MA 266 Lecture 29

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Sec 5.6 Matrix Exponentials and Linear Systems

- Consider the *single* equation linear system IVP:

- The corresponding *solution*:

- Now consider the n equations linear system IVP:

- We expect the *solution* to be of the form:

- We can verify ________________ is the solution of ________________.

Series definition of $e^{\mathbf{A}t}$

How to compute $e^{\mathbf{A}t}$?

1. Series definition*.
2. Fundamental matrix solution $\mathbf{\Phi}(t)$.
3. n linearly independent eigenvectors.

Properties of $e^{\mathbf{A}t}$

1. If $\mathbf{D} = \mathrm{diag}(d_1, \ldots, d_n)$, then

2. If $\mathbf{0}$ is the $n \times n$ zero matrix, then

3. If $\mathbf{A}$ and $\mathbf{B}$ are two $n \times n$ matrices that __________________, i.e.,

   then

4. The inverse of $e^{\mathbf{A}}$

Computing $e^{\mathbf{A}t}$ using the *series* definition

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \ldots + \mathbf{A}^n\frac{t^n}{n!} + \ldots$$

Q: When can we use this series?

Definition -**Nilpotent matrix**

- A $n \times n$ matrix is said to be *nilpotent* if

Example 1. *Show that the following matrix* $\mathbf{A}$ *is nilpotent and then use this fact to find the matrix exponential* $e^{\mathbf{A}t}$*.*

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution.

- so $\mathbf{A}^n = \mathbf{0}$ for ______________. It therefore follows from the series definition of the matrix exponential that

Example 2. *Solve the following IVP*

$$\mathbf{x}' = \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}}_{=\mathbf{A}} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

Solution.

- Show that $\mathbf{A} = \mathbf{I} + \mathbf{B}$.

- Show $\mathbf{B}$ is nilpotent.

- Compute $e^{\mathbf{A}t}$.

- Solve the IVP:

Fundamental Matrix Solutions

- The *general* solution of the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$:

  can be written in the form

  where

- To solve the IVP for a given initial condition

  it suffices to find ____________

- Conclusion:

Example 3. *Compute the matrix exponential* $e^{\mathbf{A}t}$ *for the system*

$$x_1' = 5x_1 - 3x_2, x_2' = 2x_1.$$

Solution.

- Characteristic equation:

- **Case 1:**

- **Case 2:**

- The fundamental matrix is then

- The matrix exponential $e^{\mathbf{A}t}$:

n linearly independent eigenvectors

Example 4. *Compute the matrix exponential* $e^{\mathbf{A}t}$ *for the system*

$$x_1' = 5x_1 - 3x_2, x_2' = 2x_1.$$

Solution with the alternative method.