Lecture 8: Kitaev's Quantum Double Model (Part I)

We follow Shawn Cui's notes quite closely and borrow much of his notation.

In the next two lectures we'll introduce Kitaev's quantum double model.

Kitaev Quantum Double Model

The construction should feel very analogous to our procedure for constructing the $\mathbb{Z}_2$ toric code (and you'll see it is a direct generalization). The discussion will follow the same flow: we'll fix a lattice, decorate its edges with some quantum degrees of freedom, write down some local operators and a Hamiltonian, (and then in the next lecture, hopefully) understand the ground states and explore the excited states.

The quantum double of a finite group G

But before we present the lattice model we should get acquainted with the *quantum double algebra of a finite group DG* so that we recognize it when we see it later. Recall that an *algebra* over $\mathbb{C}$ is a vector space that is also a ring.

Definition 2.1: Group algebra

The group algebra $\mathbb{C}[G]$ is the complex vector space with basis $|g\rangle$, $g \in G$ and multiplication on basis elements induced by group multiplication $|g\rangle \cdot |h\rangle = |gh\rangle$. Multiplication on all of $\mathbb{C}[G]$ comes from extending the multiplication to arbitrary linear combinations of basis elements. The multiplicative unit is $|e\rangle$, where e is the identity element of G.

Definition 2.2: Dual of group algebra

he algebra of linear functions of a finite group $\widehat{\mathbb{C}[G]}$ is the complex vector space with indicator function basis $|I_g\rangle$, $g \in G$ with multiplication of basis elements given by $|I_g\rangle \cdot |I_h\rangle = \delta_{g,h}|I_g\rangle$, where δ is the Kronecker delta function.
By indicator function, we mean the function

$$I_g : G \to \mathbb{C}$$
$$h \mapsto \begin{cases} 1 & h = g \\ 0 & h \neq g \end{cases}$$

The multiplicative unit is $\sum_g |I_g\rangle$.

The algebra of functions on G is the linear dual of the group algebra, hence the notation.

Both the group algebra and its dual are *Hopf algebras*, which means they have more going on than just an associative multiplication and unit (they also have a compatible *comultiplication* and *counit*, as well as a special map called an *antipode*. [13] But for now we will be able to keep busy with just the algebra structure.

We build a "doubled" algebra that has a copy of each of $\mathbb{C}[G]$ and $\widehat{\mathbb{C}[G]}$ sitting inside of it. As a vector space, $DG = \widehat{\mathbb{C}[G]} \otimes \mathbb{C}[G]$. We introduce new notation for the basis vectors and put $D_{h,g} = |I_h\rangle \otimes |g\rangle$.

$$\begin{cases} D_{(e,g_1)} D_{(e,g_2)} = D_{(e,g_1 g_2)} \\ D_{(h_1,e)} D_{(h_2,e)} = \delta_{h_1,h_2} D_{(e,h_1)} \\ D_{(e,g)} D_{(h,e)} = D_{(ghg^{-1},e)} D_{(e,g)} \end{cases}$$

The first two equations are just the multiplication of the group algebra and its dual. The third will look ad hoc for now; we'll just take it as given. There are also two expressions for the multiplicative identity in DG, namely $1 = D_{(e,e)}$ and $1 = \sum_{h \in G} D_{(h,e)}$.

Just like $\mathbb{C}[G]$ and its dual, DG is a Hopf algebra and has additional structure that we won't worry about for now.

State space

For simplicitly we'll build our system on a square lattice $\mathcal{L}$, but the lattice can have completely general connectivity and it should be clear at every step how our discussion generalizes to an arbitrary lattice. We won't put any boundary conditions on our lattice just yet, but if you want you can think of it as living on the sphere S^2, so that there is no boundary.

As with the toric code, we denote by V, E, and F the set of vertices, edges, and faces of our lattice. A deviation from the toric code is that we will need to assign an orientation to each $e \in E$; while this orientation is arbitrary we'll see it is also necessary. We will also introduce a new set of *sites* S consisting of a plaquette and a bounding vertex, that is, $S = \{(p, v) \mid p \in F, v \in \partial p\}$.

At every edge $e \in E$, we put a $|G|$-dimensional qudit $\mathcal{H}_e = \mathbb{C}[G]$ with computational basis $|g\rangle$ for $g \in G$. The total Hilbert space of states is $\bigotimes_{e \in E} \mathcal{H}_e$ and has $\dim(\mathcal{H}_{total}) = |G|^{|E|}$. We will work with the induced basis of g-bit strings of the form $|g_{e_1} g_{e_2} \cdots g_{|E|}\rangle = |g_{e_1}\rangle \otimes |g_{e_2}\rangle \otimes \cdots \otimes |g_{|E|}\rangle\rangle$.

We can visualize a basis state as a coloring of the edges of $\mathcal{L}$ by elements of G.

Local operators

For every $g \in G$, there is an operator $A_g(v)$ that acts on a g-bit string as follows. For each $e_i \in \text{star}(v)$, multiply $|g_{e_i}\rangle$ by $|g\rangle$ on the left if the edge e_i is pointed away from v, and multiply $|g_{e_i}\rangle$ by $|g^{-1}\rangle$ on the right if the edge is oriented pointing towards v.

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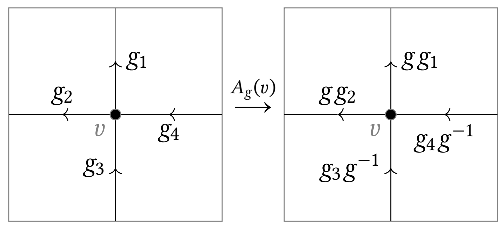
[13] We absolutely care about Hopf algebras in this subject: Kitaev's quantum double model can be even further generalized beyond what we're currently discussing, and instead of building a lattice Hamiltonian using a finite group one can use a finite-dimensional Hopf algebra H, and it will give rise to a topological phase where the anyons fuse like irreps in Rep(DH), where DH is something more general than DG.



$$|\cdots g_1 g_2 g_3 g_4 \cdots\rangle \mapsto |\cdots g g_1\ g g_2\ g_3 g^{-1}\ g_4 g^{-1} \cdots\rangle$$

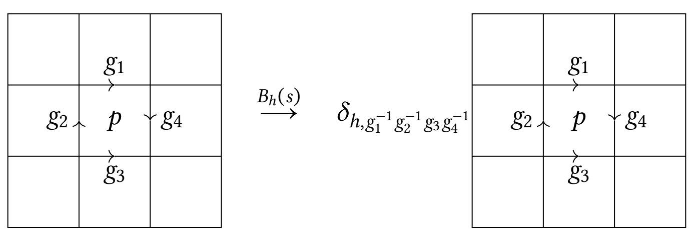
Note that sequential application of these "gauge transformation" operators for different group elements is the same as applying a single operator with the product of those group elements: $A_{g_1} A_{g_2} = A_{g_1 g_2}$.

Next we define a vertex operator by averaging over all the A_g:

$$A(s) := A(v) = \frac{1}{|G|} \sum_g A_g(v).$$

You will check it is a projector in an exercise.

We also have plaquette operators $B_h(s)$ for each $h \in G$ which projects on the states with "flux" h through a plaquette with respect to a vertex, where the flux is the product of the group elements (or their inverses) on the boundary of the plaquette as we traverse its edge counterclockwise starting at v. If an edge e_i is oriented counterclockwise we pick up a g_i, and if it's oriented clockwise we pick up a g_i^{-1}:



It is immediate that $B_h(s)$ is a projector, and that they satisfy $B_{h_1}(s) B_{h_2}(s) = \delta_{h_1,h_2} B_{h_1}(s)$.

Note that when $h = e$, $B(s)$ only depends on p and not v – it doesn't matter what vertex we start at because the factors in the left hand side of the equation $g_1^{(o(e_1))} g_2^{(o(e_2))} g_3^{(o(e_3))} g_4^{(o(e_4))} = e$ can be cyclically permuted.[14]

In the exercises you will check that $A_g(s)$ and $B_h(s)$ do not commute, but satisfy $A_g(s) B_h(s) = B_{ghg^{-1}}(s) A_g(v)$. This relation should look familiar.

We've shown that the A_g and B_h satisfy the same relations as the generators of the quantum double algebra. This says that the algebra of local operators at a site $s = (p, v)$ generated by the A_g and B_h form an N-dimensional matrix representation of the quantum double algebra DG, where $N = |G|^{|E|}$, via the assignment

[14] As was foretold, our notation is overloaded: note that e is being used to denote an edge $e \in E$ as well as the identity element $e \in G$.

$$\begin{aligned}\rho_s : DG &\to GL(N, \mathbb{C})\\ D_{e,g} &\mapsto A_g(v)\\ D_{h,e} &\mapsto B_h(p).\end{aligned}$$

Hamiltonian

Putting all the pieces together, we have the quantum double Hamiltonian

$$H = -\sum_{v \in V} A(v) - \sum_{p \in F} B(p).$$