MA 266 Lecture 36

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Transforms of Periodic Functions

Definition - *Periodic Function*

- A nonconstant function $f(t)$ defined for $t \geq 0$ is said to be **periodic** if there is a number $p > 0$ such that
$$f(t+p) = f(t),$$
for all $t \geq 0$. The least positive value of p (if any) is called the **period** of f.

Theorem - *Transforms of Periodic Functions*

- Let $f(t)$ be periodic with period p and piecewise continuous for $t \geq 0$. Then the transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > 0$ and is given by
$$F(s) = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t) dt.$$

Example 1. *The square-wave function is given by $f(t) = (-1)^{[t/a]}$ of period $p = 2a$, where $[x]$ denotes the greatest integer not exceeding x. Find its Laplace transform.*

Solution.

- Applying the above theorem yields
$$\begin{aligned}
F(s) = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t) dt &= \frac{1}{1-e^{-2as}} \left(\int_0^a e^{-st} dt - \int_a^{2a} e^{-st} dt \right) \\
&= \frac{1}{1-e^{-2as}} \left(\left[-\frac{1}{s} e^{-st} \right]_0^a - \left[-\frac{1}{s} e^{-st} \right]_a^{2a} \right) \\
&= \frac{(1-e^{-as})^2}{s(1-e^{-2as})} \\
&= \frac{1-e^{-as}}{s(1+e^{-as})}.
\end{aligned}$$

Sec 7.6 Impulses and Delta Functions

- Consider a force $f(t)$ that acts only during a very short time interval $a \leqq t \leqq b$, with $f(t) = 0$ outside this interval.

- A typical example would be a quick surge of voltage applied to an electrical system.

- In such a situation, the effect of $f(t)$ depends on the value of

- The number p is called the _ the impulse force $f(t)$ over the interval $[a, b]$.

Modeling Impulsive Forces

- Suppose for simplicity that $f(t)$ has impulse 1 and acts during some brief time interval beginning at time $t = a \geqq 0$.

- Then we can select a fixed number $\epsilon > 0$ that approximates the length of this time interval and replace $f(t)$ with the specific function

- If $b \geqq a + \epsilon$, then the impulse of $d_{a,\epsilon}$ over $[a, b]$ is

Dirac Delta Function

- The limit in the equation

  gives

Meaning of $\delta_a(t)$

- The following computation motivates the meaning that we will attach here to the symbol $\delta_a(t)$.
- If $g(t)$ is a continuous function, then the mean value theorem for integrals implies that

  for some point $\bar{t}$ in $[a, a + \epsilon]$.
- It follows that

  by continuity of g at $t = a$.
- If $\delta_a(t)$ *were* a function in the strict sense of the definition, and if we could interchange the above limit and integral, we therefore could conclude that

The Laplace Transform of $\delta_a(t)$

- If we take $g(t) = e^{-st}$ in our definition of $\delta_a(t)$, the result is

- We therefore *define* the Laplace transform of the delta function to be

Note that $\mathcal{L}\{\delta(t)\}$ is

External Force

- Consider the response of a linear differential equation to a unit impulse at the instant time $t = a$:

- We will call $x(t)$ a solution provided that

- Because

- Using the Laplace Transform

- Taking the limit as $\epsilon \to 0$

- Note that this is precisely the same result we would obtain by using:

Example 2. ***(Spring - 2019;p11)*** *Solve* $y'' + 4y = 2\delta(t - \pi)$ *with the initial conditions* $y(0) = 0$ *and* $y'(0) = 0$*.*

Solution.

Example 3. *(**Spring - 2017;p16**) Solve the IVP*

$$y'' + 2y' - 15y = \delta(t-1),\ \ y(0) = y'(0) = 0.$$

Solution.

System Analysis and Duhamel's Principle

- Consider the linear system

- The constant coefficients a, b, and c are determined by the physical parameters of the system and are independent of $f(t)$.

- Then the transform of our differential equation is

- so

Transfer Function

- The function

  is called the *transfer function* of the system.

- Thus the transform of the response to the input $f(t)$ is the product of $W(s)$ and the transform $F(s)$.

Weight function and Duhamel's Principle

- The function

  is called the *weight function* of the system.

- From the fact that $X(s) = W(s)F(s)$ we see by convolution that

- This formula is *Duhamel's principle* for the system.

Example 4. *Apply Duhamel's principle to write an integral formula for the solution of the IVP:*

$$x'' + 4x = f(t);\ x(0) = x'(0) = 0.$$

Solution.

Example 5. *Apply Duhamel's principle to write an integral formula for the solution of the IVP:*

$$x'' + 6x' + 9x = f(t);\ \ x(0) = x'(0) = 0.$$

Solution.