & \text{if } k < m\omega^2 \end{cases}$$

Example 1. *Find the steady state periodic solution of the differential equation:*

$$x'' + 3x' + 5x = -4\cos 5t.$$

- Trial particular solution:

- Replacing into the differential equation gives:

- The undetermined coefficients are:

- Write as *steady* periodic solution:

- Amplitude and angle:

Example 2. *Find the transient solution and the steady periodic solution of the initial value problem:*

$$x'' + 8x' + 25x = 200\cos t + 520\sin t, \qquad x(0) = -30, x'(0) = -10.$$

- Steady periodic solution:

- Transient solution:

Resonance - Forced Undamped Oscillations

- Allowing ____________ to approach ____________.
- Recall the particular solution:

$$x_p(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t.$$

- If ω becomes approximately equal to ω_0, the amplitude A of x_p becomes *large*.
- It is sometimes useful to rewrite $x_p(t)$ in the form:

- _________ is the *static displacement* of a spring with k due to a *constant* force F_0.

- _________ is the *amplification factor* defined as:

> The phenomenon of *resonance*—the increase without bound (as $\omega \to \omega_0$) in the amplitude of oscillations of an undamped system with natural frequency ω_0 in response to an external force with frequency $\omega \approx \omega_0$.

Example 3. ***Pure Resonance*** *- Find the particular solution $x_p(t)$ of the following undamped system:*

$$x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t.$$

- *Complementary solution:*

- *Trial Particular solution:*

Bridge Crossings and Resonance

- *In practice, a mechanical system with very little damping can be destroyed by resonance vibrations.*
- *Any complicated structure such as a bridge has many natural frequencies of vibration.*
- *The resulting resonance vibrations can be of such large amplitude that the bridge will collapse.*

Practical Resonance - Forced Damped Oscillations

- Consider the damped system:

$$mx'' + cx' + kx = F_0 \cos \omega t$$

- Note that if ____________, then the "forced amplitude" $C(\omega)$:

  always remains finite.

- However, the forced amplitude may attain a maximum for some value of ω, in which case we speak of *practical resonance*.

Two cases:

1. If ω if $c \geq \sqrt{2km}$:

2. But if $c < \sqrt{2km}$:

Example 4. *Find the amplitude $C(\omega)$ and the practical resonance frequency ω of the following forced mass-spring-dashpot system:*

$$x'' + 10x' + 650 = 100\cos\omega t.$$

- Particular solution:

- Solving for the coefficients A and B

- The amplitude $C(\omega)$ of the steady periodic forced oscillations with freq. ω :

- Find the practical resonance by solving $C'(\omega) = 0$: