MA 266 Lecture 28

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Sec 5.5 Multiple Eigenvalue Solutions

Generalized Eigenvectors

- The vector $\mathbf{v}_2$ in the equation

$$(\mathbf{A} - \lambda \mathbf{I})^2 \mathbf{v}_2 = \mathbf{0}$$

is an example of a generalized eigenvector.

Rank of a Generalized Eigenvector

- If λ is an eigenvalue of the matrix $\mathbf{A}$, then a *rank r generalized eigenvector* associated with λ is a vector $\mathbf{v}$ such that

- A rank 1 generalized eigenvector is an ordinary eigenvector because

- The vector $\mathbf{v}_2$ in the equation

$$(\mathbf{A} - \lambda \mathbf{I})^2 \mathbf{v}_2 = \mathbf{0}$$

is a rank 2 generalized eigenvector (and not an ordinary eigenvector).

Chains of Generalized Eigenvectors

- The multiplicity 2 method described earlier boils down to finding a pair ____________ of generalized eigenvectors, one of rank 1 and one of rank 2, such that

- Higher multiplicity methods involve longer "chains" of generalized eigenvectors.

Length k Chain

- A *length k chain of generalized eigenvectors based on the eigenvector* $\mathbf{v}_1$ is a set

  of k generalized eigenvectors such that

- Because $\mathbf{v}_1$ is an ordinary eigenvector, $(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_1 = \mathbf{0}$. Therefore,

Length 3 Chain

- Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a length 3 chain of generalized eigenvectors associated with the multiple eigenvalue λ of the matrix $\mathbf{A}$.

- It is easy to verify that three linearly independent solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are given by

Example 1. *In this example, the eigenvalues are given. Find the general solution of*

$$\mathbf{x}' = \underbrace{\begin{pmatrix} 5 & -1 & 1 \\ 1 & 3 & 0 \\ -3 & 2 & 1 \end{pmatrix}}_{=\mathbf{A}} \mathbf{x}; \quad \lambda = 3, 3, 3.$$

Solution

- The eigenvector equation is

$$(\mathbf{A} - 3\mathbf{I})\mathbf{v} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

  The second row implies that $a = 0$. Then, $c = b$. Thus, to within a constant multiple, the eigenvalue $\lambda = 3$ has only one single eigenvector (with $b \neq 0$)

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ b \\ b \end{pmatrix}.$$

  So the defect of $\lambda = 3$ is ____________.

- To apply the method described for triple eigenvalues, we first calculate

$$(\mathbf{A} - 3\mathbf{I})^2 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix} =$$

  and

$$(\mathbf{A} - 3\mathbf{I})^3 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix}$$

- Beginning with ____________________, for instance, we calculate

$$\mathbf{v}_2 = (\mathbf{A} - 3\mathbf{I})\mathbf{v}_3 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix}$$

$$\mathbf{v}_1 = (\mathbf{A} - 3\mathbf{I})\mathbf{v}_2 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix}$$

- The linearly independent solutions are:

Sec 5.3 Gallery of Solns for Linear Systems

Example 2. *Solve the linear system:*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Graph the phase portrait/diagram as a varies from $-\infty$ to ∞, showing the qualitatively different cases.

Solution

- Matrix multiplication yields

- The solution is

- The phase portrait for different values of a are shown next. In each case, $y(t)$ decays exponentially fast.

- **Case** $a < -1$:

- **Case** $a = -1$

- **Case** $a \in (-1, 0)$

- **Case** $a = 0$

- **Case** $a > 1$

Example 3. *Solve the linear system:*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}}_{=\mathbf{A}} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Graph the corresponding phase portrait/diagram.

Solution

- The characteristic equation is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \lambda^2 + \lambda - 6 = 0.$$

- Hence, the eigenvalues of $\mathbf{A}$ are

- The eigenvector equation is:

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- For $\lambda = 2$, this yields

$$(\mathbf{A} - 2\mathbf{I})\mathbf{v} = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- The corresponding non-trivial eigenvector is

- Similarly, for $\lambda = -3$, this yields

$$(\mathbf{A} + 3\mathbf{I})\mathbf{v} = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- The corresponding non-trivial eigenvector is

- The general solution is

- The *phase portrait is*

Example 4. *Determine the nature of the eigenvalues and eigenvectors associated to the following phase portrait.*

Example 5. *Determine what happens when the eigenvalues of the* 2×2 *linear system are complex numbers.*

Solution

- Let's write the eigenvalues as

- **Case** if $p = 0$, the general solution is

$$\mathbf{x}(t) = c_1(\mathbf{a}\cos qt - \mathbf{b}\sin qt) + c_2(\mathbf{b}\cos qt + \mathbf{a}\sin qt).$$

- **Case** if $p \neq 0$, the general solution is

$$\mathbf{x}(t) = e^{pt}(c_1(\mathbf{a}\cos qt - \mathbf{b}\sin qt) + c_2(\mathbf{b}\cos qt + \mathbf{a}\sin qt)).$$

Example 6. *Determine what happens when the eigenvalues of the 2×2 linear system are equal.*

Solution

- **Complete eigenvalue.** If there are two independent eigenvectors, then they span the plane and so *every vector is an eigenvector with this same eigenvalue.* To see this, let

- **Defective eigenvalue.** The eigenspace corresponding to λ is one-dimensional.