

MULTIPLICITY-FREE

GIVEN FIXED ANYON MODEL (UMTC) IN THE FORM

$$\{ \mathcal{L}, N_c^{ab}, [F_d^{abc}]_{ef}, [R_c^{ab}] \}$$

LABEL SET
OF ANYON TYPES
 $1, a, b, c, \dots$

FUSION
COEFFICIENTS

$$a \otimes b = \bigoplus_{c \in \mathcal{L}} N_c^{ab} c$$

F-SYMBOLS

$$\begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagup \quad \diagdown \\ e \quad d \quad f \end{array} = \sum_{f \in \mathcal{L}} [F_d^{abc}]_{ef} \begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagup \quad \diagdown \\ f \quad \quad c \end{array}$$

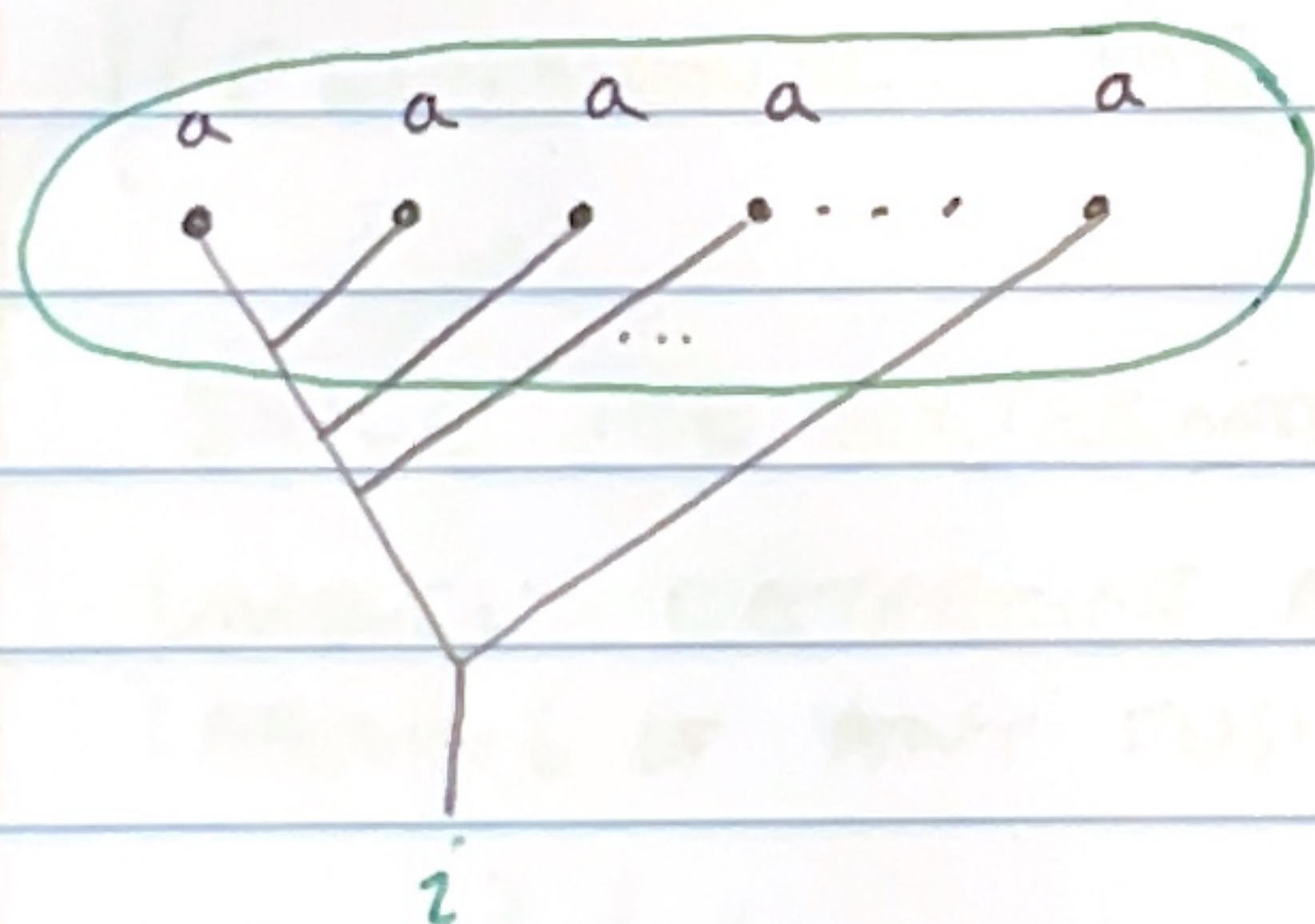
R-SYMBOLS

$$\begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ c \end{array} = R_c^{ab} \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ c \end{array}$$

CAN CONSIDER THE QUANTUM GATES REALIZED BY
ANYON EXCHANGE

- ① FIX AN ANYON TYPE $a \in \mathcal{L}$
- ② FIX A NUMBER $n \in \mathbb{N}$ (THE # OF ANYONS)
- ③ FIX A "TOTAL CHARGE" $i \in \mathcal{L}$ (NEED $i \in a^{\otimes n}$ IN ORDER TO MAKE SENSE)
- ④ FIX A FUSION TREE BASIS (WE'LL START WITH THE "LEFT-ASSOCIATED" BASIS)

WHICH WE WILL VISUALIZE BY

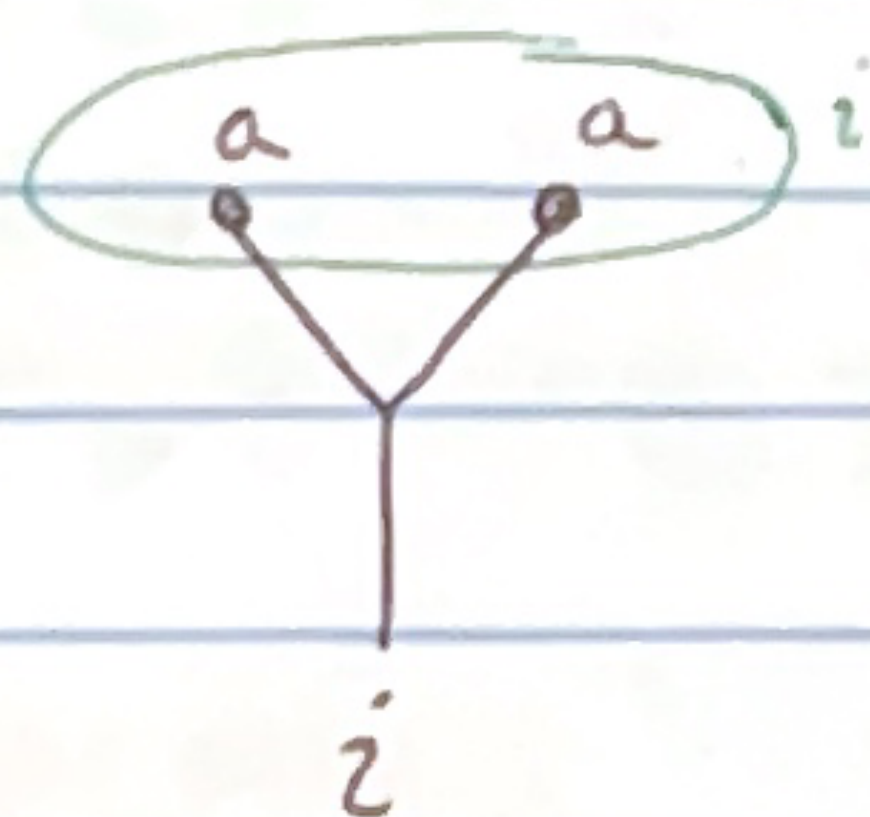


MORE PRECISELY, THE HILBERT SPACE OF STATES ASSOCIATED TO A
COLLECTION OF n ANYONS OF TYPE a WITH TOTAL CHARGE i
IS GIVEN BY

$$\mathcal{H}_{a,n,i} := \bigoplus \left\{ \begin{array}{l} \text{ADMISSIBLY LABELED (LEFT-ASSOCIATED)} \\ \text{FUSION GRAPHS} \end{array} \right\}$$

NEED $\dim(H_{a,n,i}) > 1$ IN ORDER TO STORE QUANTUM INFORMATION, SO NOTE THAT

① WE NEED $n \geq 3$, SINCE



IS THE ONLY ADMISSIBLY LABELED TRIVALENT FUSION GRAPH WITH THESE THESE EXTERNAL EDGE LABELS

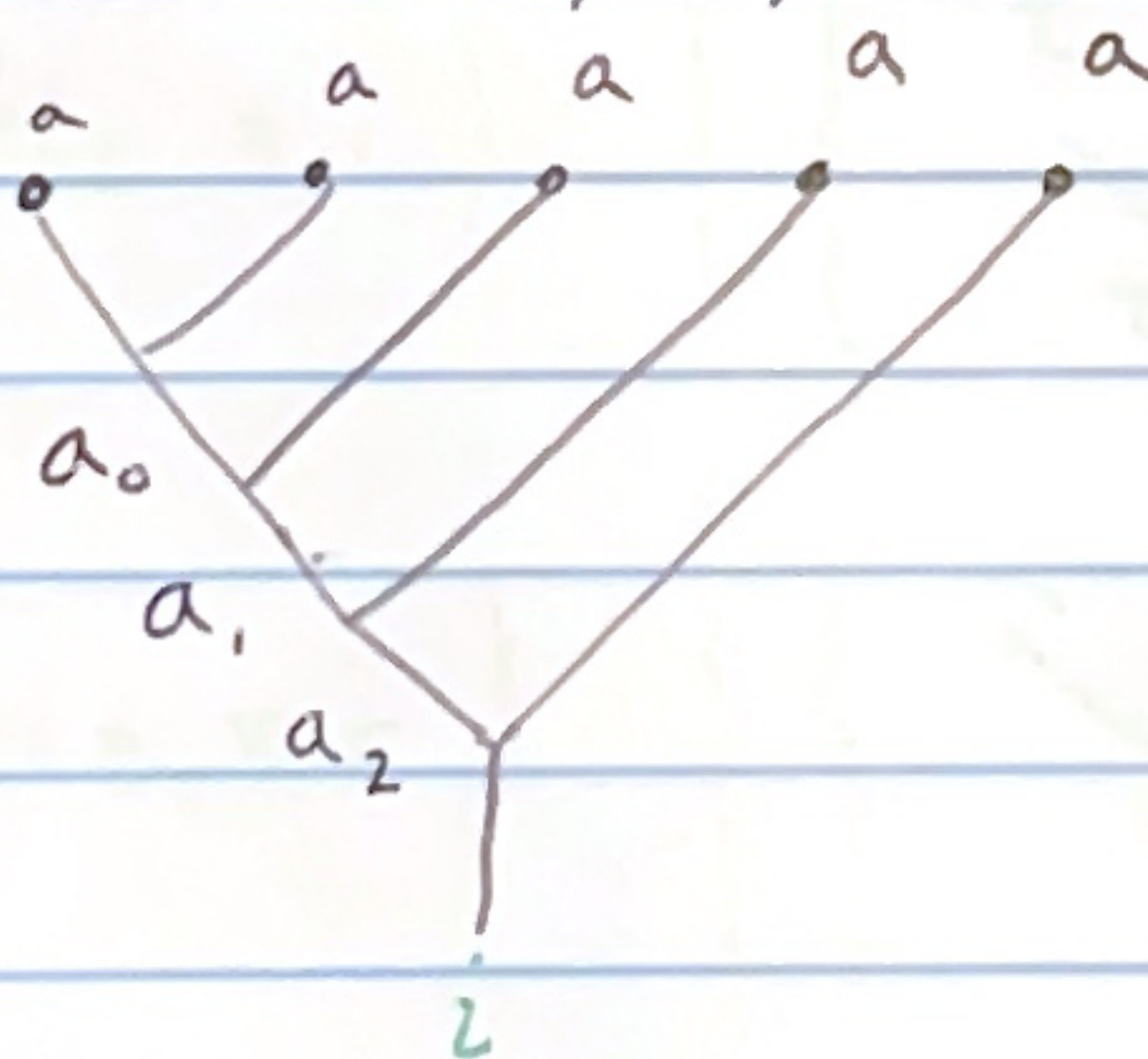
i.e. $\dim(H_{a,2,i}) = 1 \quad \forall a, i \in \mathcal{L}$

② WE NEED "NON-ABELIAN" ANYONS

RECALL AN ANYON a IS ABELIAN IF $\exists c \in \mathcal{L}$ S.T.
 $a \otimes b = c \quad \forall b \in \mathcal{L}$
 (EQUIVALENTLY THE QUANTUM DIMENSION $d_a = 1$)

SINCE THE EXTERNAL EDGE LABELS $\underbrace{a, a, \dots, a}_n, i$ UNIQUELY DETERMINE AN ADMISSIBLE LABELING OF ANY FUSION GRAPH WITH THOSE LABELS

i.e. $\exists ! a_0, a_1, \dots, a_{n-2}$ S.T



IS AN ADMISSIBLY LABELED GRAPH

SO $\dim(H_{a,n,i}) = 1 \quad \forall n$ IF a ABELIAN

EXAMPLE: FIBONACCI THEORY

$\mathcal{L} = \{1, \tau\}$
 VACUUM ANYON $\rightarrow$ 1
 FIBONACCI ANYON $\rightarrow$ τ

MANY WAYS TO ENCODE FUSION RULES:

$$\begin{cases} 1 \otimes 1 = 1 \\ 1 \otimes \tau = \tau \\ \tau \otimes 1 = \tau \\ \tau \otimes \tau = 1 \oplus \tau \end{cases}$$

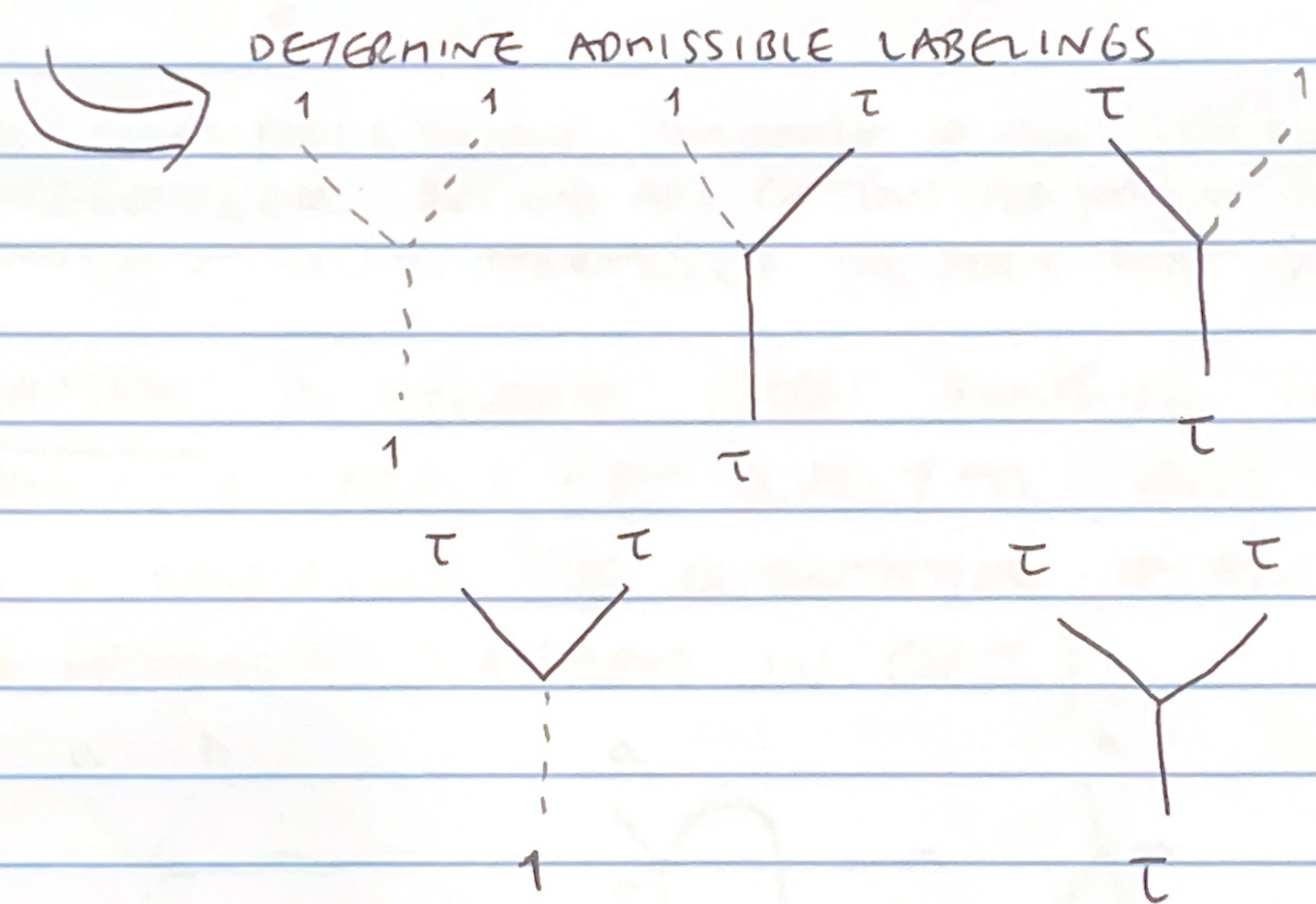
$\otimes$	1	τ
1	1	τ
τ	τ	$1 \oplus \tau$

$$\begin{aligned} N_1^{11} &= 1, N_\tau^{11} = 0 \\ N_1^{1\tau} &= 0, N_\tau^{1\tau} = 1 \\ N_1^{\tau 1} &= 0, N_\tau^{\tau 1} = 1 \\ N_1^{\tau\tau} &= 1, N_\tau^{\tau\tau} = 1 \end{aligned}$$

"FUSION RULES"

"FUSION TABLE"

"FUSION COEFFICIENTS"



$$\dim(H_{\tau, 3, 1}) = \left| \left\{ \begin{array}{c} \tau \quad \tau \quad \tau \\ \diagdown \quad \diagup \\ \tau \end{array} \right\} \right| = 1$$

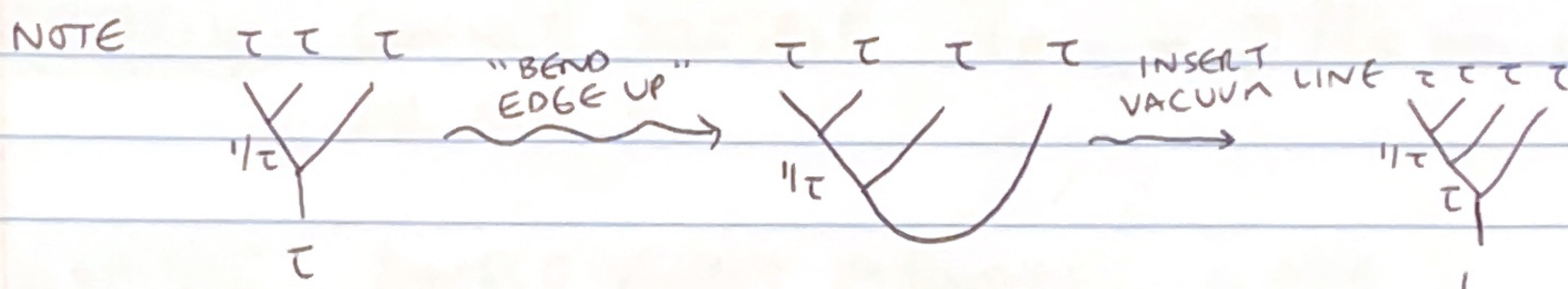
$$\dim(H_{\tau, 3, \tau}) = \left| \left\{ \begin{array}{c} \tau \quad \tau \quad \tau \\ \diagdown \quad \diagup \\ \tau \end{array}, \begin{array}{c} \tau \quad \tau \quad \tau \\ \diagup \quad \diagdown \\ \tau \end{array} \right\} \right| = 2$$

THIS IS (ONE WAY TO ENCODE) A FIBONACCI QUBIT!

THINK $|0\rangle := \begin{array}{c} \tau \quad \tau \quad \tau \\ \diagdown \quad \diagup \\ \tau \end{array}, \quad |1\rangle := \begin{array}{c} \tau \quad \tau \quad \tau \\ \diagup \quad \diagdown \\ \tau \end{array}$

$$\dim(H_{\tau, 4, 1}) = \left| \left\{ \begin{array}{c} \tau \tau \tau \tau \\ \diagup \quad \diagdown \\ \tau \end{array} \right\} \right| = 2$$

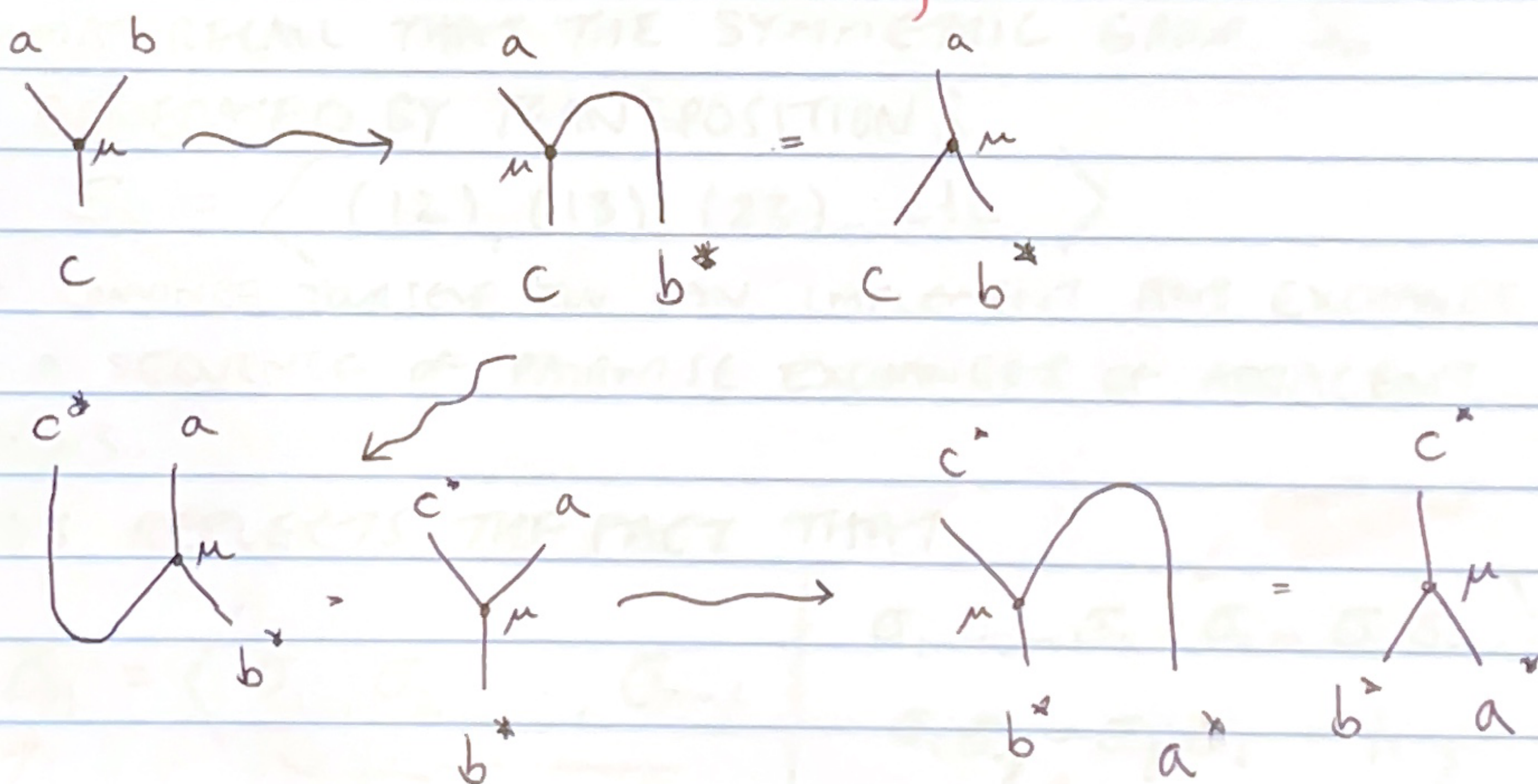
THIS IS ANOTHER WAY TO REALIZE THE SAME STATE SPACE.



THIS COMES FROM A RIGOROUS ISOMORPHISM OF HOM SPACES ON THE CATEGORY LEVEL, BUT WE ARE CONTENT FOR NOW WITH SEEING THE CONSEQUENCES IN DIAGRAMS OF THE FACT THAT $H_{\tau, 3, \tau} \cong H_{\tau, 4, 1}$

CAUTION: IN GENERAL, WHEN "BENDING" UP OR DOWN PARTICLE LABELS GET DUALIZED (THIS IS EXACTLY LIKE REVERSING THE ORIENTATION OF AN EDGE OF A FEYNMAN DIAGRAM IN QFT)

"FROBENIUS RECIPROCITY"



SYMMETRIES OF FUSION RING: $N_c^{ab} = N_{c^*}^{b^*a^*}$ FOR EXAMPLE
 $N_a^{cb^*} = N_{b^*}^{c^*a}$

EXERCISE: SHOW $\dim(H_{\tau, n, \tau})$ GROWS LIKE FIBONACCI NUMBERS i.e.

1, 1, 2, 3, 5, 8, 13, ...

EXERCISE: CONVINCE YOURSELF $H_{\tau, n, \tau} \cong H_{\tau, n, 1}$ FOR ALL n

EXAMPLE: SINGLE-QUBIT FIBONACCI GATES
WE'LL WORK WITH $H_{\tau, 3, \tau}$ FOR SIMPLICITY.



THE GATES COME FROM EXCHANGING THE ANYONS, WHICH TRACES OUT BRAIDS IN 2+1 D SPACETIME

YOU MAY RECALL THAT THE SYMMETRIC GROUP S_n IS GENERATED BY TRANSPOSITIONS.

$$S_n = \langle (12), (13), (23), \text{etc} \rangle$$

AND CONVINCE YOURSELF YOU CAN IMPLEMENT ANY EXCHANGE AS A SEQUENCE OF PAIRWISE EXCHANGES OF ADJACENT ANYONS.

THIS REFLECTS THE FACT THAT

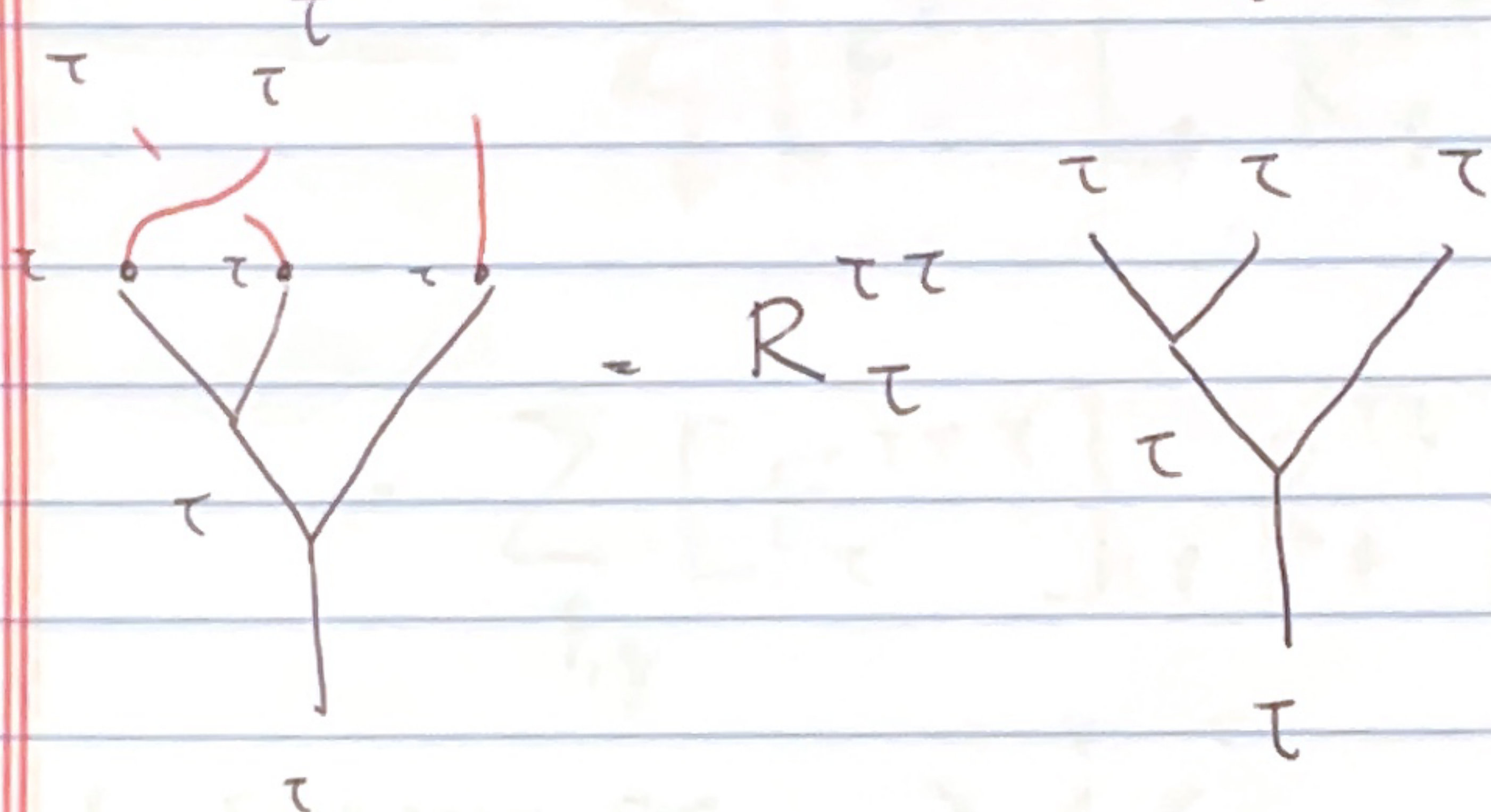
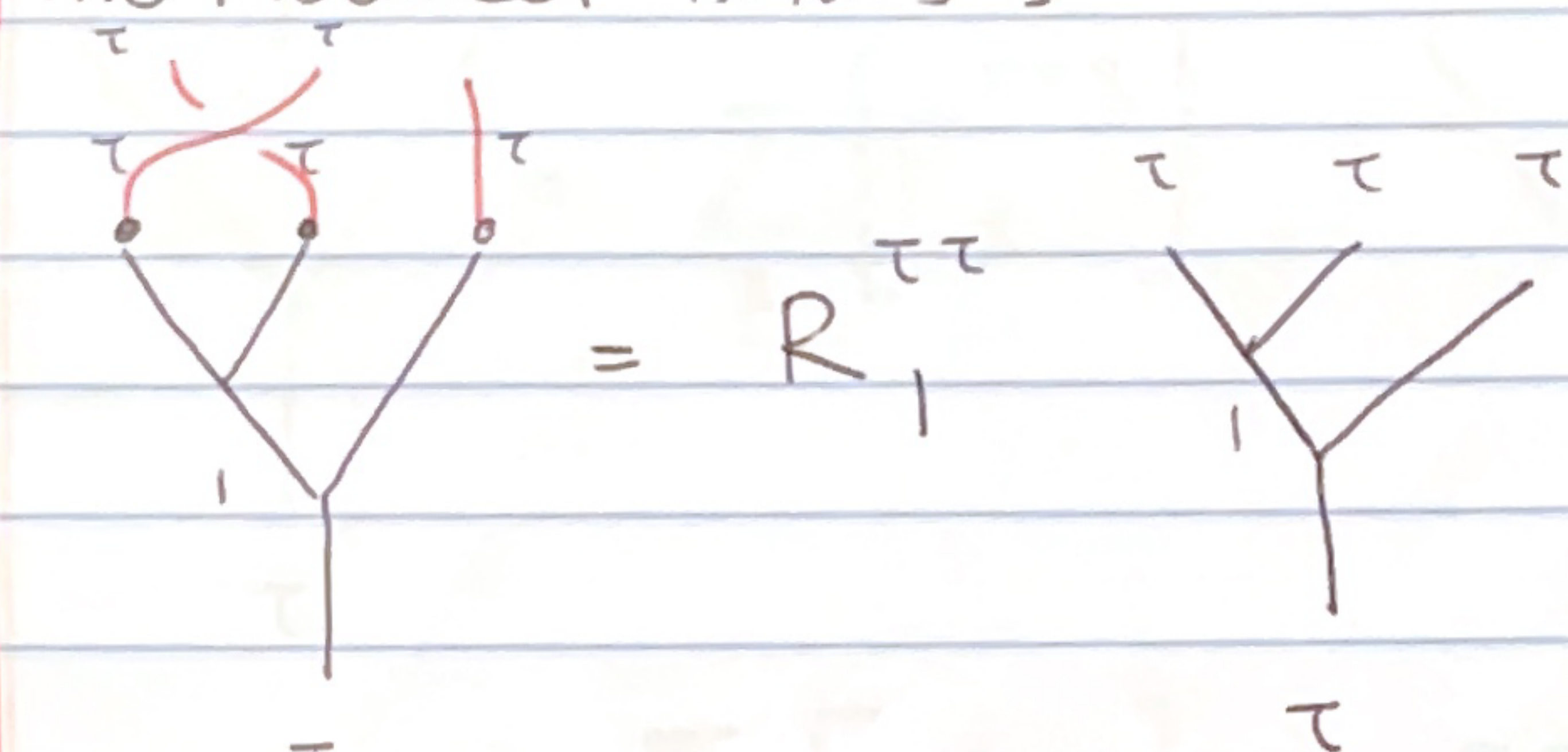
$$B_n = \left\langle \underbrace{\sigma_1, \sigma_2, \dots, \sigma_{n-1}}_{\text{BRAID GENERATORS}} \mid \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1 \end{array} \right\rangle$$

n-STRAND BRAID GROUP (pointing to B_n)

BRAID RELATION (pointing to $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$)

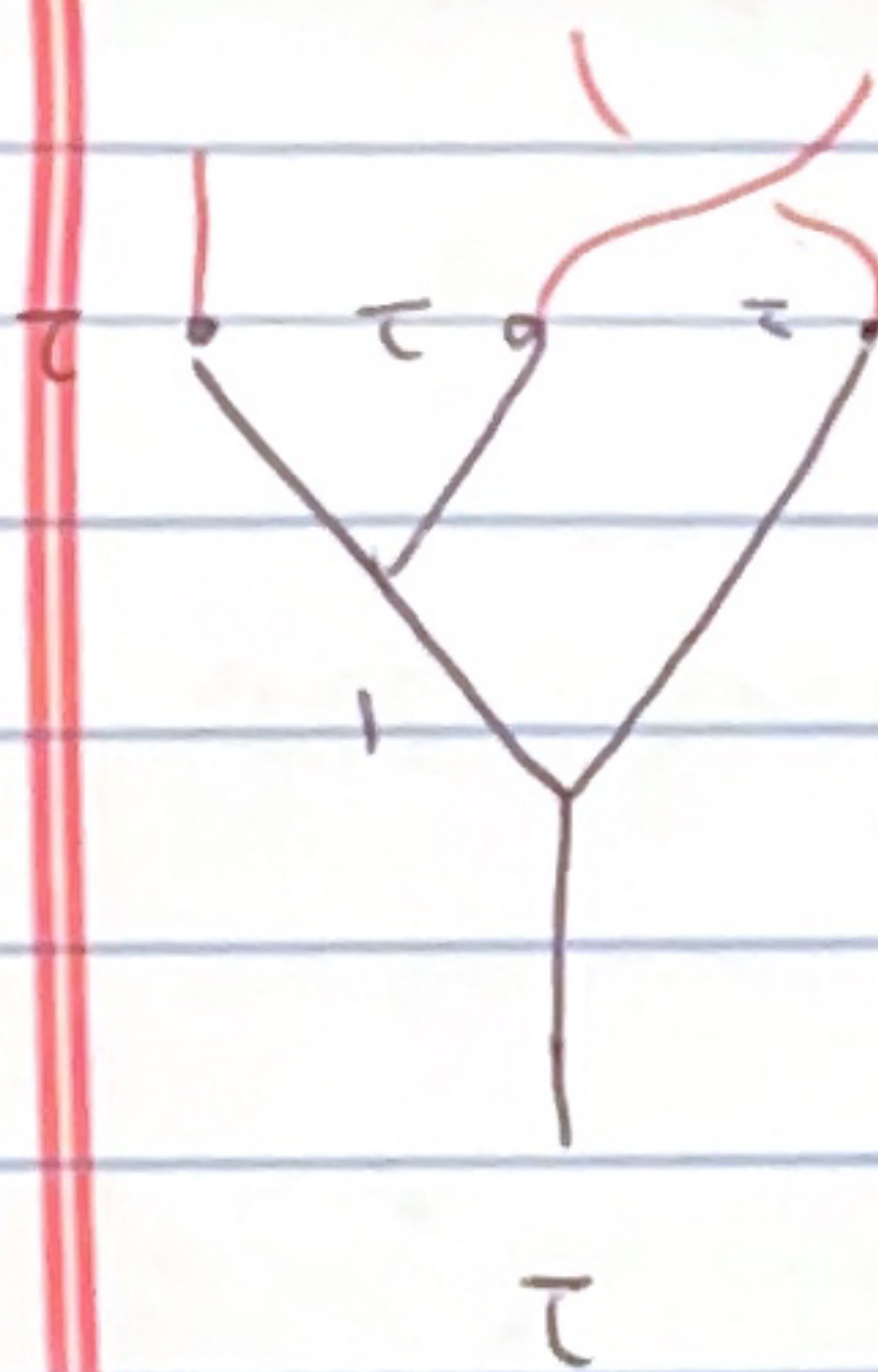
FAIR COMMUTATION (pointing to $\sigma_i \sigma_j = \sigma_j \sigma_i$)

HOW DO OUR STATES EVOLVE WHEN WE EXCHANGE THE FIRST TWO FIBONACCI ANYONS?

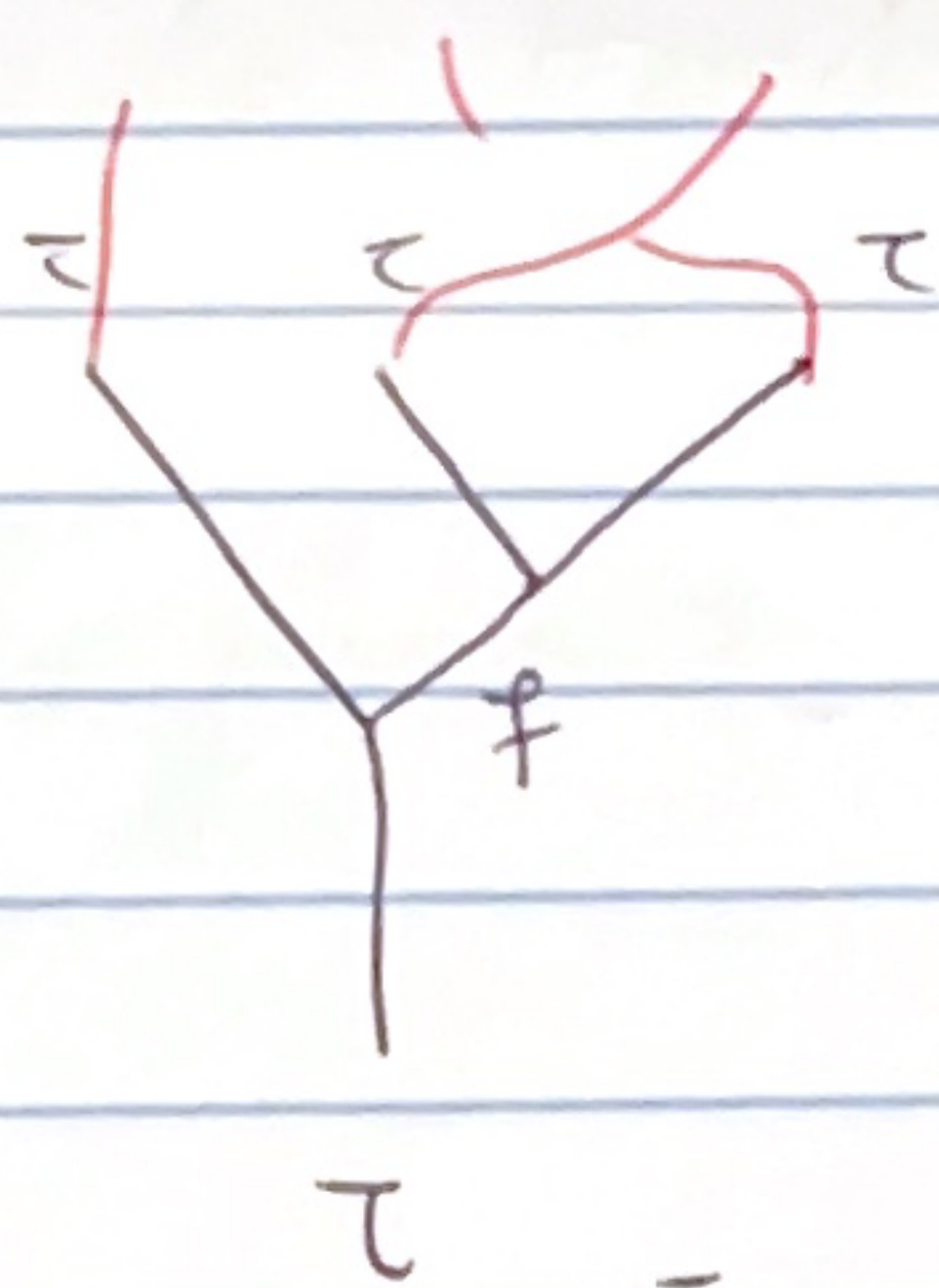


$$\begin{aligned} \hookrightarrow \sigma_1 &\xrightarrow{\text{OR}} \begin{pmatrix} R_1^{\tau\tau} & 0 \\ 0 & R_{\tau}^{\tau\tau} \end{pmatrix} \\ \text{X} &\xrightarrow{\quad} \begin{aligned} |0\rangle &\rightarrow R_1^{\tau\tau} |0\rangle \\ |1\rangle &\rightarrow R_{\tau}^{\tau\tau} |1\rangle \end{aligned} \end{aligned}$$

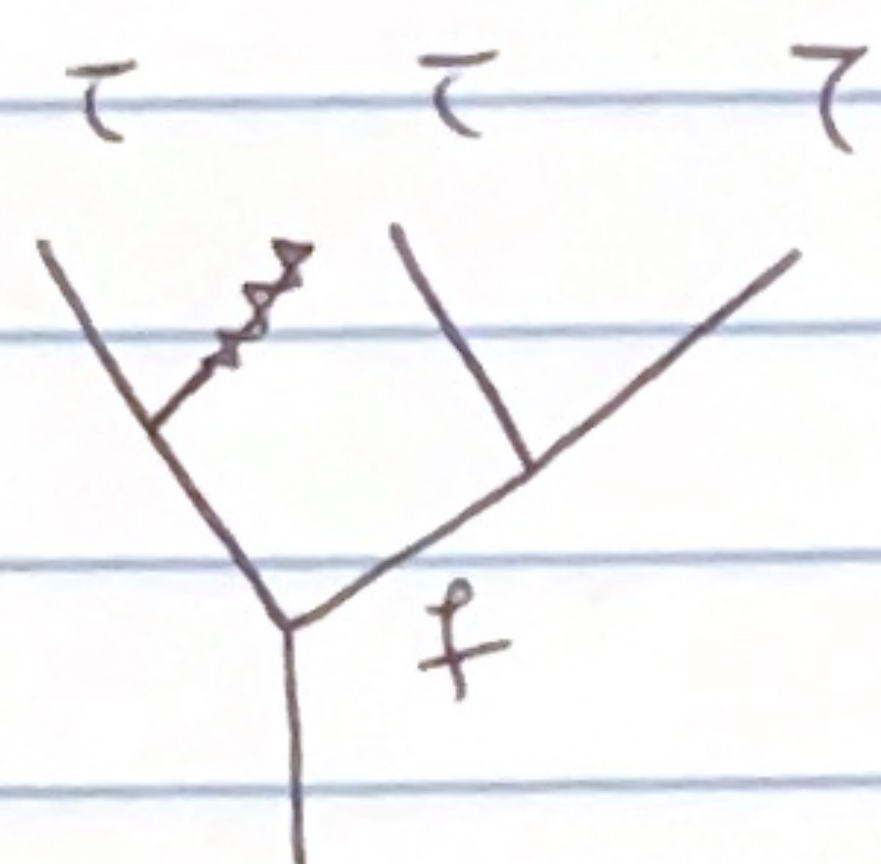
WHAT ABOUT THE SECOND TWO?



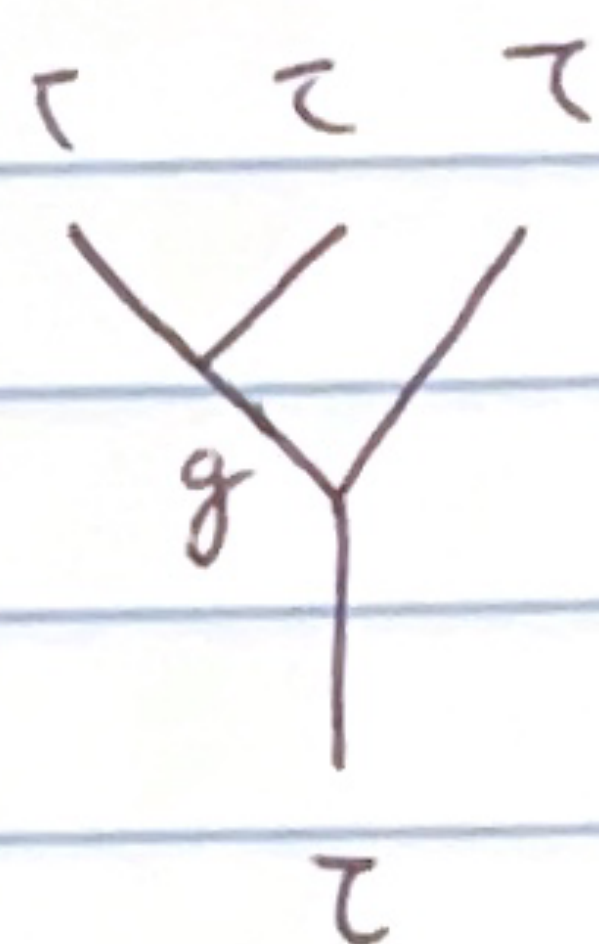
$$= \sum_f \left[F_{\tau}^{\tau\tau\tau} \right]_{1,f}$$



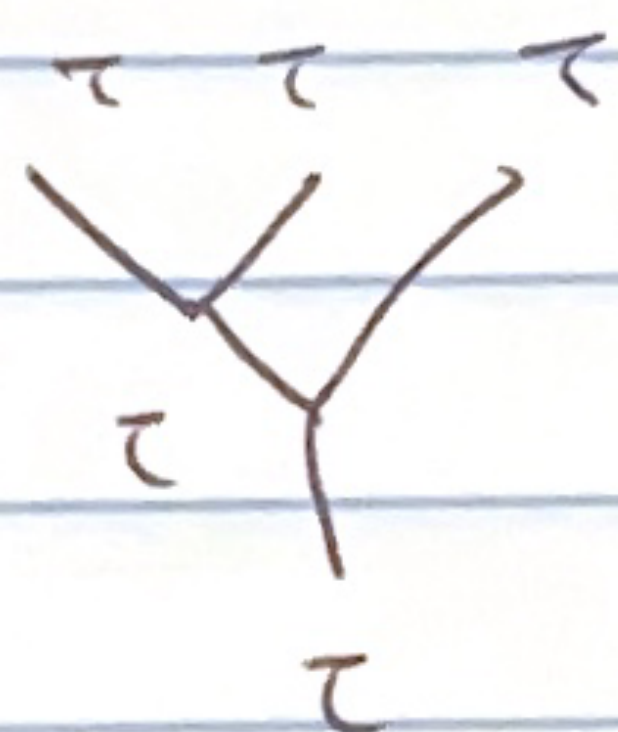
$$= \sum_f \left[F_{\tau}^{\tau\tau\tau} \right]_{1,f} R_{\tau}^{\tau\tau}$$



$$= \sum_{f,g} \left[F_{\tau}^{\tau\tau\tau} \right]_{1,f} R_{\tau}^{\tau\tau} \left[F_{\tau}^{\tau\tau\tau} \right]_{f,g}$$



& SIMILARLY FOR



$\Rightarrow$

σ_2

$\longrightarrow$

$$= \begin{bmatrix} \left[F_{\tau}^{\tau\tau\tau} \right]_{1,1} R_{\tau}^{\tau\tau} \left[F_{\tau}^{\tau\tau\tau} \right]_{1,1}^* & \left[F_{\tau}^{\tau\tau\tau} \right]_{1,2} R_{\tau}^{\tau\tau} \left[F_{\tau}^{\tau\tau\tau} \right]_{2,1}^* \\ \left[F_{\tau}^{\tau\tau\tau} \right]_{1,2} R_{\tau}^{\tau\tau} \left[F_{\tau}^{\tau\tau\tau} \right]_{2,1}^* & \left[F_{\tau}^{\tau\tau\tau} \right]_{1,2} R_{\tau}^{\tau\tau} \left[F_{\tau}^{\tau\tau\tau} \right]_{2,2}^* \\ \left[F_{\tau}^{\tau\tau\tau} \right]_{2,1} R_{\tau}^{\tau\tau} \left[F_{\tau}^{\tau\tau\tau} \right]_{1,1}^* & \left[F_{\tau}^{\tau\tau\tau} \right]_{2,1} R_{\tau}^{\tau\tau} \left[F_{\tau}^{\tau\tau\tau} \right]_{1,2}^* \\ \left[F_{\tau}^{\tau\tau\tau} \right]_{2,2} R_{\tau}^{\tau\tau} \left[F_{\tau}^{\tau\tau\tau} \right]_{1,1}^* & \left[F_{\tau}^{\tau\tau\tau} \right]_{2,2} R_{\tau}^{\tau\tau} \left[F_{\tau}^{\tau\tau\tau} \right]_{1,2}^* \end{bmatrix}$$

$$= \left[F_{\tau}^{\tau\tau\tau} \right] \begin{pmatrix} R_{\tau}^{\tau\tau} & 0 \\ 0 & R_{\tau}^{\tau\tau} \end{pmatrix} \left[F_{\tau}^{\tau\tau\tau} \right]^{\dagger}$$

$$= "FRF^{\dagger}"$$