MA 266 Lecture 25

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Sec 5.1 Linear Systems

Conclusion - Theorem: General Solutions of Homogeneous Linear Systems

- It suffices to find ____________ *linearly independent* solution vectors:

- The linear combination

  with arbitrary coefficients will then be a general solution of our system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

Example 1. *Write a general solution of the following Linear System:*

$$\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -3 & 4 \end{pmatrix} \mathbf{x}.$$

- *The general solution obtained was:*

$$x(t) = c_1 \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} + c_2 \begin{pmatrix} 2e^{-2t} \\ e^{-2t} \end{pmatrix}.$$

Example 2. *Find a particular solution of the following Linear System Initial Value Problem (IVP):*

$$\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -3 & 4 \end{pmatrix} \mathbf{x}, \qquad x_1(0) = 0, x_2(0) = 5.$$

- *Recall the general solution is:*

$$x(t) = c_1 \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} + c_2 \begin{pmatrix} 2e^{-2t} \\ e^{-2t} \end{pmatrix}.$$

- *Use the initial conditions:*

Sec 5.2 Eigenvalue Method for Homogeneous Systems

Q: How to find the n needed *linearly independent* solution vectors?

- To find these *linearly independent* solution vectors, we proceed by analogy with the characteristic root method for solving a single homogeneous equation with constant coefficients.

Form of the Solution Vectors

- It is reasonable to anticipate solution vectors of the form

  where ______________________________________ are appropriate scalar constants.

Matrix Form

- Consider the Homogeneous System in matrix form:

- Verify the trial solution:

- We cancel the nonzero scalar factor $e^{\lambda t}$ to get

Conclusion:

- __________________________ will be a nontrivial solution of

  provided that $\mathbf{v}$ is a *nonzero* vector and λ is a constant such that the *matrix product* $\mathbf{Av}$ *is a scalar multiple of the vector* $\mathbf{v}$.

- The question now is this: How do we find $\mathbf{v}$ and λ?

Finding v and λ

- To answer this question, we rewrite the equation $\mathbf{Av} = \lambda\mathbf{v}$ in the form

- By *a standard theorem of linear algebra*, it has a nontrivial solution if and only if the determinant of its coefficient matrix vanishes; that is, if and only if

The Eigenvalue Method

The Eigenvalue Method

- For solving the homogeneous system:

- consists of finding ____________ so that

  and next solving

  with this value of ____________ to obtain ________________________.

- Then ________________________ will be a solution vector.
- The name of the method comes from the following definition.

Definition Eigenvalues and Eigenvectors

- The number λ (either zero or nonzero) is called an **eigenvalue** of the $n \times n$ matrix $\mathbf{A}$ provided that

$$|\mathbf{A} - \lambda \mathbf{I}| = 0.$$

- An **eigenvector** associated with the eigenvalue λ is a *nonzero* vector $\mathbf{v}$ such that $\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$, so that

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}.$$

Example 3. *Find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$$

- Eigenvalues:

- Eigenvectors:

Characteristic Equation

- The equation

$$|\mathbf{A} - \lambda \mathbf{I}| =$$

is called the **characteristic equation** of the matrix $\mathbf{A}$.

- Its roots are the eigenvalues of $\mathbf{A}$.
- Expanding this determinant, we evidently get an nth-degree polynomial of the form

$$(-1)^n \lambda^n + b_{n-1}\lambda^{n-1} + \cdots + b_1\lambda + b_0 = 0.$$

- By the fundamental theorem of algebra, this equation has n roots.
- Possibly some are complex, possibly some are repeated.

- Thus an _______________ matrix has _________ eigenvalues (counting repetitions).

Outline of the Eigenvalue Method

Steps for the Eigenvalue Method

- In outline, this method for solving the $n \times n$ homogeneous constant-coefficient system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ proceeds as follows:

  1. Solve the characteristic equation for the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix $\mathbf{A}$.
  2. Attempt to find n *linearly independent* eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ associated with these eigenvalues.
  3. Step 2 is not always possible, but when it is, we get n linearly independent solutions:

- In this case the *general solution* of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is a linear combination

  of these n solutions.

Distinct Real Eigenvalues

- If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real and distinct, then we substitute each of them in turn in the equation

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

  and solve for the associated eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_n$.

- Then it can be proved that the particular solution vectors

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}, \quad \mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}, \quad \ldots, \quad \mathbf{x}_n(t) = \mathbf{v}_n e^{\lambda_n t}.$$

  are always linearly independent.

Example 4. *Use the eigenvalue method to find the general solution of:*

$$x_1' = 2x_1 + 3x_2,\ \ x_2' = 2x_1 + x_2.$$

- Linear system in matrix form:

- Characteristic equation:

- Eigenvector equation:

- General solution:

Example 5. *Use the eigenvalue method to find the general solution of:*

$$x_1' = 4x_1 + x_2,\ x_2' = 6x_1 - x_2.$$

- Linear system in matrix form:

- Characteristic equation:

- Eigenvector equation:

- General solution:

Example 6. *Find the solution of the IVP:*

$$x_1' = 9x_1 + 5x_2,\ x_2' = -6x_1 - 2x_2,\ x_1(0) = 1, x_2(0) = 0.$$

- Linear system in matrix form:

- Characteristic equation:

- Eigenvector equation:

- General solution:

- Particular solution:

Example 7. *The amounts $x_1(t)$ and $x_2(t)$ of salt in two brine tanks satisfy the differential equations*

$$\frac{dx_1}{dt} = -k_1 x_1 + k_2 x_2, \quad \frac{dx_2}{dt} = k_1 x_1 - k_2 x_2,$$

where $k_i = r/V_i$. Find the general solution assuming that $r = 10$ (gal/min), $V_1 = 25$ (gal), and $V_2 = 40$ (gal).