MA 266 Lecture 30

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Sec 5.6 Matrix Exponentials and Linear Systems

Example 1. *Compute the matrix exponential* $e^{\mathbf{A}t}$ *for the system*

$$x_1' = 5x_1 - 3x_2, x_2' = 2x_1.$$

Solution.

- The characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 5\lambda + 6 = 0.$$

  So, the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 2$ and $\lambda_2 = 3$.

- **Case 1:** $\lambda_1 = 2$. The associated eigenvector equation is

$$(\mathbf{A} - 2\mathbf{I})\mathbf{v}_1 = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \mathbf{v}_1 = \mathbf{0}.$$

  The eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The associated solution is:

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{2t} = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}.$$

- **Case 2:** $\lambda_2 = 3$. The associated eigenvector equation is

$$(\mathbf{A} - 3\mathbf{I})\mathbf{v}_2 = \begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix} \mathbf{v}_2 = \mathbf{0}.$$

  The eigenvector is $\mathbf{v}_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}$. The associated solution is:

$$\mathbf{x}_2(t) = \mathbf{v}_2 e^{3t} = \begin{pmatrix} \frac{e^{3t}}{2} \\ \frac{e^{3t}}{3} \end{pmatrix}.$$

- The fundamental matrix is then

$$\mathbf{\Phi}(t) = \begin{pmatrix} e^{2t} & e^{3t}/2 \\ e^{2t} & e^{3t}/3 \end{pmatrix}.$$

- So,

$$\mathbf{\Phi}(0) = \begin{pmatrix} 1 & 1/2 \\ 1 & 1/3 \end{pmatrix} \text{ and } \mathbf{\Phi}^{-1}(0) = \begin{pmatrix} -2 & 3 \\ 6 & -6 \end{pmatrix}.$$

- The matrix exponential $e^{\mathbf{A}t}$ is

$$e^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(0) = \begin{pmatrix} -2e^{2t} + 3e^{3t} & 3e^{2t} - 3e^{3t} \\ -2e^{2t} + 2e^{3t} & 3e^{2t} - 2e^{3t} \end{pmatrix}.$$

Suppose we have n linearly independent eigenvectors.

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}.$$

Then, using the series definition of $e^{\mathbf{A}t}$, we have

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}t + \frac{1}{2!}\mathbf{V}\mathbf{\Lambda}^2\mathbf{V}^{-1}t^2 + \dots \\ &= \mathbf{V}\left(\mathbf{I} + \mathbf{\Lambda}t + \frac{1}{2!}(\mathbf{\Lambda}t)^2 + \dots\right)\mathbf{V}^{-1} \\ &= \mathbf{V}\begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix}\mathbf{V}^{-1} \end{aligned}$$

Example 2. *Compute the matrix exponential $e^{\mathbf{A}t}$ for the system*

$$x_1' = 5x_1 - 3x_2, x_2' = 2x_1.$$

Solution with the alternative method.

$$\mathbf{V} = \begin{pmatrix} 1 & 1/2 \\ 1 & 1/3 \end{pmatrix} \text{ and } \mathbf{V}^{-1} = \begin{pmatrix} -2 & 3 \\ 6 & -6 \end{pmatrix}.$$

So,

$$\begin{aligned} e^{\mathbf{A}t} = \mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1} &= \begin{pmatrix} 1 & 1/2 \\ 1 & 1/3 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 6 & -6 \end{pmatrix} \\ &= \begin{pmatrix} e^{2t} & e^{3t}/2 \\ e^{2t} & e^{3t}/3 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 6 & -6 \end{pmatrix} \\ &= \begin{pmatrix} -2e^{2t} + 3e^{3t} & 3e^{2t} - 3e^{3t} \\ -2e^{2t} + 2e^{3t} & 3e^{2t} - 2e^{3t} \end{pmatrix}. \end{aligned}$$

Sec 5.7 Nonhomogeneous Linear Systems

- Given the *nonhomogeneous* linear system

- The general solution of the *nonhomogeneous* system is

  where

  1. ________________________________ is the general solution of the associated *homogeneous* system $\mathbf{x}' = \mathbf{A}x$.

  2. _________ is a particular solution of the nonhomogeneous system.

Undetermined Coefficients

Undetermined Coefficients

- Suppose $\mathbf{f}(t)$ is a *linear combination of products* of
  1. Polynomials
  2. Exponential functions
  3. Sines and cosines
- Make a guess of the *particular* solution $\mathbf{x}_p$.
- Then, we determine the undetermined *vector* coefficients by substitution in the original *nonhomogeneous* equation.

Example 3. *Apply the method of undetermined coefficients to find a particular solution of the system*

$$x' = 2x + 3y + 5,\ \ y' = 2x + y - 2t.$$

Solution

- The matrix form:

- The nonhomogeneous term $\mathbf{f}$ is ________________________. So it is reasonable to select the particular solution of the form:

- Upon substitution of $\mathbf{x} = \mathbf{x}_p$ in the nonhomogeneous system, we get

- The *particular* solution is then

Example 4. *Consider the system*

$$\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 15 \\ 4 \end{pmatrix} te^{-2t}.$$

Solution.

- The complementary solution is:

$$\mathbf{x}_c(t) = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}.$$

- The trial particular solution is:

Variation of Parameters

- **Problem:** Find a *particular* solution $\mathbf{x}_p$ of the nonhomogeneous linear system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$$

given that we have already found the general solution of the homogeneous system

$$\mathbf{x}_c(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_n\mathbf{x}_n(t).$$

- Using the *Fundamental matrix of solutions*, the general solution can be written as

$$\mathbf{x}_c(t) = \mathbf{\Phi}(t)c$$

- **Idea:** we seek a particular solution of the form

- The derivative of the particular solution is

- Substitution of $\mathbf{x}_p$ and $\mathbf{x}_p'$ into the nonhomogenoeus equation yields

- Observe

- Thus

Theorem - Variation of Parameters.

If $\mathbf{\Phi}(t)$ is a fundamental matrix for the homogeneous system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$, then a *particular solution* of the nonhomogeneous system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$$

is given by

$$\mathbf{x}_p(t) = \mathbf{\Phi}(t) \int \mathbf{\Phi}(t)^{-1}\mathbf{f}(t)dt.$$

- Consider the constant-coefficient case $\mathbf{A}(t) = \mathbf{A}$ of the nonhomogeneous IVP

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t), \quad \mathbf{x}(0) = \mathbf{x}_0.$$

- Here, we can use as a fundamental matrix

- Then, the above theorem yields

- Thus, the general solution of the homogeneous system is

Example 5. *Use the method of variations of parameters to solve the IVP*

$$\mathbf{x}' = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

Solution.

- The corresponding *matrix exponential* is

$$e^{\mathbf{A}t} = \begin{pmatrix} 1+3t & -t \\ 9t & 1-3t \end{pmatrix}.$$

Example 6. *Use the method of variations of parameters to solve the IVP*

$$\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \sec t \\ 0 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Solution.

- The corresponding *matrix exponential* is

$$e^{\mathbf{A}t} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$