Lecture 10: Kramers-Wannier duality

There are many different incarnations of what is called *Kramers-Wannier duality*, which is the smallest example of non-invertible symmetry. We'll see that this is very closely related to – but not always precisely given by – the idea of symmetry operators satisfying a non-invertible fusion rule.

The flavor of Kramers-Wannier duality most accessible to us is a 1d Hamiltonian lattice model. We follow Section 3.3 in Shu-Heng Shao's TASI lecture notes very closely.

The *transverse-field Ising lattice model* is defined on a 1d lattice with periodic boundary conditions. We put a qubit $\mathcal{H}_j = \mathbb{C}^2$ on the jth site of the lattice where for sites $j = 1, 2, \ldots, N$. The $N + 1$st site is identified with the 1st site, and so on, so that we can think of the 1d lattice as living on the circle S^1.[17] Similarly, $X_{N+1} = X_1$ and $Z_{N+1} = Z_N$.

$$\begin{array}{ccccccccccc} \bullet & \bullet & \bullet & \cdots & \bullet & \cdots & \bullet & \bullet & \bullet & \cdots \\ 0 & 1 & 2 & & j & & N & 1 & 2 & \end{array}$$

The total Hilbert space is $\mathcal{H}_{total} = \bigotimes_{j=1}^{N} \mathcal{H}_j$, with $\dim(\mathcal{H}) = 2^N$. We work in the Hadamard basis of $\mathcal{H}_{total}$, i.e. the eigenbasis of the Pauli X operators acting on each of the N sites.

Recall a basis state looks like a string of N plusses and minuses $|+--+\cdots+\rangle = |+\rangle \otimes |-\rangle \otimes |-\rangle|+\rangle \otimes \cdots \otimes |+\rangle$, where the local basis is related to the standard qubit basis by $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

The Hamiltonian of this model is given by

$$\boxed{H = -g \sum_{j=1}^{N} X_j - \sum_{j=1}^{N} Z_j Z_{j+1}.}$$

Previously when we've written down Hamiltonians, the name of the game is to find their ground state space. Today though we're more interested in the symmetries of this lattice model.

Unitary symmetry operators

Traditionally we think of symmetries as being implemented by unitary operators, which are in particular invertible. In order for a unitary to qualify as a symmetry on the lattice, one may require it to act in certain ways (locally, or on-site).

In this discusion we will not worry about these kinds of details and be content to think of a conventional symmetry as some unitary U acting on $\mathcal{H}_{total}$ that commutes with the Hamiltonian H, i.e. $[U, H] = UH - HU = 0$. Since U is invertible, commuting with the Hamiltonian is equivalent to saying that H is fixed under conjugation, $UHU^{-1} = H$ (this is how symmetries act on operators).

[17] This is just like putting our 2d lattice on a torus, but one dimension down!

$\mathbb{Z}_2$ "spin flip" symmetry

Our model has a $\mathbb{Z}_2$ symmetry enacted by the operator

$$\eta = \prod_{j=1}^{N} X_j.$$

To see that η is a symmetry, we compute $[\eta, H] = \eta H - H\eta = 0$, since η commutes with the X terms in H and commutes with each $Z_i Z_{i+1}$ term. To see the latter, note that $\prod_j X_j$ commutes with $Z_i Z_{i+1}$ whenever $j \neq i$, $j \neq i_{i+1}$, and then

$$(X_i X_{i+1})(Z_i Z_{i+1}) = (X_i Z_i)(X_{i+1} Z_{i+1}) = (-Z_i X_i)(-Z_{i+1} X_{i+1}) = (Z_i X_i)(Z_{i+1} X_{i+1}) = (Z_i Z_{i+1})(X_i X_{i+1}).$$

Alternatively, we compute

$$\begin{cases} \eta X_i \eta^{-1} = X_i \\ \eta Z_i \eta^{-1} = -Z_i \end{cases}$$

and thus

$$\eta H \eta^{-1} = H.$$

Since $\eta^2 = 1$ we see this is an invertible $\mathbb{Z}_2$ symmetry.

$\mathbb{Z}_N$ lattice translation symmetry

Since our 1d lattice with lives on a circle, there is a symmetry that arises from cyclic permutation of the N sites.

This lattice translation symmetry is implemented by the operator

$$T = \prod_{j=1}^{N-1} T_{j,j+1}, \text{ where}$$

$$T_{j,j+1} = \frac{1}{2}\left(X_j X_{j+1} + Y_j Y_{j+1} + Z_j Z_{j+1} + 1\right)$$

In the exercises you will show directly that $[T, H] = 0$. We compute that

$$\begin{cases} T X_i T^{-1} = X_{i+1} \\ T Z_i T^{-1} = Z_{i+1} \end{cases}$$

(and also $T^k X_i (T^{-1})^k = X_{i+k}$ and $T^k Z_i (T^{-1})^k = Z_{i+k}$ for $k < N$.

and hence

$$T X_i T^{-1} = -\sum_{j=1}^{N} X_{j+1} - \sum_{j=1}^{N} Z_{j+1} Z_{j+2} \overset{l \to j+1}{=} -\sum_{l=1}^{N} X_l - \sum_{l=1}^{N} Z_l Z_{l+1} = H.$$

Since $T^N = 1$ we see this is an invertible $\mathbb{Z}_N$ symmetry.

We can check that $\eta T = T\eta$, and so the invertible symmetries form the group $\mathbb{Z}_2 \times \mathbb{Z}_N$.

Noninvertible Kramers-Wannier symmetry operator D

When $g = 1$ and we specialize to the *critical* Ising model, there is another kind of symmetry that interchanges the terms in the Hamiltonian

$$\begin{aligned} X_i &\mapsto Z_i Z_{i+1} \\ Z_i Z_{i+1} &\mapsto X_i \end{aligned}$$

and leaves H fixed overall.

Suppose that this transformation could be implemented by a unitary – and in particular invertible – symmetry operator U, so that

$$\begin{cases} U X_i U^{-1} = Z_i Z_{i+1} \\ U Z_i Z_{i+1} U^{-1} = X_{i+1} \end{cases}.$$

Such an operator would necessarily act trivially on η, since

$$U \eta U^{-1} = \prod_{j=1}^{N} U X_j U^{-1} = \prod_{j=1}^{N} Z_j Z_{j+1} = (Z_1 Z_2)(Z_2 Z_3) \cdots (Z_{N-1} Z_N)(Z_N Z_1) = 1$$

But then $U \eta U^{-1} = 1 \implies \eta = U^{-1} U = 1$, a contradiction.

There exists, however, a unitary U_{KW} that *almost* achieves this symmetry, namely it sends $X_i \mapsto Z_i Z_{i+1}$ and $Z_i Z_{i+1} \mapsto X_{i+1}$ except at the endpoints of our spin chain. It acts on the terms of the Hamiltonian as

$$\begin{cases} U_{KW} X_i U_{KW}^{-1} = Z_i Z_{i+1} & 1 \leq i \leq N-1 \\ U_{KW} Z_i Z_{i+1} U_{KW}^{-1} = X_{i+1} & 1 \leq i \leq N-1, \end{cases}$$

but there are obstructions to the symmetry being implemented

$$\begin{cases} U_{KW} X_N U_{KW}^{-1} = \eta Z_N Z_1 \\ U_{KW} Z_N Z_1 U_{KW}^{-1} = \eta X_1. \end{cases}$$

This is definitely not a symmetry, since

$$U H U^{-1} = \sum_{j=1}^{N-1} X_j - \sum_{j=1}^{N-1} Z_j Z_{j+1} - \eta X_1 - \eta Z_n Z_1 \neq H.$$

We won't be concerned with the exact form of U_{KW} since its derivation is a bit beyond our scope for today, but it may interest you to know it doesn't look too out of the ordinary.

Up to a phase,

$$U_{KW} = \prod \left(\frac{1 + i X_j}{\sqrt{2}} \right) \left(\frac{1 + i Z_j Z_{j+1}}{\sqrt{2}} \right) \left(\frac{1 + i X_N}{\sqrt{2}} \right).$$

Happily there is an operator D that commutes with H and implements the symmetry in the sense that

$$\begin{cases} DX_i = Z_i Z_{i+1} D \\ DZ_i Z_{i+1} = X_{i+1} D \end{cases}$$

for all values of j.

You can check that

$$D = U_{KW}(1+\eta)$$

has this property using what we already know about how U_{KW} and η act on our local operators.

But it is neither unitary nor invertible, so we can't talk about D^{-1}. The quickest way to see this is to observe that there are some basis states in the kernel of D: e.g. if $N = 5$ the operator η sends

$$|+-+--\rangle \mapsto (-1)^3|+-+--\rangle,$$

and hence $(1+\eta)|+-+--\rangle = |+-+--\rangle - |+-+--\rangle = 0$. In general we see that D sends basis states with odd parity to 0.

We interpret D as a *non-invertible symmetry* of the lattice model.

Algebra of symmetry operators on the lattice

In summary, we have symmetry generators η, T, and D. They satisfy some relations, which we list without proof, that appear to be fusion rules at first glance.

$$\begin{cases} \eta^2 = 1 \\ T^N = 1 \\ \eta T = T\eta \\ \eta D = D\eta \\ DT = TD \\ D^2 = (1+\eta)T \end{cases}.$$

We note that D is not self dual, since 1 doesn't appear in D^2. This means we don't have enough information at this point to determine whether these correspond to a fusion ring. It turns out that these symmetries do not give rise to a fusion category symmetry on the lattice precisely, although we do not give details here.

What is true is that in the *thermodynamic limit* where we take $N \to \infty$ and limit to a continuum field theory, the translation symmetry operator T limits to 1 in a strong enough sense that we can disregard it.

The continuum theory realized by the limit of the critical Ising model is the *Ising conformal field theory (CFT)*, and it has symmetry generators

$$\begin{cases} \eta^2 = 1 \\ \eta D = D\eta \\ D^2 = 1+\eta \end{cases}$$

which you hopefully recognize as satisfying the Ising fusion rules introduced previously.

Summary

We say that this 1d lattice model exhibits a non-invertible symmetry. While the symmetry operators didn't exactly form a fusion ring on the lattice, they formed the Ising fusion ring in the thermodynamic ($N \to \infty$) limit. What we're seeing is a piece of an Ising fusion category symmetry of the (1+1)D Ising CFT.

Up until today we had been studying 2d lattice models, and there we saw fusion rings play two different roles: first as organizing the fusion rules of the quantum degrees of freedom that decorated the lattice, and second as the fusion rules of the emergent quasiparticles.

Here the role of the fusion ring is to organize the *categorical symmetries* of a (1+1)D QFT. After we finally see fusion categories and not just fusion rings, we'll return to the subject of non-invertible symmetry and explain the relationship between these different roles that fusion categories play in different settings and dimensions.