Unit 2: Lecture 1

Introduction

Many of the groups that we encounter naturally in physics are *matrix groups*, where the group multiplication is just matrix multiplication, like GL(N), O(N), and U(N).

The idea of representation theory is to *represent* elements of a group as matrices and their product via matrix multiplication.[9]

The genesis of the importance of group theory and representation theory in theoretical physics dates goes back about a century to the work of Hermann Weyl, who understood that elementary particles are classified by *unitary irreducible representations* of the symmetry groups of quantum field theories, and resulted in the Nobel prize-winning work of Wigner classifying the unitary representations of the Poincaré group. The role that representation theory plays in the areas of physics we're exploring in this course has some connections and analogies to that story, but it's really a story in it's own right.

Definition 2.1: Abstract

A N-dimensional representation of a group G is a homomorphism

$$\rho : G \to GL(V)$$

where V is an N-dimensional vector space over the field $\Bbbk$.

If we pick a basis of V, so that we can express any linear transformation $T \in GL(V)$ as an $N \times N$ matrix, then we can give a more concrete definition.

Definition 2.2: Concrete

A *matrix representation* is a homomorphism

$$\rho : G \to GL(N, \Bbbk).$$

If $\Bbbk = \mathbb{R}$ or $\mathbb{C}$ then the we call the representation real or complex, respectively. If $\rho(G) \subset O(N)$ or $U(N)$ then we call the representation orthogonal or complex, respectively. A representation $\rho : G \to GL(N, \mathbb{C})$ is called *faithful* if $\ker(\rho)$ is trivial, i.e. if ρ is injective. At times conflate the defining homomorphism ρ of a representation with the vector space V the group acts on, or give a name to the representation.

Wait a second...

We're here for physics, so we know we mostly only care about $\Bbbk = \mathbb{C}$, since quantum mechanics happens over $\mathbb{C}$. We also like to be able to do concrete calculations, so you might wonder why we're bothering with this distinction between abstract representations and matrix representations. The distinction – whether we have chosen a basis of a vector space – will be exactly the same distinction we see later between a fusion category and a *skeletal* fusion

[9] For infinite groups and representations which are not finite-dimensional the story is more involved, but we'll only be interested in finite-dimensional representations in this course.

category. These distinctions are often conflated in the physics literature, and this can cause a good deal of confusion for newcomers. It will be worth our while to appreciate the distinction so that we can be empowered to do physics using a variety of computational, categorical, and even topological methods!

Examples of group representations

Let's start with some examples of representations of the smallest nontrivial group.

Example 2.1: Representations of $\mathbb{Z}_2$

The trivial representation

$$\begin{aligned} \rho : \mathbb{Z}_2 &\to GL(1, \mathbb{C}) \\ g &\mapsto 1 \end{aligned}$$

The "sign" representation

$$\begin{aligned} \rho : \mathbb{Z}_2 &\to GL(1, \mathbb{C}) \\ g &\mapsto -1 \end{aligned}$$

The permutation representation

$$\begin{aligned} \rho : \mathbb{Z}_2 &\to GL(2, \mathbb{C}) \\ g &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Representations can be added

Notice that we can add representations together to get higher-dimensional ones by taking the direct sum.[10]

[10] Of course you know from linear algebra when and how you can add and multiply matrices together, but we should review how to take the *direct sum* and *tensor product* of matrices since in order to do concrete calculations they require we establish a shared convention. Let A and B be two square matrices, say $n \times n$ and $m \times m$. Then $A \oplus B$ is the $(n+m)\times(n+m)$-matrix block-diagonal matrix which we denote by $\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$, where the 0 in the upper right corner is understood to be the $n \times m$ matrix of all zeros and the one in the bottom left is the $m \times n$ matrix of all zeros.

Example 2.2: Direct sum of representations of $\mathbb{Z}_2$

The direct sum of the permutation and sign representations:

$$\begin{aligned} \rho : \mathbb{Z}_2 &\to GL(3, \mathbb{C}) \\ g &\mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

It seems quite obvious, but it's worth pointing out that the dimensions of representations add under the direct sum.

Definition 2.3

A representation V of G is called reducible if it can be written as a direct sum (up to equivalence, which we'll state more precisely shortly). Otherwise it is called irreducible, and we call V an *irrep* of G.

Theorem 2.1: Maschke's Theorem (in characteristic zero)

Let V be a complex representation of a finite group G. Then if V has a subrepresentation U, there is some other subrepresentation W of V such that $V = U \oplus W$.

It follows that every representation of a finite group G is a direct sum of irreps. This property is called semisimplicity and it's one of the hallmarks of a fusion category. We can also multiply representations together by taking the tensor product.

Example 2.3

The tensor product of the permutation and sign representations:

$$\begin{aligned} \rho : \mathbb{Z}_2 &\to GL(2, \mathbb{C}) \\ g &\mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

In representation theory in general one might be interested in how a given reducible representation decomposes into irreps. A special case of this problem is to understand how the tensor product of irreps decomposes into irreps.

Fusion table for irreps of $\mathbb{Z}_2$

We'll see later why it turns out that the trivial representation, let's call it 1, and the sign representation, call it s, are the only two irreps of $\mathbb{Z}_2$ (up to equivalence). You'll see in the exercises why the permutation representation of $\mathbb{Z}_2$ is reducible, even though the matrix image of the generator isn't identically equal to a direct sum of matrices.

We can assemble the data of how the irreps of $\mathbb{Z}_2$ break down into irreps when they are tensored together with a *fusion table*.

Example 2.4: Fusion table of $\mathbb{Z}_2$

ou can check that the fusion table for $\mathbb{Z}_2$ irreps looks like this:

$\otimes$	1	s
1	1	s
s	s	1

The first row and column of this table will always be determined by the properties of the trivial representation, just like the first row and column of a multiplication of a group. In this case the only nontrival *fusion rule* is $s \otimes s = 1$.

Notice that the fusion table for the irreps of $\mathbb{Z}_2$ looks the same as the group multiplication table for $\mathbb{Z}_2$. This is not a coincidence and is true for all finite abelian groups.

So let's see some examples of representations of a nonabelian group. Of course we have the trivial (one-dimensional) representation of S_3, but there's also a nontrivial one-dimensional representation.

Example 2.5: Sign or alternating representation of S_3

Define $\rho : S_3 \to GL(1, \mathbb{C})$ by sending a permutation σ to the parity of a transposition decomposition

$$\begin{aligned} \mathrm{id} &\mapsto 1 \\ (12) &\mapsto -1 \\ (23) &\mapsto -1 \\ (13) &\mapsto -1 \\ (123) &\mapsto 1 \\ (132) &\mapsto 1 \end{aligned}$$

Notice that the definition we gave works for all n, and that it will always give an irreducible representation.

Example 2.6: Permutation representation of S_3

Define $\rho : S_3 \to GL(3, \mathbb{C})$ by:

$$\mathrm{id} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(13) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$(23) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(123) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$(132) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The permutation representation is not irreducible, but it does have an irreducible two-dimensional subrepresentation. Rather than identifying this irreducible subrepresentation we'll construct it directly by leveraging the isomorphism $S_3 \cong D_6$.

Recall that D_6 is the group of rigid motions of the equilateral triangle. If we think of the triangle as being embedded in the plane with the centroid at the origin, then we can think about effecting the rotation and reflection generators r and s using matrices acting on $\mathbb{R}^2$. Under the isomorphism $S_3 \cong D_6$ we identify r with (123) and s with (23).

Example 2.7: Standard representation of S_3

Define $\rho : S_3 \to GL(3, \mathbb{C})$ on generators by:

$$\mathrm{id} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$(123) \mapsto \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$
$$(23) \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

You can multiply in S_3 to generate the matrix images of the other permutations.

Now let's call the trivial representation, alternating representation, and standard representation of S_3 1, ψ, and std, respectively. Just like for $\mathbb{Z}_2$, we can assemble the data of how these irreps decompose when we take their tensor product, or *fuse* them.

Example 2.8: Fusion table of S_3

e won't discuss how to compute these tensor product decompositions; the goal for now is the appreciate this idea of a fusion table and how it arises very naturally in representation theory.

$\otimes$	1	ψ	std
1	1	ψ	std
ψ	ψ	1	std
std	std	std	$1 \oplus \psi oplus \mathrm{std}$

Notice that entries of the fusion table may now be a sum of irreps instead of a single irrep! This shows how fusion tables are definitely more general than multiplication tables for a group.

Intertwiners of representations and the category of representations of a finite group

Remember how when we introduced groups we placed a big emphasis on groups *and* structure-preserving maps (homomorphisms) between them? We were thinking about group theory as telling us about what's happening inside the category **Grp**.

Now we have a new algebraic gadget (representations) and so have to also think about their structure-preserving maps (intertwiners).

Definition 2.4: Abstract

An intertwiner ϕ between two representations $\rho : G \to GL(V)$ and $\psi : G \to GL(W)$ of G is a linear map $\phi : V \to W$ such that $\phi \circ \rho(g) = \psi(g) \circ \phi$. This defining condition can be expressed in a *commutative diagram*.

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ {\scriptstyle \rho(g)}\downarrow & & \downarrow{\scriptstyle \psi(g)} \\ V & \xrightarrow[\phi]{} & W \end{array}$$

In other words, ϕ "commutes" with the action of G. For this reason sometimes intertwiners are also called *G-equivariant* linear maps. If ϕ is a vector space isomorphism then we say the representations ρ and ψ are equivalent.

Definition 2.5: Concrete

An intertwiner between two matrix representations of G is the matrix representing a linear transformation $V \to W$ such that the same condition holds, namely a matrix T such that $T\rho(g) = \psi(g)T$ for all $g \in G$.
Two matrix representations are equivalent if there is an invertible matrix PT such that $T\rho(g)T^{-1} = \psi(g)$ for all $g \in G$. Note that this is exactly saying there is a change of basis matrix that relates all of the matrix representations of g for all g simultaneously.

Just like groups and their homomorphisms, representations of G and their intertwiners form a category, called **Rep**(G). When G is finite, **Rep**(G) is a fusion category! We've seen that we can add and multiply representations (objects in **Rep**(G)) – this reflects that the category possesses some additional structure, namely a direct sum $\oplus$ and a tensor product $\otimes$. We'll learn a lot more about **Rep**(G) shortly and about more general fusion categories as time goes on.

One thing about **Rep**(G) we can appreciate right now is that it is what is called *semisimple*, because every representation decomposes into a direct sum of *irreducible representations* by Maschke's theorem.

When we talk about classifying the representations of a finite group G, what we really mean is classifying its irreps, or the *simple* objects in **Rep**(G).

We'll finish up this discussion by covering some facts about the representation theory of finite groups that we need to have at least a nodding acquaintance with.

But first, recall the *conjugation action* of G on itself

$$\begin{aligned} G \times G &\to G \\ (g, h) &\mapsto g^{-1}hg \end{aligned}$$

that partitions G into *conjugacy classes*, where two elements h and k are in the same conjugacy class (i.e. *conjugate*) if there exists $g \in G$ such that $k = g^{-1}hg$.

Theorem 2.2

A finite group G has r distinct irreducible representations, where r is the number of conjugacy classes of G.

Theorem 2.3

Let $n_1, n_2, \ldots, n_r$ be the dimensions of the irreps of a finite group G. Then

$$\sum_{i=1}^{r} n_i^2 = |G|$$

.

Since every element of a finite abelian group A is in a conjugacy class all by itself, it follows from the first theorem above that there are $|A|$ many irreps of A. Then the dimension constraint in the second theorem implies that each of these irreps must be one-dimensional.

Revisiting the definition of a representation, this tells us that understanding the irreps of a finite abelian group A is the same as understanding homomorphisms $\rho : A \to \mathbb{C}^\times$.