MA 266 Lecture 35

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Theorem - *Integration of Transforms*

- Suppose that $f(t)$ is piecewise continuous for $t \geq 0$. In addition, assume $f(t)$ satisfies
$$\lim_{t \to 0^+} \frac{f(t)}{t} \text{ exists and it is finite,}$$
and that $|f(t)| \leq Me^{ct}$ as $t \to +\infty$. Then
$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)d\sigma = \int_s^\infty \mathcal{L}\{f(t)\}d\sigma,$$
for $s > c$. Equivalently,
$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t\mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)d\sigma\right\}.$$

Example 1. *Find the Laplace Transform of* $f(t) = \frac{e^t - e^{-t}}{t}$.

Solution.

- We first check
$$\lim_{t \to 0^+} \frac{e^t - e^{-t}}{t} = \lim_{t \to 0^+} \frac{e^t + e^{-t}}{1} = 2 < +\infty.$$
- Also,
$$\mathcal{L}\{e^t - e^{-t}\} = \frac{1}{s-1} - \frac{1}{s+1}.$$
- Using the above Theorem, we have
$$\mathcal{L}\left\{\frac{e^t - e^{-t}}{t}\right\} = \int_s^\infty \frac{1}{\sigma - 1} - \frac{1}{\sigma + 1} d\sigma = \lim_{\sigma \to \infty} \ln\left(\frac{\sigma - 1}{\sigma + 1}\right) - \ln\left(\frac{s-1}{s+1}\right).$$
- Thus,
$$\mathcal{L}\left\{\frac{e^t - e^{-t}}{t}\right\} = \ln\left(\frac{s+1}{s-1}\right).$$

Sec 7.5 Periodic and Piecewise Continuous Input Fun's

- Engineering - discontinuous functions.

Definition - *Unit Step Function*

- The corresponding Laplace transform is

- Laplace: $e^{-as} \implies$ Time domain: $t \to t - a$.

Theorem - *Translation on the t-axis*

- If $\mathcal{L}\{f(t)\}$ exists for $s > c$, then

  and

- for $s > c + a$.

- Note that

Example 2. ***(Fall 2019; p17)*** *Find the Laplace transform of*

$$f(t) = \begin{cases} 0 & \textit{when } 0 < t < 1, \\ e^{2t}t^2 & \textit{when } t \geq 1. \end{cases}$$

Solution.

Example 3. ***(Fall 2018; p17)*** *Find the Laplace transform of*

$$f(t) = \begin{cases} 0 & \textit{when } t < \pi, \\ t - 2\pi & \textit{when } \pi \le t < 2\pi, \\ 0 & \textit{when } t \ge 2\pi. \end{cases}$$

Solution.

Example 4. *Find the inverse Laplace transform* $\mathcal{L}^{-1}\{F(s)\}$ *of*

$$F(s) = \frac{2s(e^{-\pi s} - e^{-2\pi s})}{s^2 + 4}.$$

Solution.

Example 5. *(**Spring 2019; p15**) Find the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$ of*

$$F(s) = \frac{10e^{-s}}{s^2 - 5s + 6} + \frac{2}{s^2 - 2s + 5}.$$

Solution.

Example 6. *Find the solution of the IVP*

$$y'' + 4y = \cos 2t - u(t - 2\pi)\cos 2(t - 2\pi).$$

Solution.

Transforms of Periodic Functions

Definition - *Periodic Function*

- A nonconstant function $f(t)$ defined for $t \geq 0$ is said to be **periodic** if there is a number $p > 0$ such that

- for all $t \geq 0$. The least positive value of p (if any) is called the **period** of f.

Theorem - *Transforms of Periodic Functions*

- Let $f(t)$ be periodic with period p and piecewise continuous for $t \geq 0$. Then the transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > 0$ and is given by

Example 7. *The square-wave function is given by* $f(t) = (-1)^{[t/a]}$ *of period* $p = 2a$, *where* $[x]$ *denotes the greatest integer not exceeding* x. *Find its Laplace transform.*

Solution.