Lecture 13: Gauge Transformations

Let N_c^{ab} be the fusion coefficients of a fusion ring on the label set L.

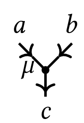
Two sets of F-symbols $[F_d^{abc}]_{(m,\alpha,\beta),(n,\gamma,\delta)}$ and $[\tilde{F}_d^{abc}]_{(m,\alpha,\beta),(n,\gamma,\delta)}$ on describe the same fusion category with fusion rules N_c^{ab} if they are related by a *gauge transformation.*

A gauge transformation will consist of a collection of invertible matrices Γ_c^{ab} for each $a, b, c \in L$. When (N, F) describes a unitary fusion category, we will see that the Γ_c^{ab} should be unitary matrices.

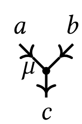
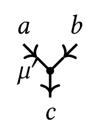
We will now give a derivation of the data of a gauge transformation by exploring the *gauge degrees of freedom* in writing down a skeletal description of a fusion category with respect to a fixed fusion rule. When we introduce the formal definition of a fusion category, where the right notion of "isomorphism" is that of a *monoidal autoequivalence functor*, we will be able to return to this subject and understand what we are doing as "writing down a monoidal autoequivalence functor in coordinates".

Gauge degrees of freedom on trivalent fusion spaces

Recall that our skeletal category consists of Hom spaces with bases given by admissibly labeled fusion trees. These bases were induced from an initial choice of basis for $V_c^{ab} = \mathrm{Hom}(a \otimes b, c)$.

$$V_c^{ab} = \mathrm{span}_{\mathbb{C}}\left(\left\{\right.\right.$$

$$\left.\left.\right\}_{\mu=1,2,\ldots,N_c^{ab}}\right)$$

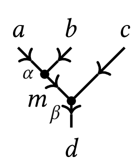
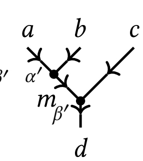
We could equally well have chosen some other basis, whose vectors would be linear combinations of these trivalent fusion graphs:


$$\mapsto \qquad [\Gamma_c^{ab}]_{\mu,\mu'}$$


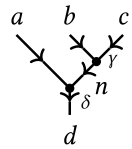
At the moment all we require of the $[\Gamma_c^{ab}]_{\mu,\mu'}$ is that they assemble into a change-of-basis matrix, so that in particular the matrix $\Gamma_c^{ab} \in \mathrm{GL}(N_c^{ab}, \mathbb{C})$ for each triple $a, b, c \in L$. We want our trivalent splitting basis of V_{ab}^c to be dual to that of V_c^{ab}, so there are no additional degrees of freedom that arise from considering trivalent Hom spaces.

Of course, this basis change propagates downstream and induces changes of bases for $\mathrm{Hom}(a \otimes b \otimes c, d)$.

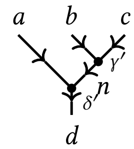
Now the induced left-associated basis transforms by


$$\mapsto \qquad \sum_{\alpha',\beta'} [\Gamma_m^{ab}]_{\alpha\alpha'}[\Gamma_d^{mc}]_{\beta\beta'}$$


Similarly, the right-associated basis transforms



$$\mapsto \quad \sum_{\gamma',\delta'} [\Gamma^{an}_d]_{\delta\delta'} [\Gamma^{bc}_n]_{\gamma\gamma'}$$



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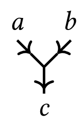
Gauge Transformed F-symbols

Comparing our two gauge-transformed bases of $\mathrm{Hom}(a \otimes b \otimes c, d)$ we see that the original F-symbols transform as

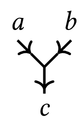
$$[F^{abc}_d]_{(m,\alpha,\beta),(n,\gamma,\delta)} \quad \mapsto \quad \sum_{\alpha',\beta',\gamma',\delta'} [\Gamma^{ab}_m]_{\alpha'\alpha} [\Gamma^{mc}_d]_{\beta'\beta} [F^{abc}_d]_{(m,\alpha',\beta'),(n,\gamma',\delta')} [\left(\Gamma^{an}_d\right)^{-1}]_{\delta,\delta'} [\left(\Gamma^{bc}_n\right)^{-1}]_{\gamma,\gamma'}$$

Multiplicity-free case

When the category is multiplicity-free the gauge transformations are greatly simplified. Now the gauge degree of freedom is just the ability to rescale a basis element of V^{ab}_c



$$\mapsto \quad \Gamma^{ab}_c$$



where $\Gamma^{ab}_c \in \mathbb{C}^\times$, or in the unitary case $\Gamma^{ab}_c \in U(1)$.

The F-symbols then transform by a ratio of four gauge symbols,

$$[F^{abc}_d]_{m,n} \quad \mapsto \quad \frac{\Gamma^{ab}_m \Gamma^{mc}_d}{\Gamma^{an}_d \Gamma^{bc}_n} [F^{abc}_d]_{m,n}.$$

In the exercises you will show that the gauge-transformed F-symbols automatically satisfy the pentagon equations.

Gauge Invariants

We have established that there are many degrees of freedom involved in describing a fusion category via F-symbols. But often times you will look at some tables of data in a physics paper and see that the F-symbols take a particularly nice form. This is because someone has made a wise choice in *fixing a gauge* so that the F-symbols are easy to work with and are defined over a nice subfield of $U(1)$.

However, with the exception of special *gauge invariant* F-symbols, the F-symbols themselves do not have an intrinsic value. That is, most F-symbols are not invariants of our fusion category under monoidal autoequivalence.

Certain values of the F-symbols will be manifestly gauge invariant though. For example, the symbol $[F_b^{abc}]_{b,b}$ is gauge invariant, since it transforms like

$$[F_b^{abc}]_{b,b} \quad \mapsto \quad \frac{\Gamma_b^{ab}\Gamma_b^{bc}}{\Gamma_b^{ab}\Gamma_b^{bc}}[F_b^{abc}]_{b,b} = [F_b^{abc}]_{b,b}.$$

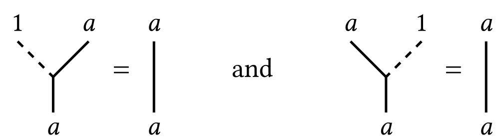
2.0.1 Frobenius-Schur indicators

Now we can return to our discussion of Frobenius-Schur indicators, and show why their definition as the sign of a certain F-symbol $[F_a^{aa^*a}]_{1,1}$ only has gauge-invariant meaning for self-dual objects.

Under a gauge transformation,

$$[F_a^{aa^*a}]_{1,1} \quad \mapsto \quad \frac{\Gamma_1^{aa^*}\Gamma_a^{1a}}{\Gamma_a^{a1}\Gamma_1^{a^*a}}[F_a^{aa^*a}]_{1,1}.$$

You'll recall that we already declared an identification of the pictures



For this reason we'll want to make the gauge choice $\Gamma_a^{1a} = \Gamma_a^{a1} = 1$ for all $a \in L$. One can actually prove that there is no loss of generality in doing so.

That leaves us with

$$[F_a^{aa^*a}]_{1,1} \quad \mapsto \quad \frac{\Gamma_1^{aa^*}}{\Gamma_1^{a^*a}}[F_a^{aa^*a}]_{1,1}.$$

There is nothing enforcing that $\Gamma_1^{aa^*} = \Gamma_1^{a^*a}$, so in general the F-symbol is not gauge invariant. However, if $a = a^*$, these gauge symbols cancel, and we see that the value of $[F_a^{aaa}]_{1,1}$ is independent of gauge, and in particular its sign (the Frobenius-Schur indicator of a self-dual object) is an invariant of the category.

Invariants of fusion categories from skeletal data

Because most F-symbols are not gauge invariant, there are not a ton of simple but also meaningful invariants of unitary fusion categories that aren't actually just invariants of the underlying fusion ring (like the rank, Frobenius-Perron dimensions, global dimension, etc.) outside of the Frobenius-Schur indicators.[19] However, by taking clever combinations of (monomials in) F-symbols one can cook up more gauge invariants.

We'll see that for unitary *modular* fusion categories there will be many more invariants that will be accessible to us and also have a more direct physical interpretation in terms of an anyon model.

[19] Since the quantum dimensions are positive and equal to the Frobenius-Perron dimensions by fiat, they don't contain additional information in the unitary case even though they needed the higher structure to be defined.