MA 266 Lecture 26

Christian Moya, Ph.D.

Sec 5.2 Eigenvalue Method for Homogeneous Systems

The Eigenvalue Method

- To solve the $n \times n$ homogeneous constant-coefficient linear system:

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

1. Solve the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

for the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix $\mathbf{A}$.

2. Attempt to find n *linearly independent* eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ associated with these eigenvalues using

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}.$$

3. Step 2 is not always possible, but when it is, we get n linearly independent solutions:

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}, \mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}, \ldots, \mathbf{x}_n(t) = \mathbf{v}_n e^{\lambda_n t}.$$

- In this case, the *general solution* of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is a linear combination

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_1\mathbf{x}_n(t).$$

of these n solutions.

Distinct Real Eigenvalues

- If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are **real and distinct**, then we substitute each of them in turn in the equation

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

  and solve for the associated eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$.

- Then, the particular solution vectors

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}, \quad \mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}, \quad \ldots, \quad \mathbf{x}_n(t) = \mathbf{v}_n e^{\lambda_n t}.$$

  are always **linearly independent**.

Example 1. *Find the solution of the IVP:*

$$x_1' = 9x_1 + 5x_2, \;\; x_2' = -6x_1 - 2x_2, \;\; x_1(0) = 1, x_2(0) = 0.$$

Solution

- The matrix form of the system is

$$\mathbf{x}' = \underbrace{\begin{pmatrix} 9 & 5 \\ -6 & -2 \end{pmatrix}}_{=:\mathbf{A}} \mathbf{x}.$$

- The characteristic equation of the coefficient matrix is

$$\begin{aligned} \det \begin{pmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{pmatrix} &= (9-\lambda)(-2-\lambda) + 30 \\ &= \lambda^2 - 7\lambda + 12 = (\lambda-4)(\lambda-3) = 0, \end{aligned}$$

  so we have the *distinct real* eigenvalues $\lambda_1 = 4$ and $\lambda = 3$.

- For the coefficient matrix $\mathbf{A}$, the eigenvector equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ is

$$\begin{pmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- **Case 1:** $\lambda_1 = 4$. Substitution of the first eigenvalue $\lambda_1 = 4$ in $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ yields

$$\begin{pmatrix} 5 & 5 \\ -6 & -6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- The choice $a = 1$ yields $b = -1$, and thus $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

- **Case 2:** $\lambda_1 = 3$. Exercise - Answer: $\mathbf{v}_2 = \begin{pmatrix} 5 \\ -6 \end{pmatrix}$

- The *general solution*:

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t}$$

  therefore takes the form:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 5 \\ -6 \end{pmatrix} e^{3t}$$

- The resulting scalar equations are

$$\begin{aligned} x_1(t) &= c_1 e^{4t} + 5c_2 e^{3t} \\ x_2(t) &= -c_1 e^{4t} - 6c_2 e^{3t}. \end{aligned}$$

- When we impose the initial conditions $x_1(0) = 1$ and $x_2(0) = 0$, we get

$$\begin{pmatrix} 1 & 5 \\ -1 & -6 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

  that is readily solved (in turn) for $c_1 = 6$ and $c_2 = -1$. Thus, finally, the solution of the initial value problem is:

$$\mathbf{x}(t) = 6 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} - \begin{pmatrix} 5 \\ -6 \end{pmatrix} e^{3t}$$

  or equivalently

$$\begin{aligned} x_1(t) &= 6e^{4t} - 5e^{3t} \\ x_2(t) &= -6e^{4t} + 6e^{3t}. \end{aligned}$$

Example 2. *The amounts $x_1(t)$ and $x_2(t)$ of salt in two brine tanks satisfy the differential equations*

$$\frac{dx_1}{dt} = -k_1 x_1 + k_2 x_2,$$
$$\frac{dx_2}{dt} = k_1 x_1 - k_2 x_2,$$

where

$$k_i = \frac{r}{V_i}, \qquad i = 1, 2.$$

Find the general solution assuming that $r = 10$ (gal/min), $V_1 = 25$ (gal), and $V_2 = 40$ (gal).

Solution

- If $r = 10$ (gal/min), $V_1 = 25$ (gal), and $V_2 = 40$ (gal), then

- The matrix form of the system is

- The characteristic equation is

- Thus, the coefficient matrix $\mathbf{A}$ has

- **Case 1.** $\lambda =$ _____________. Substituting

- **Case 2.** $\lambda =$ _____________. Exercise - Answer

- The *general solution*:

$$\mathbf{x}(t) = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t}$$

therefore takes the form:

Distinct Complex Eigenvalues

- Even if some eigenvalues are *complex*, so as long as they are *distinct* the eigenvalue method yields n *linearly independent* solutions.

- **Problem:** The eigenvectors associated to *complex eigenvalues* are ____________________.

- Thus, we will have *complex-valued* solutions.

- Suppose by solving characteristic equation

- we get the pair of complex-conjugate eigenvalues

Eigenvectors

- $\mathbf{v}$ is the eigenvector associated to λ, so that

- Similarly, $\bar{\mathbf{v}}$ is the eigenvector associated to the complex conjugate $\bar{\lambda}$, so that

- $\mathbf{v}$ defined componentwise:

Complex-Valued Solution

- The *complex-valued* solution associated with λ and $\mathbf{v}$ is then

  that is,

Real-Valued Solution

- Because the *real* and *imaginary* parts of a *complex-valued* solution are also solutions, we thus get two *real-valued* solutions

- It is easy to check that the same two real-valued solutions result from taking real and imaginary parts of $\bar{\mathbf{v}}e^{\bar{\lambda}t}$

Procedure for Finding Real-Valued Solutions

1. Find explicitly a single *complex-valued* solution $\mathbf{x}(t)$ associated with the complex eigenvalue λ;
2. Then, find the *real* and *imaginary* parts $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ to get two *independent* real-valued solutions corresponding to the complex conjugate eigenvalues λ and $\bar{\lambda}$.

Example 3. *Find the solution of the IVP:*

$$x_1' = 2x_1 - 5x_2,\ x_2' = 4x_1 - 2x_2,\ x_1(0) = 2, x_2(0) = 3.$$

Solution

- The matrix form of the system is

- The characteristic equation of the coefficient matrix is

  so we have the complex eigenvalues:

- Substituting

- The corresponding complex-valued solution

- The real and imaginary parts of $\mathbf{x}(t)$ are the *real-valued* solutions

- The real-valued *general solution* is

- The resulting scalar equations are

- When we impose the initial conditions

- Thus, finally, the solution of the IVP is:

Example 4. *Find the the general solution of:*

$$x_1' = x_1 - 5x_2,\ x_2' = x_1 + 3x_2.$$

Solution

- The matrix form of the system

- The corresponding characteristic equation

- Substituting

- The corresponding complex-valued solution

- The real and imaginary parts of $\mathbf{x}(t)$ are the *real-valued* solutions

- The real-valued *general solution* is

Example 5. *Find the particular solution of the system*

$$\begin{aligned}\frac{dx_1}{dt} &= 3x_1 + x_3\\ \frac{dx_2}{dt} &= 9x_1 - x_2 + 2x_3\\ \frac{dx_3}{dt} &= -9x_1 + 4x_2 - x_3\end{aligned}$$

that satisfies the initial conditions $x_1(0) = 0$, $x_2(0) = 0$, *and* $x_3(0) = 17$.

- The coefficient matrix $\mathbf{A}$ is

- The characteristic equation

- **Case 1.**

- **Case 2.**

- Finally, given the initial conditions yield

- Thus, the solution to the IVP is