Lecture 6

In the process of reviewing some basic aspects of the representation theory of finite groups we saw that the data of how the tensor product of irreps of a finite group decompose into a direct sum of irreps can be encoded in what we called a fusion table. These fusion tables looked a lot like the multiplication table for a finite group (and for abelian groups they simply are), except that each entry in the table could be a sum of the row and column labels instead of just a single one of them. Fusion tables make sense beyond finite groups and their representations, and we can draw them for any *fusion ring*. A fusion ring is the most basic part of a fusion category so we'll devote some time to understanding them on their own terms.

This entire lecture is going to be about *fusion rules*, which boils down to understanding equations that look like

$$a \times b = \sum_{c \in L} N_c^{ab} c$$

where the N_c^{ab} are just nonnegative integers and L is a finite set with $a, b, c \in L$. We call L the *label set* and N_c^{ab} the *fusion coefficients*. You've seen much more complicated equations in your life, these are pretty nice.

Remark 5. *There is something important we should address. Depending on where you're reading about fusion rules, you may see $a, b, c \in L$ called particles or particle types. This is valid when they are describing the fusion of point-lile particles or quasiparticles like anyons. But it is *not* always correct to interpret the elements of the label set as particles. Depending on the context they may instead be interpreted (rather abstractly) as quantum degrees of freedom, as symmetry operators, or as very general topological defects.*

We are interested in each of these contexts in this course, so we will try to be very general when discussing fusion rules for now. Since they are most intuitive to understand when intepreting them as particles, this is the language that we'll default to at times as a conceptual aid to learn definitions.

Generally speaking though we should try to be very careful delineating between the various roles that fusion rings (and later categories) play.

Examples of fusion rules

Example 2.1: Toric code anyon fusion rules

Recall that there are four 2-particle excitations in the toric code model, which we denoted by $\{1, e, m, f\}$.
We saw that they fuse like $\mathbb{Z}_2 \times \mathbb{Z}_2$, so (a generating list of) their fusion rules look like

$$\begin{cases} e \times e = 1 \\ m \times m = 1 \\ f \times f = 1 \\ e \times m = m \times e = f \end{cases} .$$

Example 2.2: Group rings as fusion rings

If G is a finite group, then the group multiplication gives fusion rules

$$\begin{cases} g \times h = gh \end{cases} .$$

Example 2.3: Fusion ring of Rep(G)

If G is a finite group, we've seen that the decomposition of the tensor product of irreps into irreps gives fusion rules. For example, for S_3 we have:

$$\begin{cases} \mathrm{sgn} \times \mathrm{sgn} = 1 \\ \mathrm{sgn} \times \mathrm{std} = \mathrm{std} \times \mathrm{sgn} = \mathrm{std} \\ \mathrm{std} \times \mathrm{std} = 1 + \mathrm{sgn} + \mathrm{std} \end{cases}$$

Remark 6. *(Feel free to ignore this for now if it doesn't make sense) We say that a fusion rule categorifies if there is a fusion category whose (isomorphism classes of) simple objects fuse like the elements of the label set L. We'll be able to make this more precise once we've given a careful definition of a fusion category.*

*The group fusion rules categorify to the fusion categories Vec(G) called G-graded vector spaces. And, as you already know, the Rep(G) fusion rules categorify to **Rep**(G).* [12] *We'll see Vec(G) and Rep(G) are definitely non-isomorphic fusion rings: the former has invertible fusion rules and the latter does not. However, they are Morita equivalent. It will be a while before we have built up enough theory to explain what that means. For now, what this means for us is that if we were to try to build a lattice Hamiltonian using the quantum degrees of freedom afforded by these categories, we would actually get the same topological phase of matter. It will also mean that both **Vec**(G) and **Rep**(G) will generate the same Turaev-Viro-Barrett-Westbury TQFT.*

[12] Both these fusion rules admit some "twisted" categorifications, as you'll show in the exercises for Unit 3.

We will talk about this more at length next time, but there are four different – but not terribly different – ways to communicate the data of a fusion ring: by listing the fusion coefficients, by listing the fusion matrices, by giving the fusion table, or by listing the fusion rules.

Definition 2.1

Given fusion rules, the *fusion matrices* N_a for $a \in L$ are the matrices defined by

$$(N_a)_{bc} = N_c^{ab}.$$

The smallest example of a fusion ring that does not come from finite groups in one of the two ways mentioned above is called the *Fibonacci fusion ring*. We will give the fusion matrices and fusion table along with fusion rules in the next few examples so you can get the hang of the different (but admittedly very closely related) ways to encode fusion ring data.

Example 2.4: Fibonacci fusion rules

Fusion matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Fusion rules

$$\begin{cases} \tau \times \tau = 1 + \tau \end{cases}$$

Fusion table

	1	τ
1	1	τ
τ	τ	$1+\tau$

These fusion rules come from anyon models in that setting τ is called a Fibonnaci anoyn.

Example 2.5: Ising fusion rules

It is standard to write the basis of the Ising fusion ring as $L = \{1, \sigma, \psi\}$.

Fusion matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Fusion rules

$$\begin{cases} \sigma \times \sigma = 1 + \psi \\ \sigma \times \psi = \psi \times \sigma = \sigma \\ \psi \times \psi = 1 \end{cases}$$

Fusion table

	1	σ	ψ
1	1	σ	ψ
σ	σ	$1+\psi$	σ
ψ	ψ	σ	1

These fusion rules category to anyon models. We call ψ the Majorana fermion and σ the Ising anyon.

Fusion rings

Fusion rules on a fixed label set L have more going on than just the equation above, they form a special kind of ring called a *fusion ring*. So let's very briefly recall what a ring is.

Definition 2.2

A ring $(R, +, \cdot)$ is a set R with two binary operations called addition and multiplication such that

1. $(R, +)$ is an abelian group
2. the multiplication $\cdot$ is associative
3. the left and right distributive properties hold:

$$r \cdot (s + t) = r \cdot s + r \cdot t$$
$$(r + s) \cdot t = r \cdot t + s \cdot t$$

All the rings we care about will be *unital* rings: there exists a multiplicative identity $1 \in R$ with $1 \cdot r = r \cdot 1 = r$ for all $r \in R$. When we say ring from here on out we mean a unital ring. You should be able to guess what a ring homomomorphism and ring isomorphism must be.

Definition 2.3

A *fusion ring* is a unital ring $(F, +, \times)$ which is free as a $\mathbb{Z}$-module with finite basis[a] $L \subset F$ together with an involution $\star : L \to L$ called *duality* that lifts to an anti-involution on F satisfying

- $1 \in L$ and $1^* = 1$,
- $a \otimes b = \sum_{c \in L} N_{ab}^{c} c$ with $N_{ab}^{c} \in \mathbb{Z}_{\geq 0}$ for all $a, b \in L$
- $N_{ab}^{1} = N_{ba}^{1} = \delta_{a^* b}$ for all $a, b \in L$.

[a]"Free as a $\mathbb{Z}$-module over L just means that every element in F can be written as a finite integer linear combination of elements of L

One can show that there are some additional constraints on the N_c^{ab} that follow from this definition, sometimes called *Frobenius reciprocity*:

$$N_c^{ab} = N_b^{a^* c} = N_a^{cb^*}.$$

Other similar identities follow from these, and we will give a conceptual derivation of these in the next lecture.

From now on when we want to talk about a fusion ring we will just specify the label set L and the fusion coefficients N_c^{ab} for $a, b, c \in L$.

It should be clear that commutativity of the fusion rules (the multiplication among basis elements) implies commutativity of the full fusion ring. We care about both commutative and noncommutative fusion rings.

Remark 7. *Fusion rings that describe anyons are always commutative.*

We haven't seen an example of noncommutative fusion ring with a multi-fusion channel yet. You'll explore one in the Unit 2 Exercises called the Haagerup fusion category, sometimes denoted $\mathcal{H}_3$.

Definition 2.4

The *rank* of a fusion ring is the number of elements in the label set L.

There is some additional terminology we should introduce. A fusion ring is called *multiplicity-free* if $N_c^{ab} \in \{0, 1\}$ for all $a, b, c \in L$. Otherwise we say the fusion ring *has multiplicity*. The set (or you could take the multi-set) of nonzero N_c^{ab} as you range over $a, b, c \in L$ is sometimes called its *multiplicities*. An element of a fusion ring $a \in L$ is called *self-dual* if $a^* = a$, i.e. if it is fixed by the duality involution on L. Otherwise a is called *nonself-dual*.

Since the fusion matrices are nonnegative integer matrices there is a nice theorem that applies from linear algebra called the Frobenius-Perron theorem. We will not state it very carefully here, and in fact it really requires slightly stronger hypotheses than are given below.

Theorem 2.1

Let A be a matrix with nonnegative integer entries. Then A has a positive real eigenvalue λ which is larger than all of its other eigenvalues.

This largest eigenvalue of a nonnegative integer matrix A is called its Frobenius-Perron eigenvalue.

Definition 2.5

The Frobenius-Perron dimension aka FP dimension of $a \in L$ is the Frobenius-Perron eigenvalue of N_a.

Note that $\dim(1) = 1$ in any fusion ring.

Example 2.6: I

ing has rank 3 and Frobenius-Perron dimensions $1, \sqrt{2}, 1$.

Definition 2.6: A

isomorphism of fusion rings $(F, L, \times)$ and $(\tilde{F}, \tilde{L}, \tilde{\times})$ is a bijection of $\phi : L \to \tilde{L}$ such that

$$N_{\phi(c)}^{\phi(a)\phi(b)} = N_c^{ab}.$$

A fusion ring isomorphism is basically a relabeling of the label set that preserves the fusion rules.

Example 2.7

When G is a finite nonabelian group, the **Vec**(G) and **Rep**(G) fusion rings we saw earlier are nonisomorphic.

The rank, multiplicities, and FP dimensions are all fusion ring invariants. So is e.g. the number of self-dual elements.

Remark 8. *The assignment of FP dimensions to elements of a label set is a ring homomorphism from the fusion ring to* $(\mathbb{C}, +, \cdot)$.

This fact can be used to turn the linear algebra problem of computing the eigenvalues of the fusion matrices into the problem of solving a system of quadratic equations in several variables. Sometimes the latter is easy enough to solve by inspection for small examples, and this is typically how folks read off the dimensions if they aren't already listed. Let's see an example:

Example 2.8

Recall the $\frac{1}{2}E_6$ fusion rules

$$\begin{cases} x^2 = 1 + 2x + y \\ xy = yx = x \\ y^2 = 1 \end{cases}$$

and put d_x, d_y for the FP dimensions of x and y. Then they satisfy

$$\begin{cases} d_x^2 = 1 + 2d_x + d_y \\ d_x d_y = d_y d_x = d_x \\ d_y^2 = 1 \end{cases}$$

The last equation tells us $d_y = 1$, and then the first equation tells us that

$$d_x^2 - 2d_x - 2 = 0.$$

The positive solution is $d_x = 1 + \sqrt{3}$.
For most of us this is faster than factoring the characteristic polynomial of the fusion matrix

$$N_x = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

but to each their own!

Next time we'll see lots more examples, and talk about the classification of fusion rings, start drawing some pictures to help us visualize fusion rules, and talk about fusion rings as they appear in anyon models.