PHYS 570/MA 595: Unit 3 Exercises, Due October 9, 2025

Exercises will be added to this list but no new problems will be added after Thursday October 2.

Tuesday September 23 and Thursday September 25

1. In Kitaev's quantum double model for a finite group G, the vertex gauge transformation operators $A_g(s)$ and the operators $B_h(s)$ that project onto sites with flux h do not commute for all $g, h \in G$.
   Explain why they satisfy $A_g(s)B_h(s) = B_{ghg^{-1}}(s)A_g(s)$. (5 points)
   This completes the calculation that shows that the local operators at a site s form a representation of the quantum double algebra DG.

2. Recall that $A_g(s)$ only depends on the vertex v at the site s, and so we can write $A_g(v) = A_g(s)$.
   Verify that the quantum double vertex operator $A(v) = \frac{1}{|G|}\sum_{g\in G} A_g(v)$ is a projector, i.e. $A(v)^2 = A(v)$. (5 points)

3. Consider the quantum double model on a square lattice as in class. When you plug in $G = \mathbb{Z}_2$, **describe** the relationship of the vertex and plaquettes operators $A(v)$ and $B(p)$ to the vertex and plaquette operators A_v and B_p that appeared in Kitaev's toric code Hamiltonian from Lectures 4 and 5. (5 points)

Tuesday September 30

4. Consider a periodic 1d lattice with N sites with a qubit on each site. Show that the operator $S_{i,j}$ that swaps the ith and jth qubits $S_{i,j} = \frac{1}{2}\left(X_iX_j + Y_iY_j + Z_iZ_j + 1\right)$ acts on the Pauli X operator on site i to the Pauli X operator on site j, i.e. **show** that

   $$S_{i,j}X_j(S_{i,j})^{-1} = X_i. \qquad \text{(6 points)}$$

   It will be helpful to remember the relations in the single-qubit Pauli group from the Unit 1 Exercises, which can be summarized with the notation $\sigma_i\sigma_j = \delta_{ij}I + \mathrm{i}\epsilon_{ijk}\sigma_k$ where $\sigma_1 = X$, $\sigma_2 = Y$, and $\sigma_3 = Z$ and ϵ_{ijk} is the Levi-Civita symbol.

5. Let $H = -g\sum_{j=1}^N X_j - \sum_{j=1}^N Z_jZ_{j+1}$ be the transverse-field Ising Hamiltonian, and put $T_{j,j+1} = S_{j,j+1}$ where S is the operator defined in the previous problem. **Argue** that the $\mathbb{Z}_N$ lattice translation symmetry operator

   $$T = \prod_{j=1}^{N-1} T_{j,j+1}$$

   commutes with H. (4 points)