PHYS 570/MA 595: Unit 4 Exercises, Due October 23, 2025

Sometimes important definitions, derivations, and examples will be deferred to the exercises, but they are not meant to be hard. Exercises will be added to this list but no new problems will be added after Thursday October 16.

Thursday October 3

1. A 3-cochain on G with coefficients in $U(1)$ is a function $G \times G \times G \to U(1)$. The set of 3-cochains $C^3(G, U(1))$ forms a group under pointwise multiplication. A 3-cocycle on G valued in $U(1)$ is a function $\omega : G \times G \times G \to U(1)$ such that

$$\omega(gh, k, l)\omega(g, h, kl) = \omega(g, h, k)\omega(g, hk, l)\omega(h, k, l)$$

for all $g, h, k, l \in G$. **Check** that 3-cocycles on G form a subgroup of $C^3(G, U(1))$, call it $Z(G, U(1))$. (3 points)

2. Let G be a finite group and consider the group ring $\mathbb{C}[G]$ as the fusion ring with basis $L = G$ and $N_k^{gh} = \delta_{k,gh}$ for all $g, h, k \in G$. **Show** that unitary[1] solutions to the pentagon equations for G-fusion rules over are 3-cocycles on G with $U(1)$ coefficients. (5 points)

Thursday October 9

3. A 3-cocycle ω is a 3-coboundary if there exists a function $\phi : G \times G \to U(1)$ such that

$$\omega(g, h, k) = \phi(g, hk)\phi(h, k)\phi(gh, k)^{-1}\phi(g, h)^{-1}$$

for all $g, h, k \in G$. Two 3-cocycles ω and $\tilde{\omega}$ differ by a coboundary if there exists a 3-coboundary ϕ such that

$$\omega(g, h, k) \cdot \tilde{\omega}(g, h, k)^{-1} = \phi(g, hk)\phi(h, k)\phi(gh, k)^{-1}\phi(g, h)^{-1}.$$

Check that differing by a 3-coboundary is an equivalence relation on G 3-cocycles with $U(1)$ coefficients. (3 points)

4. **Show** that two unitary solutions to the pentagons for G-fusion rules are gauge equivalent if they differ by a 3-coboundary. (5 points)
The group of equivalence classes of 3-cocycles modulo 3-coboundaries in called the third cohomology group of G with coefficients in $U(1)$ and is denoted by $H^3(G, U(1))$. These four exercises show that fusion categories with G-fusion rules are classified by $H^3(G, U(1))$.

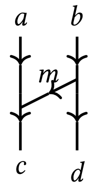
5. Suppose that some *multiplicity-free* F-symbols $[F_d^{abc}]_{m,n}$ are solutions to the pentagon equations. **Show** that an arbitrary gauge-transformation of these F-symbols also satisfy the pentagon equations. (4 points)

(continued on next page)

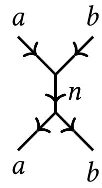
[1] Here by unitary solutions all we mean are F-matrices which are unitary matrices.

Thursday October 16

6. **Derive** the formula for the following version of the two "I=H" moves in a multiplicity-free unitary fusion category without tetrahedral symmetry.[2] (6 points)



$$= \sqrt{\frac{d_m d_n}{d_b d_c}}[F_n^{amd}]_{c,b}$$



7. Please **fill out** this brief anonymous survey to let me know how the course is going for you. In order to keep the survey completely anonymous but closed to course participants it is password protected. The password is posted on our Brightspace page in the "Mid-Semester Survey Module".

   ☐ I have taken the survey (4 points)

[2]Due to the volume of pictures in this unit I am a bit slow in posting TeXed notes, so please refer to board photos from class from previous lectures for our conventions for diagrammatic moves.