Lecture 9: Kitaev's Quantum Double Model (Part II)

As in the previous lecture we follow Shawn Cui's notes quite closely and borrow much of his notation.

Recall the definition of the quantum double Hamiltonian from last lecture

$$H = -\sum_{v \in V} A(v) - \sum_{p \in F} B(p)$$

where the vertex operator $A(v) = \frac{1}{|G|} A_g(v)$ is the group average of the vertex gauge transformations at v and $B(p) = B_e(p)$ projects onto states with zero flux through the plaquette p. Recall that the algebra of local operators at a site $s = (p, v)$ generated by the A_g and B_h form an N-dimensional matrix representation of the quantum double algebra DG, where $N = |G|^{|E|}$, via the assignment

$$\begin{aligned} \rho_s \,:\, DG &\to GL(N, \mathbb{C}) \\ D_{e,g} &\mapsto A_g(v) \\ D_{h,e} &\mapsto B_h(p). \end{aligned}$$

In order to give a careful derivation of the excitations in this model we would need to understand violations of the terms in teh Hamiltonian and introduce ribbon operators which are capable of creating pairs of excitations from a ground state. Since we won't have time to go through that carefully in class, we will have to get to the punchline.

Kitaev tells us (on page 18 of the 2003 Annals of Physics aricle *Fault-tolerant quantum computation by anyons*) that we can interpret the irreducible representations of DG as the particle types that will emerge. Recall from last lecture that the irreps of DG are counted by pairs (C, χ) where C is a conjugacy class of G and χ is an irrep of $Z(g)$ for a representative $g \in G$. For now, we'll just have to take this on faith. But we'll see that this assertion is corroborated by the calculation of the ground state degeneracy of the model on the torus, which will be the main focus of this lecture.

Let's proceed assuming that we've established that H is a commuting projector Hamiltonian (verifying some of the details will be deferred to exercises) and hence the ground state space is given by

$$\mathcal{H}_{ground} = \{|\psi\rangle \in \mathcal{H}_{total} \,|\, A(v)|\psi\rangle = |\psi\rangle, B(p)|\psi\rangle = |\psi\rangle \text{ for all } v \in V, p \in F\}.$$

In fact, something stronger is true which is not completely obvious at first: eigenstates of $A(v)$ give eigenstates of $A_g(V)$ for all g. This comes from the fact that $A_g(v)A(v) = A(v)$ for all g.[15]

Then if $A(v)|\psi\rangle = \lambda|\psi\rangle$, on the one hand we have

$$A_g(v)A(v)|\psi\rangle = A_g\lambda|\psi\rangle$$

[15] This happens because a group G always acts freely on itself by left multiplication.

and on the other hand this is equal to

$$A(v)|\psi\rangle = \lambda|\psi\rangle,$$

which shows that $\psi\rangle$ is an eigenstate of A_g with eigenvalue 1 for any $g \in G$.

Therefore we also have

$$\mathcal{H}_{ground} = \{|\psi\rangle \in \mathcal{H}_{total} \,|\, A_g(v)|\psi\rangle = |\psi\rangle, B(p)|\psi\rangle = |\psi\rangle \text{ for all } v \in V, g \in G, p \in F\}.$$

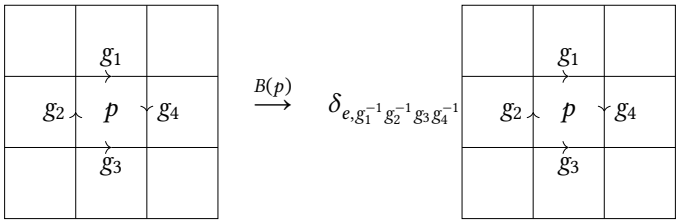
Before digging into $\mathcal{H}_{ground}$, we can make one observation. Recall from the previous lecture that the algebra of local operators at a site generated by the $A_g(s)$ and $B_h(s)$ define a representation of the quantum double algebra DG on $\mathcal{H}_{total}$. If we restrict this representation to $\mathcal{H}_{ground}$ it becomes a trivial representation.

To build an understanding of what ground states look like we must understand what it means for a state to be a +1 eigenstate of $A_g(v)$ and $B(p)$ for fixed v and p. To emphasize the analogy of this analyze with our treatment of the toric code, we'll start with the plaquette operators.

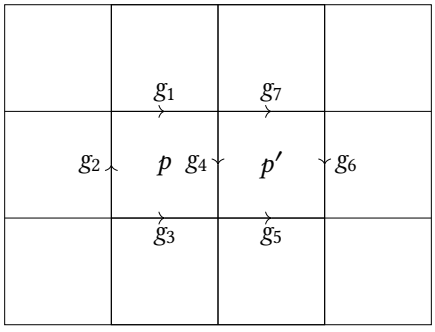
Action of plaquette operators on basis states

From the definition of $B(p) = B_e(s)$ we see that a basis state $|\cdots g_{e_1} g_{e_2} g_{e_3} g_{e_4} \cdots\rangle$ is a +1 eigenstate of $B(p)$ if $g_{e_1}^{o(e_1)} g_{e_2}^{o(e_2)} g_{e_3}^{o(e_3)} g_{e_4}^{o(e_4)} = e \in G$, where $o(e_i) \in \{\pm 1\}$ is the orientation of the edge e_i.

For example,



Now consider two neighboring plaquettes p and p' which share an edge. Locally a basis state looks like the following picture, where we've chosen an indexing so that $e_4 \in \partial p \cap \partial p'$.



If a state is a +1 eigenstate of both $B(p)$ and $B(p')$, then

$$g_{e_1}^{o(e_1)} g_{e_2}^{o(e_2)} g_{e_3}^{o(e_3)} g_{e_4}^{o(e_4)} = e \text{ and } g_{e_4}^{-o(e_4)} g_{e_5}^{o(e_5)} g_{e_6}^{o(e_6)} g_{e_7}^{o(e_7)} = e.$$

Note that the orientation of the edge e_4 is reversed from its orientation in p', since it will be oriented clockwise in one plaquette and counterclockwise in the other.

Combining these equations we get

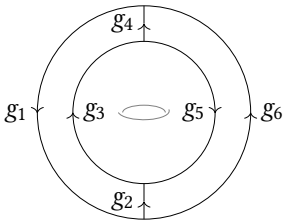
$$g_{e_1}^{o(e_1)} g_{e_2}^{o(e_2)} g_{e_3}^{o(e_3)} g_{e_5}^{o(e_5)} g_{e_6}^{o(e_6)} g_{e_7}^{o(e_7)} = e,$$

which says that the total flux through the cell $p \cup p'$ is zero.

This pattern will continue as we keep adding neighboring plaquettes, provided that there's no topology. Stated differently, a basis state $|\psi\rangle$ will be a $+1$ eigenstate of all $B(p)$ simultaneously if the flux through every contractible loop on $\mathcal{L}$ vanishes.

Here's an example that shows that the flux being zero through contractible loops does not imply that the non-contractible loops is trivial.

Imagine that our neighboring plaquettes encircle a handle.

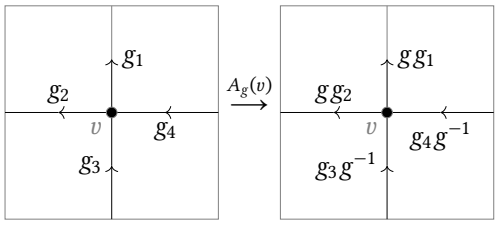


The zero flux condition around the left plaquette enforces $g_1 g_2 g_3 g_4 = e$ and $g_2^{-1} g_6 g_4^{-1} g_5 = e$. Now if we look at the flux through a noncontractible loop like the ones formed by the edges e_1 and e_6 or g_3 and g_5 we see that they can have nonzero flux. For example, $g_6 g_1 = g_2 g_5^{-1} g_3^{-1} g_2^{-1}$, and we cannot in general conclude that the right hand side is trivial.

To summarize so far, we've established that a g-bit string basis states are $+1$ eigenstates of $B_e(p)$ for all p if the coloring of the edges of $\mathcal{L}$ by elements of G conspire so that every plaquette has zero flux. When this happens all contractible closed loops on the lattice will have zero flux.

Action of vertex operators on basis states

Recall that $A_g(v)$ acts on a basis state $|\cdots g_{e_1} g_{e_2} g_{e_3} g_{e_4} \cdots\rangle$ by multiplying each group element on the left by g if $o_v(e_i) = 1$ and on the right by g^{-1} if $o_v(e_i) = -1$, like we saw with the example



$$|\cdots g_1 g_2 g_3 g_4 \cdots\rangle \mapsto |\cdots g\,g_1\; g\,g_2\; g_3 g^{-1}\; g_4 g^{-1} \cdots\rangle$$

So certainly our basis states are not generally eigenstates of the $A_g(v)$ for a fixed g individually. But if we take an equal weight superposition of states which can be related by the action

of $A_g(v)$ operators for some collection of vertices such a state would be invariant under the action of $A_g(v)$ for all $v \in V$.

Just like with the toric code, we want to define an equivalence relation $\sim$ (on the "zero flux" +1 eigenstates of $B_e(p)$) where we identity $|\psi\rangle \sim |\psi'\rangle$ if $|\psi'\rangle$ if they are equal up to the application of a vertex gauge transformations A_g at various vertices on the lattice. Then a ground state is (up to a normalization), given by

$$|[\psi]\rangle = \sum_{|\psi'\rangle \sim |\psi\rangle} |\psi'\rangle.$$

Dimension of ground state space

But how many of these ground states are there? This depends on the topology of the surface where our lattice lives, which we've been agnostic about up until now. We'll state the result and give a very rough idea of the form that it takes.

Theorem 2.1

The ground state degeneracy of the Kitaev quantum double model for a finite group G is given by

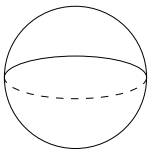
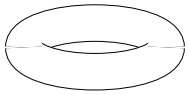
$$\dim\left(\mathcal{H}(\Sigma_g)\right) = \#Hom(\pi(\Sigma_g), G)\text{up to the conjugation action by } G$$

We will explain where the conjugation action comes in when we hone in on $\Sigma_1 = T^2$ shortly, but we won't give further details here. Instead we'll try to give some indications *why* $\pi_1(\Sigma_g)$ and its maps to G are relevant for this calculation.

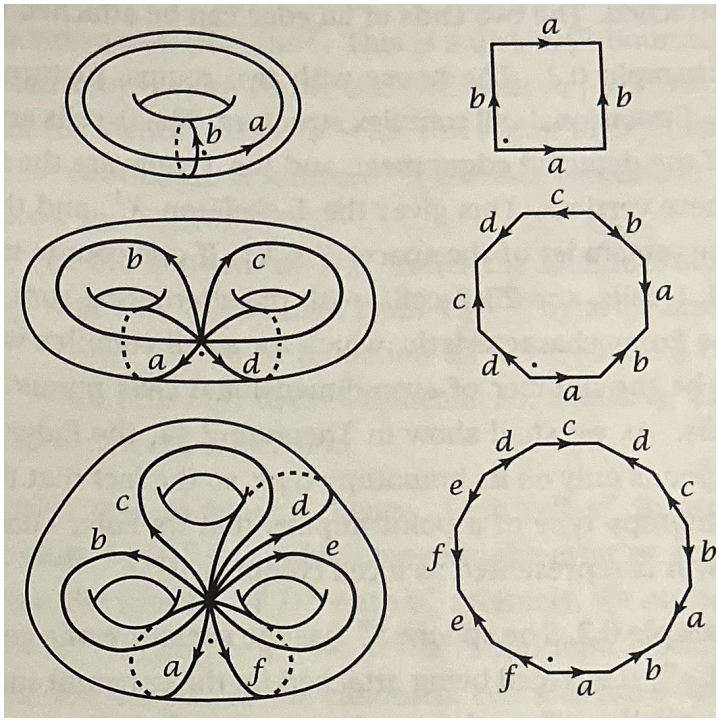
Towards this end, we've talked about "putting your lattice on a higher genus surface," but how do you actually do this?

Topology Interlude

The classification of closed, oriented surfaces from topology tells us what space manifolds we can put our theory on. Our choices are control by genus, and there is a genus g surface Σ_g for each genus.

$\Sigma_0 = S^2 =$  $\Sigma_1 = T^2 =$  $\Sigma_2 = \ldots$ $\Sigma_3 = \ldots$

The easiest way to do "put your theory on Σ_g" is by using a *fundamental domain* of a genus g surface. The easiest way to describe them is as a regular $4g$-gon with sides identified in adjacent groups of four, creating one genus at a time by gluing the edges in an alternating pattern. We're already comfortable with the $g = 1$ case. When $g = 2$ and $g = 3$, we glue up an octagon by identifying the sides like indicated below in this figure I clipped from Hatcher's *Algebraic Topology* `https://pi.math.cornell.edu/hatcher/AT/AT+.pdf`.



Here's a really nice Math Stack Exchange post where people have shared sequences of pictures to help you visualize the $g = 2$ case:

`https://math.stackexchange.com/questions/479371/how-to-construct-a-genus-2-surface-from-8-gon`

So if you put your lattice on one of these polygons with the boundary edges identified in the right manner you'll have successful put your theory on a Σ_g.

Fundamental group

We will focus on giving a high level understanding of the fundamental group of a surface instead of giving the full details of its definition.

We can think of the fundamental group as a functorial assignment of groups to topological spaces (with basepoint):

$$\begin{aligned} \pi_1 \,:\, \mathbf{Top} &\to \mathbf{Grp} \\ X &\mapsto \pi_1(X) \\ X \xrightarrow{f} Y &\mapsto \pi_1(X) \xrightarrow{\pi_1(f)} \pi_1(Y) \end{aligned}$$

In the exercises you showed that isomorphisms are preserved under functors, so you showed that if $X \simeq Y$ are homeomorphic topological spaces, then $\pi_1(X) \cong \pi_1(Y)$ as groups. In other words, $\pi_1(X)$ is an invariant of topological spaces, and surfaces in particular.

It is okay to just think about the fundamental group of a surface as some group we assign to it that "counts the number of holes", but it will help to have at least a rough idea of the definition.

Definition 2.1

The fundamental group $\pi_1(X, x)$ of a space X with respect to a basepoint x is the set of homotopy classes of loops based at x with multiplication inherited from concatenation of paths.

If X is path connected, like when $X = \Sigma_g$, changing the basepoint from x to $\tilde{x}$ gives an isomorphic group $\pi_1(X, \tilde{x})$ which is related to $\pi_1(X, x)$ by conjugation with a path p from x to $\tilde{x}$.

Since every closed loop on the sphere S^2 is contractible, $\pi_1(S^2) = 1$. A presentation of the fundamental group of a genus g surface for $g > 1$ can be remembered using the same alternating pattern used to glue up their fundamental domains,

$$\pi_1(\Sigma_g) = \langle a_1, a_2, \dots, a_g, b_1, b_2, \dots b_g \,|\, [a_i, g_i] = e \text{ for } i = 1, 2, \dots g \rangle.$$

Intuition behind general formula for ground state degeneracy

Returning to the general formula for the ground state degeneracy on a genus g surface, we know explain why our lattice model is a natural source of maps $\pi_1(\Sigma_g) \to G$. Suppose that our lattice $\mathbb{L}$ lives on Σ_g, and consider a path $l : I \to \Sigma_g$ with $l(0) = x$ and $l(1) = x$, i.e. a closed loop. We can always use homotopy to pick an equivalent path that lives directly on the lattice $\mathbb{L}$ and do a basepoint change to ensure that l has a vertex v as basepoint. As we traverse l starting at time 0, we will encounter a sequence of edges, $e_1, e_2, \dots, e_m$. Given a basis state of $\mathcal{H}_{total}$, which we can think of an a colored of the edges E of $\mathcal{L}$ by elements of G,

Then given a loop l, we assign it the group element given by the product of all the group elements coloring the edges in the loop, $g_1, g_2, \dots g_m$, multiplying by their inverse when the orientation of the edge is against the direction we traverse l.

This map will induce a well-defined isomorphism on homotopy classes precisely for the ground states, but that takes a bit more work to show than we have time to cover. Instead, we will compute the ground state degeneracy explicitly when $g = 1$.

Ground state degeneracy on the torus

We already know from our discussion of the toric code how to put boundary conditions on $\mathcal{L}$ to put our system on the torus. In this case, $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$ and the ground state degeneracy is counted by homomorphisms $\phi : \mathbb{Z} \times \mathbb{Z} \to G$ (up to the conjugation action of G).

If an assignment ϕ of group elements to generators

$$\begin{aligned} \phi &: \mathbb{Z} \times \mathbb{Z} \to G \\ (1, 0) &\mapsto g \\ (0, 1) &\mapsto h \end{aligned}$$

is a homomorphism then $\phi((1,1)) = \phi((1,0))\phi((0,1)) = gh$ and $\phi((1,1)) = \phi((0,1))\phi((1,0)) = hg$, i.e. g and h commute. There are no other relations in $\mathbb{Z} \times \mathbb{Z}$ besides commutativity – it is the *free abelian group on two generators*, so every pair of commuting elements $g, h \in G$ defines a homomorphism out of $\mathbb{Z} \times \mathbb{Z}$. So for a fixed g, we count $|Z(g)|$ many commuting elements h.

How many of these are there up to conjugation? Now we identify pairs of the form xgx^{-1}, xhx^{-1} for some $x \in G$ with the pair g, h. Now for each conjugacy class $C = [g]$, we count the number of conjugacy classes in $Z(g)$. Remember that the number of irreps of a finite group is given by its number of conjugacy classes.

In summary, a basis for the ground state space of the Kitaev quantum double model for a finite group G on a torus is in bijection with pairs (C, χ) where C is a conjugacy class of G and χ is an irrep of $Z(g)$ for a representative $g \in G$.

We gave a very rough idea of why the ground state degeneracy on a genus g surface is counted by $\mathrm{Hom}(\pi_1(\Sigma_g), G)$ up to conjugation.

In order to exhibit how excitations are created in dual pairs from a ground state one needs to introduce *ribbon operators*, analogous to the string operators from Lectures 4 and 5. Now our quasiparticles will live at the endpoint sites of a ribbon on the combined lattice/dual lattice.

We probably don't have time to appreciate the full details of the ribbon operators, so we will wrap up this discussion for now by zooming back out to see the big picture. [16]

The upshot is that the irreps of the quantum double algebra DG classify the kinds of quasiparticles that can live at a site, and their fusion rules are given by the fusion table of the irreps of DG.

When G is abelian – let's call it A – you can check that there are no interesting relations among the generators $D_{(b,a)}$ of the quantum double algebra $DA = \widehat{\mathbb{C}[A]} \otimes \mathbb{C}[A]$. This tells us that the representation theory of DA is given by the representation theory of $\hat{A} \times A \cong A \times A$.

In other words, when we feed an abelian group A into the Kitaev quantum double model, we will get a topological phase where the anyons fuse like $A \times A$. We saw the smallest example of this behavior with the $\mathbb{Z}_2$ toric code, where the anyons fuse like $\mathbb{Z}_2 \times \mathbb{Z}_2$. Note in particular that we will always get *abelian anyons* – anyons with fusion rules that come from a(n abelian) group – when we start with A abelian.

The smallest nonabelian group is S_3, and it gives rise to *nonabelian anyons* – anyons with fusion rules that, while commutative, are not all invertible. More generally, when G is nonabelian the quantum double will yield nonabelian anyons. This is a place where the terminology lines up neatly, but remember that nonabelian anyons are more general and need not come from nonabelian groups via the quantum double construction.

[16] Hopefully when we cover string-net models we will make up for lost time and talk about ribbon operators in detail.

Nonabelian anyons in the DS_3 model

Recall from the exercises that there are eight irreps of DS_3, which we label as follows:

Irrep of DS_3	Anyon Type
$(\{\text{id}\}, 1)$	A
$(\{\text{id}\}, \text{sgn})$	B
$(\{\text{id}\}, \text{std})$	C
$(\{(12), (13), (23)\}, 1)$	D
$(\{(12), (13), (23)\}, s)$	E
$(\{(123), (132)\}, 1)$	F
$(\{(123), (132)\}, \omega)$	G
$(\{(123), (132)\}, \omega^*)$	H

Example 2.1: DS_3 fusion rules

Let $L = \{A, B, C, D, E, F, G, H\}$. Then the tensor product of irreps of DS_3 decompose according to the following fusion table:

$\otimes$	A	B	C	D	E	F	G	H
A	A	B	C	D	E	F	G	H
B	B	A	C	E	D	F	G	H
C	C	C	$A \oplus B \oplus C$	$D \oplus E$	$D \oplus E$	$G \oplus H$	$F \oplus H$	$F \oplus G$
D	D	E	$D \oplus E$	$A \oplus C \oplus F \oplus F \oplus H$	$B \oplus C \oplus F \oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
E	E	D	$D \oplus E$	$B \oplus C \oplus F \oplus G \oplus H$	$A \oplus C \oplus F \oplus G \oplus J$	$D \oplus E$	$D \oplus E$	$D \oplus E$
F	F	F	$G \oplus H$	$D \oplus E$	$D \oplus E$	$A \oplus B \oplus F$	$H \oplus C$	$G \oplus C$
G	G	$F \oplus H$	$D \oplus E$	$D \oplus E$	$H \oplus C$	$H \oplus C$	$A \oplus B \oplus G$	$F \oplus C$
H	H	H	$F \oplus G$	$D \oplus E$	$D \oplus E$	$G \oplus C$	$F \oplus C$	$A \oplus B \oplus H$