MA 266 Lecture 34

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Sec 7.3 Translation and Partial Fractions

Rule 1 - *Linear Partial Fractions*

- The portion of the partial fraction decomposition of $R(s)$ corresponding to the linear factor $s-a$ of multiplicity n is a sum of n partial fractions, having the form

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \ldots + \frac{A_n}{(s-a)^n} = \frac{P(s)}{(s-a)^n}.$$

where $A_1, A_2, \ldots,$ and A_n are constants.

Rule 2 - *Quadratic Partial Fractions*

- The portion of the partial fraction decomposition of $R(s)$ corresponding to the irreducible quadratic factor $(s-a)^2 + b^2$ of multiplicity n is a sum of n partial fractions, having the form

$$\frac{A_1 s + B_1}{(s-a)^2 + b^2} + \frac{A_2 s + B_2}{((s-a)^2 + b^2)^2} + \ldots + \frac{A_n s + B_n}{((s-a)^2 + b^2)^n} = \frac{P(s)}{((s-a)^n + b^2)^n}.$$

where $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots$ and B_n are constants.

Example 1. *Use Laplace transforms to solve the IVP*

$$x'' + 4x' + 8x = e^{-t};\ x(0) = x'(0) = 0.$$

Solution.

- Using the Laplace transform:

$$s^2X(s) + 4sX(s) + 8X(s) = \frac{1}{s+1}$$

- Solving for $X(s)$ yields

$$X(s) = \frac{1}{(s+1)(s^2+4s+8)} = \frac{1}{(s+1)[(s+2)^2+4]}.$$

- We use partial fractions:

$$\frac{A}{(s+1)} + \frac{Bs+C}{(s+2)^2+4} = \frac{1}{(s+1)[(s+2)^2+4]}.$$

- The corresponding algebraic equations are:

$$\begin{aligned} A + B &= 0 \\ 4A + B + C &= 0 \\ 8A + C &= 1. \end{aligned}$$

- Thus, we have $A = \frac{1}{5}$, $B = -\frac{1}{5}$, and $C = -\frac{3}{5}$ and

$$\begin{aligned} X(s) &= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s+3}{(s+2)^2+4} \\ &= \frac{1}{5}\left(\frac{1}{s+1} - \frac{s+2}{(s+2)^2+4} - \frac{1}{2}\frac{2}{(s+2)^2+4}\right) \end{aligned}$$

- The solution of the IVP is then

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{5}\left(e^{-t} - \frac{e^{-2t}}{2}\left(2\cos 2t + \sin 2t\right)\right).$$

Sec 7.4 Derivatives, Integrals, and Prod. of Transforms

Definition - *The Convolution of Two Functions*

- The **convolution** ____________________ of the piecewise continuous functions f and g is defined for $t \geq 0$ as follows:

Example 2. *Find the convolution* $f(t) * g(t)$ *of* $f(t) = t$ *and* $g(t) = e^{at}$.

Solution.

Theorem - *The Convolution Property*

- Suppose that $f(t)$ and $g(t)$ are piecewise continuous for $t \geq 0$ and that $|f(t)|$ and $|g(t)|$ are bounded by Me^{ct} as $t \to +\infty$. Then the Laplace transform of the convolution $f(t) * g(t)$ exists for $s > c$; moreover

  and

Example 3. ***(Fall - 2017; p17)*** *The Laplace Transform of* $f(t) = \int_0^t \tau^2 e^{-\tau} \sin(\pi(t-\tau))$ *is*

Solution.

Example 4. ***(Fall - 2015; p14)*** *Let* $F(s) = \frac{6}{(s-3)^3}$ *and* $G(s) = \frac{5}{s^2+25}$. *What is the inverse Laplace transform of the product of* $F(s)G(s)$*?*

Solution.

Example 5. ***(Fall - 2019; p19)*** *Let $y(t)$ the solution of the IVP*

$$y'' + 4y' + 3y = \int_0^t \cos(\tau)\sinh(t-\tau)d\tau;\ y(0) = y'(0) = 1.$$

Find $Y(s) = \mathcal{L}\{y(t)\}$.

Solution.

Example 6. ***(Spring - 2018; p16)*** *Find the solution of the IVP*

$$y'' + 4y = g(t);\ y(0) = -1,\ y'(0) = 4.$$

Solution.

Theorem - *Differentiation of Transforms*

- If $f(t)$ is piecewise continuous for $t \geq 0$ and $|f(t)| \leq Me^{ct}$ as $t \to +\infty$, then

  for $s > c$. Equivalently

- Repeated application of the theorem gives

  for $n = 1, 2, 3, \ldots$

Example 7. *(**Fall - 2019; p18**) Find the Laplace Transform of $f(t) = te^t \cos 2t$.*

Solution.

Theorem - *Integration of Transforms*

- Suppose that $f(t)$ is piecewise continuous for $t \geq 0$. In addition, assume $f(t)$ satisfies

  and that $|f(t)| \leq Me^{ct}$ as $t \to +\infty$. Then

  for $s > c$. Equivalently,

Example 8. *Find the Laplace Transform of* $f(t) = \frac{e^t - e^{-t}}{t}$.

Solution.