MA 266 Lecture 33

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Sec 7.2 Transformation of Initial Value Problems

Theorem - *Transform of Integrals*

- If $f(t)$ is a piecewise continuous function for $t \geq 0$ and satisfies the condition of exponential order $|f(t)| \leq Me^{ct}$ for $t \geq T$, then

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s}$$

  for $s > c$. Equivalently,

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau = \int_0^t \mathcal{L}^{-1}\{F(s)\}d\tau.$$

Example 1. *Find the inverse transform of*

$$F(s) = \frac{1}{s(s^2-9)}.$$

Solution.

- Recall

$$\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}, \qquad (s > |k|).$$

- Applying the above theorem yields

$$\begin{aligned}\mathcal{L}\left\{\frac{1}{s(s^2-9)}\right\} &= \int_0^t \mathcal{L}\left\{\frac{1}{s^2-9}\right\} d\tau \\ &= \frac{1}{3}\int_0^t \mathcal{L}\left\{\frac{3}{s^2-9}\right\} d\tau \\ &= \frac{1}{3}\int_0^t \sinh 3\tau d\tau \\ &= \frac{1}{3}(\cosh 3t - 1).\end{aligned}$$

Sec 7.3 Translation and Partial Fractions

We start with the following elementary property of the Laplace transform

Theorem - *Translation on the s-Axis*

1. If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > c$, then $\mathcal{L}\{e^{at}f(t)\}$ exists for $s > a + c$, and

2. Equivalently,

3. Thus the translation $s \mapsto s - a$ in the transform corresponds to multiplication of the original function of t by e^{at}.

Example 2. *Apply the translation theorem to find the Laplace transform of the following function*

$$f(t) = e^{-t/2} \cos 2\left(t - \frac{1}{8}\pi\right).$$

Solution.

- Solving a linear differential equation requires, in general, computing the inverse transform of the rational function of the form

  where the degree of $P(s)$ is less than that of $Q(s)$.

- To find the inverse transform $\mathcal{L}^{-1}\{R(s)\}$, we use *partial fractions.* We provide next two rules for *partial fraction decomposition* of $R(s)$.

Rule 1 - *Linear Partial Fractions*

- The portion of the partial fraction decomposition of $R(s)$ corresponding to the linear factor $s - a$ of multiplicity n is a sum of n partial fractions, having the form

  where ________________________________ are constants.

Example 3. *Use partial fractions to find the Laplace transform of*

$$F(s) = \frac{5s - 4}{s^3 - s^2 + 2s}.$$

Solution.

Rule 2 - *Quadratic Partial Fractions*

- The portion of the partial fraction decomposition of $R(s)$ corresponding to the irreducible quadratic factor $(s-a)^2+b^2$ of multiplicity n is a sum of n partial fractions, having the form

  where ______________________________ are constants.

Example 4. *Apply the translation theorem and partial fractions to find the inverse Laplace transform of*

$$F(s) = \frac{2s-3}{9s^2-12s+20}.$$

Solution.

Example 5. *Use partial fractions to find the Laplace transform of*

$$F(s) = \frac{s^3}{(s-4)^4}.$$

Solution.

Example 6. *Use Laplace transforms to solve the IVP*

$$x'' + 4x' + 8x = e^{-t};\ x(0) = x'(0) = 0.$$

Solution.

Example 7. *Use Laplace transforms to solve the IVP*

$$x'' + \omega_0^2 x = F_0 \sin \omega t;\ \ x(0) = x'(0) = 0,$$

that determines the undamped forced oscillations of a mass on a spring.

Solution.