PHYS 570/MA 595: Unit 2 Exercises, Due September 25, 2025

Sometimes important definitions, derivations, and examples will be deferred to the exercises, but they are not meant to be hard. Exercises will be added to this list but no new problems will be added after Thursday September 18.

Thursday September 11

1. **Check** that the Hadamard matrix $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ interwines the permutation representation of $\mathbb{Z}_2$ with the direct sum of the trivial and sign representations. (2 points)

2. Is it possible for a group representation V to satisfy $V \otimes V = 1 \oplus V$? **Argue** why or why not. (2 points)

Tuesday September 16

3. **Describe** all multiplicity-free fusion rings of rank 3 which contain a non-self dual basis element. (6 points)

4. **Compute** the Frobenius-Perron dimensions of the Haagerup fusion rules on the label set $L = \{1, \omega, \omega^*, X, Y, Z\}$.

$\times$	1	ω	ω^*	X	Y	Z
1	1	ω	ω^*	X	Y	Z
ω	ω	ω^*	1	Y	Z	X
ω^*	ω^*	1	ω	Z	X	Y
X	X	Z	Y	$1+X+Y+Z$	$\omega^*+X+Y+Z$	$\omega+X+Y+Z$
Y	Y	X	Z	$\omega+X+Y+Z$	$1+X+Y+Z$	$\omega^*+X+Y+Z$
Z	Z	Y	X	$\omega^*+X+Y+Z$	$\omega+X+Y+Z$	$1+X+Y+Z$

You'll notice this fusion table has some patterns, what can you say about it? (6 points)

5. (a) Say we have two fusion rings, with label sets L_1, L_2 and fusion coefficients $N_{c_1}^{a_1 b_1}$ for $a_2, b_2, c_2 \in L_2$ and $N_{c_2}^{a_2 b_2}$ for $a_2, b_2, c_2 \in L_2$, respectively. **Check** that $L = L_1 \times L_2$ – which we will write as $L = \{a \boxtimes b\}_{a \in L_1, b \in L_2}$ – gives the free $\mathbb{Z}$-module on L the structure of a fusion ring with respect to the multiplication on L given by $(a_1 \boxtimes a_2) \times (b_1 \boxtimes b_2) = \sum_{c_1 \in L_1, c_2 \in L_2} N_{c_1}^{a_1 b_1} N_{c_2}^{a_2 b_2} c_1 \boxtimes c_2$. (4 points)

   We will see that this categorifies to to a product of categories called the Deligne product, and that this operation can be interpreted as a stacking of physical theories.

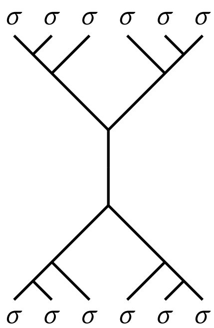
   (b) **Fill in the blanks** in the fusion table for the fusion ring Fib $\boxtimes$ Ising. (2 points)

	$1 \boxtimes 1$	$1 \boxtimes \sigma$	$1 \boxtimes \psi$	$\tau \boxtimes 1$	$\tau \boxtimes \sigma$	$\tau \boxtimes \psi$
$1 \boxtimes 1$	$1 \boxtimes 1$	$1 \boxtimes \sigma$	$1 \boxtimes \psi$	$\tau \boxtimes 1$	$\tau \boxtimes \sigma$	$\tau \boxtimes \psi$
$1 \boxtimes \sigma$	$1 \boxtimes \sigma$					
$1 \boxtimes \psi$	$1 \boxtimes \psi$					
$\tau \boxtimes 1$	$\tau \boxtimes 1$					
$\tau \boxtimes \sigma$	$\tau \boxtimes \sigma$					
$\tau \boxtimes \psi$	$\tau \boxtimes \psi$					

(c) **Explain** why $d_{a \boxtimes b} = d_a d_b$ in any Deligne product of fusion rings, where d_a is the Frobenius-Perron dimension.[1] (2 points)

Thursday September 18

6. **Enumerate** the admissible labelings of this diagram using the Ising fusion rules. Assume all edges are oriented going down the page. (3 points)



7. Let $G = S_3$. Enumerate the pairs of the form (C, χ), where C is a conjugacy class in G and χ is an irrep of the centralizer subgroup $Z(c) = \{g \in G | gc = cg\}$ for a representative $c \in C$. (3 points)

   For example, take $C = \{(123), (132)\}$ and $c = (123)$. Then $Z(c) = \langle (123) \rangle \cong \mathbb{Z}_3$, ... and so on.

[1] Recall from lecture that this notation d_a for $a \in L$ is typically reserved for the quantum dimension of an object (rather than the Frobenius-Perron dimension in either a unitary or spherical fusion category but that we're co-opting it anyway.