MA 266 Lecture 27

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Sec 5.2 Eigenvalue Method for Homogeneous Systems

Distinct Complex Eigenvalues

Formula for distinct complex eigenvalues

- *Distinct* complex conjugate eigenvalues $\lambda_{1,2} = p \pm iq$ with eigenvectors $\mathbf{v}_{1,2} = \mathbf{a} \pm i\mathbf{b}$ produce two linearly independent *real-valued* vector solutions:

$$\mathbf{x}_1(t) = e^{pt}(\mathbf{a}\cos qt - \mathbf{b}\sin qt)$$
$$\mathbf{x}_2(t) = e^{pt}(\mathbf{b}\cos qt + \mathbf{a}\sin qt)$$

Procedure for Finding Real-Valued Solutions

1. Find explicitly a single *complex-valued* solution $\mathbf{x}(t) = \mathbf{v}_1 e^{\lambda_1 t}$ associated with the complex eigenvalue λ_1 and eigenvector $\mathbf{v}_1$;

2. Then, find the *real* and *imaginary* parts $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ to get two *independent* real-valued solutions corresponding to the complex conjugate eigenvalues $\lambda_{1,2}$.

Example 1. *Find the the general solution of:*

$$\mathbf{x}' = \underbrace{\begin{pmatrix} 1 & -5 \\ 1 & 3 \end{pmatrix}}_{=\mathbf{A}} \mathbf{x}$$

Solution

- The corresponding characteristic equation

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \det \begin{pmatrix} 1-\lambda & -5 \\ 1 & 3-\lambda \end{pmatrix} \\ &= (1-\lambda)(3-\lambda) + 5 \\ &= \lambda^2 - 4\lambda + 8 = 0 \end{aligned}$$

- Using the quadratic formula, we obtain the pair of complex eigenvalues:

$$\lambda_{1,2} = p \pm iq = 2 \pm 2i.$$

- Substituting $\lambda_1 = 2 + 2i$ into the *eigenvector* equation $(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v}_1 = \mathbf{0}$ gives

$$\begin{pmatrix} 1-2i & -5 \\ 1 & -1-2i \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Note that y and z satisfy

$$-\frac{y}{5} - \frac{2y}{5} - z = 0$$

.

- Choose $y = -5$, then $z = 1 + 2i$. Hence, the eigenvector is

$\mathbf{v}_1 = \mathbf{a} + i\mathbf{b} =$

Method 1.

- Use the formulae:

$$\begin{aligned}\mathbf{x}_1(t) &= e^{pt}(\mathbf{a}\cos qt - \mathbf{b}\sin qt)\\ \mathbf{x}_2(t) &= e^{pt}(\mathbf{b}\cos qt + \mathbf{a}\sin qt)\end{aligned}$$

Method 2.

- Compute the corresponding complex-valued solution

$$\begin{aligned}\mathbf{x}(t) = \mathbf{v}_1 e^{\lambda_1 t} = \begin{pmatrix} -5 \\ 1+2i \end{pmatrix} e^{(2+2i)t} &= \begin{pmatrix} -5 \\ 1+2i \end{pmatrix} e^{2t}(\cos 2t + i\sin 2t)\\ &= e^{2t}\begin{pmatrix} -5\cos 2t - 5i\sin 2t \\ (\cos 2t + i\sin 2t) + 2i\cos 2t - 2\sin 2t \end{pmatrix}\end{aligned}$$

- The real and imaginary parts of $\mathbf{x}(t)$ are the real-valued solutions:

- The *general solution* is

$$\begin{aligned}\mathbf{x}(t) &= c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)\\ &= e^{2t}\begin{pmatrix} -5c_1\cos 2t - 5c_2\sin 2t \\ (c_1 + 2c_2)\cos 2t + (-2c_1 + c_2)\sin 2t \end{pmatrix}.\end{aligned}$$

Sec 5.5 Multiple Eigenvalue Solutions

Repeated Roots

- Suppose the characteristic equation

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

does *not* have n distinct roots, and thus has *at least one repeated root.*

Definition. An eigenvalue is of *multiplicity* k if it is a k-fold root of the characteristic equation.

- An eigenvalue of multiplicity $k > 1$ may have *fewer* than k linearly independent associated eigenvectors.
- In this case we are unable to find a "complete set" of n linearly independent eigenvectors of $\mathbf{A}$, as needed to form the general solution of the system.

Complete Eigenvalues

An eigenvalue of multiplicity k is said to be __________________________ if it has k linearly independent associated eigenvectors.

- If every eigenvalue of the matrix $\mathbf{A}$ is complete, then—because eigenvectors associated with different eigenvalues are linearly independent—it follows that $\mathbf{A}$ does have a complete set of n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ associated with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (each repeated with its multiplicity).
- In this case a general solution of

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

is still given by the usual combination

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{v}_n e^{\lambda_n t}.$$

Example 2. *Find a general solution of the system*

$$\mathbf{x}' = \underbrace{\begin{pmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{pmatrix}}_{=\mathbf{A}} \mathbf{x}.$$

Solution

- The characteristic equation of the coefficient matrix $\mathbf{A}$ is

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \det \begin{pmatrix} 9-\lambda & 4 & 0 \\ -6 & -1-\lambda & 0 \\ 6 & 4 & 3-\lambda \end{pmatrix} \\ &= (0) \cdot \det \begin{pmatrix} -6 & -1-\lambda \\ 6 & 4 \end{pmatrix} - (0) \cdot \det \begin{pmatrix} 9-\lambda & 4 \\ 6 & 4 \end{pmatrix} + (3-\lambda) \cdot \det \begin{pmatrix} 9-\lambda & 4 \\ -6 & -1-\lambda \end{pmatrix} \\ &= (5-\lambda)(3-\lambda)^2 = 0. \end{aligned}$$

- Thus $\mathbf{A}$ has

- **Case 1.** $\lambda_1 = 5$. The eigenvector equation $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_1 = \mathbf{0}$ is:

$$(\mathbf{A} - 5\mathbf{I})\mathbf{v}_1 = \begin{pmatrix} 4 & 4 & 0 \\ -6 & -6 & 0 \\ 6 & 4 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

- Each of the first two eq'ns imply $b = -a$. Then, one can reduce the third equation to

- The choice of $a = 1$ yields the eigenvector:

- **Case 2.** $\lambda_2 = 3$. Here the eigenvector equation is:

$$(\mathbf{A} - 3\mathbf{I})\mathbf{v}_2 = \begin{pmatrix} 6 & 4 & 0 \\ -6 & -4 & 0 \\ 6 & 4 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

- Here $\mathbf{v}_2$ is an eigenvector if and only if

- The above does not involve c. Thus c is arbitrary.

- Thus, we have found the complete set $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ of linearly independent eigenvectors associated to the eigenvalues $\lambda_1 = 5, \lambda_2 = 3$, and $\lambda_3 = 3$. Thus, the corresponding *general solution* is

Defective Eigenvalues

We start with an illustrative example.

Example 3. *Find the eigenvalues and the eigenvectors of the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & -4 \\ 4 & 9 \end{pmatrix}$$

Solution

- The coefficient matrix has characteristic equation

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \det \begin{pmatrix} 1-\lambda & -4 \\ 4 & 9-\lambda \end{pmatrix} \\ &= (1-\lambda)(9-\lambda) + 16 \\ &= \lambda^2 - 10\lambda + 25 = 0 \end{aligned}$$

- Thus **A** has

- The corresponding eigenvector equation is

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Hence

- Thus the multiplicity ___________________ eigenvalue _____________ has _____________ independent eigenvector. Hence

Definition. An eigenvalue λ of multiplicity $k > 1$ is called ________________________ if it is not complete.

- If λ has only $p < k$ linearly independent eigenvectors, then the number

  of "missing" eigenvectors is called the *defect* of the defective eigenvalue λ.
- If the eigenvalues of the $n \times n$ matrix $\mathbf{A}$ are not all complete, then the eigenvalue method as yet described will produce *fewer* than the needed n linearly independent solutions of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
- We therefore need to discover how to find the "missing solutions" corresponding to a defective eigenvalue λ of multiplicity $k > 1$.

The Case $k = 2$

- Suppose there is a single eigenvector $\mathbf{v}_1$ associated with the defective eigenvalue λ.
- Then at this point we have found only the single solution

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t}$$

  of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

The Second Solution

- We explore a second solution of the form

- When we substitute ______________________________ in $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we get

- We obtain the two equations

  that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ must satisfy in order for

  $$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t} = \mathbf{v}_1 t e^{\lambda t} + \mathbf{v}_2 e^{\lambda t}$$

  to give a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
- The *first* eq'n confirms that $\mathbf{v}_1$ is an eigenvector of $\mathbf{A}$ associated with eigenvalue λ.
- Then the *second* equation says that the vector $\mathbf{v}_2$ satisfies
- To solve the two equations simultaneously, it suffices to find a solution $\mathbf{v}_2$ of the single equation $(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_2 = \mathbf{0}$ such that the resulting vector $\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2$ is *nonzero.*

Algorithm Defective Multiplicity **2** Eigenvalues

1. First find a nonzero solution $\mathbf{v}_2$ of the equation

   such that

   is nonzero, and therefore is an eigenvector $\mathbf{v}_1$ associated with λ.
2. Then form the two independent solutions

   and

   of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ corresponding to λ.

Example 4. *Find a general solution of the system*

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & 9 \end{pmatrix} \mathbf{x}.$$

Solution

- In the previous example, we showed that $\mathbf{A}$ has a *defective* eigenvalue:

- Following the Algorithm, we start by calculating

- If we try

- Therefore the two solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are:

- The general solution is:

Generalized Eigenvectors

- The vector $\mathbf{v}_2$ in the equation

$$(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_2 = \mathbf{0}$$

is an example of a generalized eigenvector.

Rank of a Generalized Eigenvector

- If λ is an eigenvalue of the matrix $\mathbf{A}$, then a *rank r generalized eigenvector* associated with λ is a vector $\mathbf{v}$ such that

- A rank 1 generalized eigenvector is an ordinary eigenvector because

- The vector $\mathbf{v}_2$ in the equation

$$(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_2 = \mathbf{0}$$

is a rank 2 generalized eigenvector (and not an ordinary eigenvector).

Chains of Generalized Eigenvectors

- The multiplicity 2 method described earlier boils down to finding a pair ____________ of generalized eigenvectors, one of rank 1 and one of rank 2, such that

- Higher multiplicity methods involve longer "chains" of generalized eigenvectors.

Length k Chain

- A *length k chain of generalized eigenvectors based on the eigenvector* $\mathbf{v}_1$ is a set

  of k generalized eigenvectors such that

- Because $\mathbf{v}_1$ is an ordinary eigenvector, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0}$. Therefore,

Length 3 Chain

- Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a length 3 chain of generalized eigenvectors associated with the multiple eigenvalue λ of the matrix $\mathbf{A}$.

- It is easy to verify that three linearly independent solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are given by

Example 5. *Find three linearly independent solutions of the system*

$$\mathbf{x}' = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \mathbf{x}.$$

Solution

- The characteristic equation of the coefficient matrix $\mathbf{A}$ is

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \det \begin{pmatrix} -1-\lambda & 0 & 1 \\ 0 & -1-\lambda & 1 \\ 1 & -1 & -1-\lambda \end{pmatrix} \\ &= (-1-\lambda) \cdot \det \begin{pmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} + (1) \cdot \det \begin{pmatrix} 0 & -1-\lambda \\ 1 & -1 \end{pmatrix} \\ &= (-1-\lambda)^3 = 0. \end{aligned}$$

- Thus $\mathbf{A}$ has the eigenvalue

- The eigenvector equation is

- To apply the method described for triple eigenvalues, we first calculate

- Beginning with ________________, for instance, we calculate

- The linearly independent solutions are:

Example 6. *In this example, the eigenvalues are given. Find the general solution of*

$$\mathbf{x}' = \underbrace{\begin{pmatrix} 5 & -1 & 1 \\ 1 & 3 & 0 \\ -3 & 2 & 1 \end{pmatrix}}_{=\mathbf{A}} \mathbf{x}; \quad \lambda = 3, 3, 3.$$

Solution

- The eigenvector equation is

$$(\mathbf{A} - 3\mathbf{I})\mathbf{v} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The second row implies that $a = 0$. Then, $c = b$. Thus, to within a constant multiple, the eigenvalue $\lambda = 3$ has only one single eigenvector (with $b \neq 0$)

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ b \\ b \end{pmatrix}.$$

So the defect of $\lambda = 3$ is ____________.

- To apply the method described for triple eigenvalues, we first calculate

$$(\mathbf{A} - 3\mathbf{I})^2 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix} =$$

and

$$(\mathbf{A} - 3\mathbf{I})^3 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix}$$

- Beginning with __________________, for instance, we calculate

$$\mathbf{v}_2 = (\mathbf{A} - 3\mathbf{I})\mathbf{v}_3 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix}$$

$$\mathbf{v}_1 = (\mathbf{A} - 3\mathbf{I})\mathbf{v}_2 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix}$$

- The linearly independent solutions are: