

Lecture 11: Building a fusion category

In this lecture we'll describe the data structure of a fusion category and its associated *graphical calculus*, starting with the data of a fusion ring, namely its fusion coefficients N_c^{ab} with respect to a label set L .

Skeletal fusion categories versus fusion categories

We'll see that the data of a fusion category can be captured by a pair $(N_c^{ab}, F_d^{abc}]_{(m,\alpha,\beta),(n,\gamma,\delta)}$ consisting of fusion coefficients and *F-symbols*. A fusion category presented in this way is called a *skeletal fusion category*. A skeletal fusion category is to a fusion category as a group's multiplication table is to a group's definition as a set equipped with a binary operation satisfying some axioms.¹⁸ That is, it lets us do concrete calculations with a specific instance of a fusion category, either by hand or with the help of a computer! On the other hand, we may not always be so lucky to have our hands on *skeletal data* $(N_c^{ab}, F_d^{abc}]_{(m,\alpha,\beta),(n,\gamma,\delta)}$ for a fusion category of interest, or if the category gets too big it might be an unwieldy way to work with it. These points will be clearer in due time. For now, we emphasize that what we are introducing in this lecture is not the definition of a fusion category. A skeletal fusion category is rather a concrete, combinatorial way to encode a fusion category, which in theory always exists, but may be hard to find or simply not necessary to have depending on your purposes.

Theorem 2.1

Every fusion category is equivalent to a skeletal fusion category, i.e. admits a description of the form $(N_c^{ab}, F_d^{abc}]_{(m,\alpha,\beta),(n,\gamma,\delta)}$.

Just like how permuting the rows and columns of the multiplication table of a group will still describe an isomorphic group, there is a redundancy or degrees of freedom in a skeletal description of a fusion category which comes from fusion ring isomorphisms and compatible gauge transformations. We won't see gauge transformations until the next lecture though.

Unitary fusion categories versus spherical fusion categories

Most of the time in physics we care about (skeletal) *unitary fusion categories*, which require just a bit of extra data to specify. In a later lecture we'll discuss a slight generalization of *unitary fusion categories* called *spherical fusion categories*, which will empower us to handle fusion categories which are unitary, *anti-unitary*, or neither!

The graphical calculus arises from identifying certain morphisms in our fusion category with graphs whose edges are labeled by elements of L . Properties of morphisms or isomorphisms between different spaces of morphisms can then be encoded in diagrammatic moves that relate different graphs. The graphical calculus lets us do rigorous calculations in a (skeletal) fusion category.

¹⁸We'll continue to lean into this analogy throughout this unit.

Remark 11. You'll recall that we introduced string diagrams to encode morphisms and the axioms they satisfy in a general category way back in Lecture 1. String diagrams and the “trivalent graphs” you're about to use to encode morphisms of skeletal fusion categories are closely related, and some of that discussion applies to this lecture. However, string diagrams give a much more general kind of graphical calculus that works for fusion categories that are not skeletal, or monoidal categories which lack the kind of nice finiteness properties needed for a fusion category. We'll revisit string diagrams again when we discuss the abstract definition of a fusion category in a few lectures.

Building a fusion category from a fusion ring

Let N_c^{ab} be the fusion coefficients of a fusion ring on the label set L .

As you know, understanding a category means understanding its objects and its morphisms. We'll describe a category $\mathcal{C}$ whose objects are built out of L and morphisms are built out of the fusion and splitting spaces of dimension N_c^{ab} that we introduced to close out our discussion of fusion rings a few lectures ago.

Objects

An object X in a skeletal fusion category $\mathcal{C}$ is of the form

$$X = \bigoplus_{a \in L} n_a a$$

, where $n_a \in \mathbb{Z}_{\geq 0}$. You should think of the $\bigoplus$ symbol as notation that generalizes how we add representations in the category $\text{Rep}(G)$; the objects we're adding no longer can be interpreted as representations but they add (and multiply) like representations. It tells us we're working in a category now, and not just a ring. For skeletal fusion categories there is no real harm in just thinking of $\bigoplus$ as “+” in a fusion ring and thinking of objects as finite nonnegative integer linear combinations of elements of L .

The elements of L are called *simple objects* in $\mathcal{C}$, and they generalize the (isomorphism classes of) irreducible representations in $\text{Rep}(G)$ that form the basis of the $\text{Rep}(G)$ fusion ring. We even use the notation $\text{Irr}(\mathcal{C}) = L$. Intuitively, elements of L are simple in the sense that they can't be written as a nontrivial direct sum of elements of L , but we'll see another characterization of simplicity via morphisms shortly.

Remember that one of the big ideas of category theory is that we want to think of objects up to isomorphism – and this often gets translated into saying “it's bad to talk about objects X and Y being equal, we talk about them being isomorphic”. But a skeletal fusion category is a safe place to talk about equality of objects. This is because if $X \cong Y$ in a skeletal category, then $X = Y$. There is only one object per isomorphism class.

Definition 2.1

In a general fusion category $\mathcal{C}$, $\text{Irr}(\mathcal{C})$ is a set of “representatives” of isomorphism classes of simple objects. The *rank* of $\mathcal{C}$ is $|\text{Irr}(\mathcal{C})|$.

In a skeletal fusion category, since there is only one simple object a per isomorphism class $[a]$ of simple object, we can safely identify $\text{Irr}(C)$ with L and conflate isomorphism classes of simple objects with their representatives. This is also why we could borrow the terminology “rank” for $|L|$ when working with fusion rings.

We’ll see a fusion category has a structure $\otimes$ to multiply objects that generalizes the tensor product in $\text{Rep}(G)$, which again in the skeletal case you can think of as coming from the multiplication on the fusion ring generated by L .

Now we write the fusion rules of C as

$$a \otimes b = \bigoplus_{c \in L} N_c^{ab} c.$$

Since arbitrary objects are built out of simple ones under $\oplus$ we will be able to understand arbitrary morphisms as built out of morphisms between simple objects under $\oplus$ and $\otimes$. We will continue to use lowercase Latin letters $1, a, b, c, \dots$ for simple objects and capital $X, Y, Z, \dots$ Latin letters for more general objects.

Morphisms

All morphisms in a fusion category form finite-dimensional $\mathbb{C}$ -vector spaces, $\text{Hom}(X, Y) \cong \mathbb{C}^n$ for some n . In a skeletal fusion category specifically, all morphisms form finite-dimensional $\mathbb{C}$ -vector spaces “with basis”. Briefly, these bases will be given by admissibly labeled “trivalent graphs”, but we’ll build up to that general understanding from simple objects step-by-step.

Morphisms between simple objects

For $a \in L$ we have $\text{Hom}(a, a) = \text{span}_{\mathbb{C}}(\{id_X\})$. In particular $\text{Hom}(a, a) \cong \mathbb{C}$ is one-dimensional as a vector space.

Definition 2.2

An object X in a fusion category over $\mathbb{C}$ is simple if $\text{Hom}(X, X) \cong \mathbb{C}$.

If $a \neq b$, then $\text{Hom}(a, b) = 0$. In summary,

$$\text{Hom}(a, b) \cong \delta_{a,b} \mathbb{C}$$

for simple a and b . This is a categorical version of Schur’s Lemma in representation theory, which we sadly did not have time to cover.

Fusion and splitting spaces

Recall the definition of the fusion and splitting spaces V_c^{ab} and V_{ab}^c as $\mathbb{C}$ -linear combinations of the “trivalent fusion and splitting vertices”

$$V_c^{ab} = \text{span}_{\mathbb{C}} \left(\left\{ \begin{array}{c} a \quad b \\ \swarrow \quad \searrow \\ \mu \\ \downarrow \\ c \end{array} \right\}_{\mu=1,2,\dots,N_c^{ab}} \right) \quad \text{and} \quad V_{ab}^c = \text{span}_{\mathbb{C}} \left(\left\{ \begin{array}{c} c \\ \downarrow \\ \nu \\ \swarrow \quad \searrow \\ a \quad b \end{array} \right\}_{\nu=1,2,\dots,N_c^{ab}} \right).$$

These are our $\text{Hom}(a \otimes b, c)$ and $\text{Hom}(c, a \otimes b)$.

[clarify what happens when $N_c^{ab} = 0$]

Recall that id_1 has a special picture,

$$\text{id}_1 = \begin{array}{c} a \\ \vdots \\ a \end{array}$$

We may call these vacuum lines, borrowed from skeletal unitary *modular* fusion categories, although they may not always be able to be interpreted that way.

Now the fusion rules $1 \otimes a = a \otimes 1 = a$ of the multiplicative identity $1 \in L$ give us the pictures

$$\begin{array}{c} 1 \quad a \\ \swarrow \quad \searrow \\ a \end{array} = \begin{array}{c} a \\ | \\ a \end{array} \quad \text{and} \quad \begin{array}{c} a \quad 1 \\ \swarrow \quad \searrow \\ a \end{array} = \begin{array}{c} a \\ | \\ a \end{array}$$

This gives us the beginnings of a limited notion of *isotopy* in our graphical calculus. We can bend/straighten lines by adding/removing vacuum lines. We're starting to see where topology enters the picture.

$\text{Hom}(a \otimes b, a \otimes b)$

[transition]

[completeness relation]

F-symbols

Moving on to a three-fold product of simples, we observe that there are two equally natural choices of bases for $\text{Hom}(a \otimes b \otimes c, d)$, each of which is induced from the bases of our trivalent Hom spaces. There's a left-associated basis

$$\left\{ \begin{array}{c} a \quad b \quad c \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \alpha \quad \beta \\ \swarrow \quad \searrow \\ m \\ \downarrow \\ d \end{array} \right\}_{m \in L, 1 \leq \alpha \leq N_m^{ab}, 1 \leq \beta \leq N_d^{mc}}$$

and a right-associated basis

$$\left\{ \begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagup \quad \diagup \\ \quad \gamma \\ \diagdown \quad \diagup \\ \quad \delta \\ d \end{array} \right\}_{n \in L, 1 \leq \gamma \leq N_n^{bc}, 1 \leq \delta \leq N_d^{an}}$$

If we use a change of basis to express the left basis as a linear combination of the right basis, we'll get an equation of vectors that looks like

$$\begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagup \quad \diagup \\ \quad \alpha \\ \diagdown \quad \diagup \\ \quad m \quad \beta \\ \diagdown \quad \diagup \\ \quad \delta \\ d \end{array} = \sum_n [F_d^{abc}]_{m,n} \begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagup \quad \diagup \\ \quad \gamma \\ \diagdown \quad \diagup \\ \quad n \\ \diagdown \quad \diagup \\ \quad \delta \\ d \end{array}.$$

where $m, n, \alpha, \beta, \gamma, \delta$ index each set of basis vectors for $\text{Hom}(a \otimes b \otimes c, d)$. The F -symbols are sometimes called the 10j-symbols because of the 10 indices that appear.

[F-move]

In the multiplicity-free case, we can ignore the vertex labels and the F -move becomes

$$\begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagup \quad \diagup \\ \quad m \\ \diagdown \quad \diagup \\ \quad d \end{array} = \sum_n [F_d^{abc}]_{m,n} \begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagup \quad \diagup \\ \quad n \\ \diagdown \quad \diagup \\ \quad d \end{array}.$$

Similarly, F -symbols of a multiplicity-free fusion category these are sometimes called 6j-symbols. If we had more time we'd discuss the origin of the terminology of 6j-symbols in physics, which predates fusion categories.

Pentagon equations

To simplify what follows for anyone seeing this for the first time, we assume that our fusion rules are multiplicity-free.

[add exposition transitioning]

[add pentagon diagrams]

We see that the F -symbols must satisfy the *pentagon equations*

$$[F_e^{fed}]_{g,l} [F_e^{abl}]_{f,k} = \sum_{h \in L} [F_g^{abc}]_{f,h} [F_e^{ahd}]_{g,k} [F_k^{bcd}]_{h,l}$$

for all $a, b, c, d, e, f, g, k, l \in L$.

