Lecture 7

Recap

Here is an example with some multiplicity.

Example 2.1: $\frac{1}{2}E_6$ fusion rules

The label set $L = \{1, x, y\}$ can be equipped with the fusion rules

$$\begin{cases} x^2 = 1 + 2x + y \\ xy = yx = x \\ y^2 = 1 \end{cases}.$$

More Examples

It should be clear how to translate back and forth between the various data structures that encode fusion rules. When we're reading and writing (as opposed to computing), the best way to communicate them is to just list a generating set of fusion rules, like we do in the examples in the rest of this section.

Example 2.2: Tambara-Yamagami fusion rules

Let G be a finite group. Define $L = G \sqcup \{m\}$. Then there is a fusion ring on the label set L with fusion rules

$$\begin{cases} g \times h = gh & \text{for } g, h \in G \\ g \times m = m \times g = m & \text{for } g \in G \\ m \times m = \sum_{g \in G} g \end{cases}$$

A Tambara-Yamagami fusion ring has rank $|G| + 1$. Each $g \in L$ has FP dimension equal to 1, and the FP dimension of m is $\sqrt{|G|}$.

We will see that these fusion rules only categorify to a fusion category when G is abelian, and that they only categorify to a modular tensor category when $G = \mathbb{Z}_2$, which gives the Ising fusion rules we saw earlier.

Example 2.3: $\mathrm{Rep}(H)$ fusion rules

If H is a finite-dimensional Hopf algebra, then the decomposition of the tensor product of irreps of H gives fusion rules just like for $\mathrm{Rep}(G)$ when G was a finite group. We won't discuss any generalities here, but we will see an example of this towards the end of this lecture.

Not all fusion rings are categorifiable, i.e. not all fusion rings come from a fusion category. The smallest example of a fusion ring which doesn't come from a fusion category is rank 4:

Example 2.4

Let $L = \{1, X, Y, Z\}$ and consider the fusion rules

$$\begin{cases} XY = YX = 1 \\ X^2 = Y \\ Y^2 = X \\ XZ = ZX = Z \\ YZ = ZY = Z \\ Z^2 = 1 + X + Y + Z \end{cases} .$$

We claim but do not justify that the associated fusion ring does not categorify to a fusion category.

Even though the failure of the ring above to category is some pure algebra fact, it does tell us something about physics. Notice that it is commutative, so you might wonder whether there is some topological phase whose anyons fuse like X, Y, Z as above. But the failure of the ring to categorify tells you there is no such theory of anyons.

But we do know that if a fusion ring does categorify, it admits finitely many fusion categorifications. This result is colloquially referred to as *Ocneanu rigidity*.

Theorem 2.1: Ocneanu rigidity

There are only finitely many fusion categories with the same Grothendieck (fusion) ring.

Remark 10. *We haven't talked yet about "the fusion ring of a fusion category", since we're working on building up to fusion categories through fusion rings. But if one has a fusion category – whatever that is – you can take equivalence classes of simple objects under isomorphism and the tensor product $\otimes$ and direct sum $\oplus$ of objects in the category will induce a multiplication and addition on these equivalence classes. These give the label set and fusion ring operations.*

Computational approach to fusion rings

We saw before that there are several different ways to encode the data of a fusion ring. Here let's think of a fusion ring symbolically via its fusion coefficients $\{N_c^{ab}\}_{a,b,c\in L}$. These variables satisfy the axioms of a fusion ring if they satisfy

1. $N_c^{ab} \in \mathbb{Z}_{\geq 0}$ (nonnegativity)
2. $N_c^{a1} = \delta_{c,a}, N_c^{1b} = \delta_{c,b}$ (unitality)
3. $\sum_i N_i^{ab} N_d^{ic} = \sum_j N_d^{aj} N_j^{bc}$ (associativity)
4. $N_1^{ab} = \delta_{b,a^*}$ (duality)

5. $N_c^{ab} = N_b^{a^*c} = N_a^{cb^*}$. (duality)

In other words, we can think of a fusion ring as a collection of nonnegative integers satisfying a bunch of equations. Then we can ask computers to help us classify them by looking for solutions to these equations. This brute-force approach is especially helpful in understanding the landscape of small fusion rings without multiplicity.

Remember how we looked at a table of the classification of finite groups of small order? This list classifying fusion rings by rank is very analogous: `https://anyonwiki.github.io/pages/Lists/losmffr.html`

Theorem 2.2

Multiplicity-free fusion rings are classified up to rank 10.

Theorem 2.3

The fusion rings of modular tensor categories are known up to rank 12.

Next we're going to draw some pictures that will help us visualize the fusion rules. We'll use the language of (quasi-)particles to interpret these pictures, even thought strictly speaking they can't be interpreted that way unless the pictures correspond to a fusion category which is also unitary and modular.

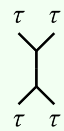
Admissibly labeled trivalent graphs

Fix $\{N_c^{ab}\}_{a,b,c\in L}$ a fusion ring. We say that a *trivalent vertex* with edges colored by elements of L is *admissibly labeled* if $N_c^{ab} \neq 0$.

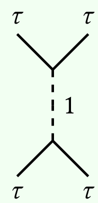
We can extend this notion of admissible labeling to graphs built by gluing trivalent verticse together; it is best to see this with an example.

Example 2.5

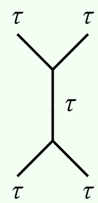
Consider the Fibonacci fusion rule $\tau \times \tau = 1 + \tau$ on the label set $L = \{1, \tau\}$. If we fix the *external edge labels* of the following diagram to be all τ,



then there are multiple ways to color the *internal edge* with elements of L so that every trivalent vertex in the diagram is admissibly labeled, namely



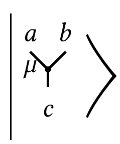
and



Notice we have drawn the edge colored by the trivial label $1 \in L$ with a dashed line. In the Fibonacci anyon model the label 1 corresponds to the vacuum anyon, and an edge labeled by 1 is a *vaccuum line.*

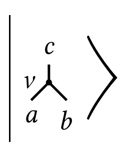
These kinds of pictures (admissibly labeled "trivalent" graphs) will be morphisms in a fusion category. We'll discuss this at much greater length in a week or two, but for now we will introduce the *fusion spaces* V_c^{ab} and *splitting spaces* V_{ab}^c.

Suppose $N_c^{ab} \neq 0$. Then these trivalent vertices define vectors



where $\mu = 1, 2, \ldots N_c^{ab}$. Take V_c^{ab} to be the complex vector space spanned by these vectors, so that $\dim(V_c^{ab}) = N^{ab}$. For now this is just a vector space but we will make it a Hilbert space later.

Similarly define vectors

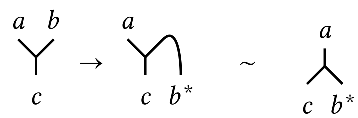


for $\nu = 1, 2, \ldots N_c^{ab}$.and use them as a basis for a $\mathbb{C}$-vector space V_{ab}^c.

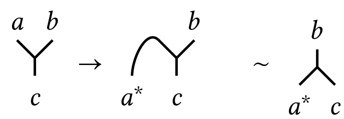
Later we will build a category C with objects $a, b, c \in C$ and morphisms $\mathrm{Hom}_C(a \otimes b, c) = V_c^{ab}$, $\mathrm{Hom}_C(c, a \otimes b) = V_{ab}^c$. Of course those are far from the only kinds of morphisms we care about, so there's more category to build. Notice that we are building Hom-sets which will have the structure of vector spaces, so we will call them Hom-spaces.

How to remember the Frobenius-reciprocity identities

For now you should this of the sequences of pictures below as a trick for remembering the identities that exhibit symmetries of the fusion coefficients with respect to duality.



"Bending" down one of the "arms" of the graph results in another admissibly labeled diagram; we are starting with a vector in V_c^{ab} and getting a vector in $V_{c^*b}^{a}$. We'll talk about what's really happening categorically later.



Similarly, N_c^{ab} is an admissible labeling if and only if $N_b^{a^*c}$ is and admissible labeling.