## PHYS 570/MA 595: Unit 5 Exercises, Due November 6, 2025

This set may look long but don't be intimidated. Exercises will be added to this list but no new problems will be added after Thursday October 30. Thank you for your patience while I catch up on TeXing lecture notes.

### Tuesday October 21 and Thursday October 30

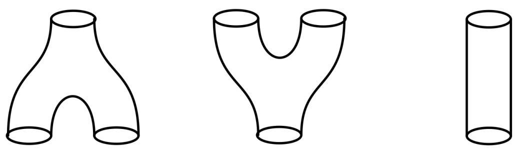
1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A natural transformation of two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ is a collection of morphisms $\phi_X : F(X) \to G(X)$ for every object $X \in \mathcal{C}$ that are are natural with respect to morphisms in the sense that the following diagram commutes for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$

$$\begin{tikzcd} F(X) \arrow[r, "\phi_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\phi_Y"'] & G(Y) \end{tikzcd}$$

**Check** that functors and natural transformations form a category, $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$. (4 points)

2. **Describe** isomorphisms in $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$. (2 point)

3. Assume $m, n \geq 1$. Give a **proof by picture** that every cobordism from the disjoint union of $n$ circles to the disjoint union of $m$ circles can be generated by composition and disjoint union of



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(5 points)

4. A Frobenius algebra $(A, \mu, \Delta, \eta, \epsilon)$ over $\mathbb{C}$ is
   - an associative algebra $(A, \mu, \eta)$:
     a $\mathbb{C}$-vector space $A$ with multiplication $\mu : A \otimes A \to A$ and unit $\eta : \mathbb{C} \to A$ satisfying
     - $\mu \circ (\mathrm{id} \otimes \mu) = \mu \circ (\mu \otimes \mathrm{id})$ *associativity*
     - $\mu \circ (\eta \otimes \mathrm{id}) = \mathrm{id} = \mu \circ (\mathrm{id} \otimes \eta)$ *unitality*
   - a coassociative coalgebra $(A, \Delta, \epsilon)$:
     a $\mathbb{C}$-vector space $A$ with comultiplication $\Delta : A \to A \otimes A$ and counit $\epsilon : A \to \mathbb{C}$ satisfying

– $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ *coassociativity*

– $(\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id} = (\epsilon \otimes \mathrm{id}) \circ \Delta$ *counitality*

satisfying the *Frobenius condition*:

$$(\mathrm{id} \otimes \mu) \circ (\Delta \otimes \mathrm{id}) = \Delta \circ \mu = (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta)$$

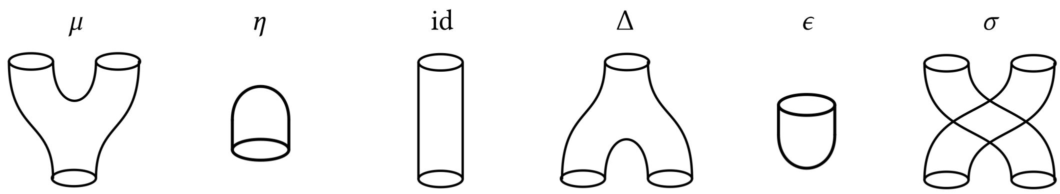
.

Furthermore, let

$$\begin{aligned} \sigma \ : A \otimes A &\to A \otimes A \\ x \otimes y &\mapsto y \otimes x \end{aligned}$$

A Frobenius algebra is commutative if $\mu \circ \sigma = \mu$ and cocommutative if $\sigma \circ \Delta = \Delta$.

By representing the Frobenius algebra maps by elementary cobordisms,




representing composition of maps by vertical stacking, and $\otimes$ by disjoint union, **draw** pictures of the defining equations of a commutative and cocommutative Frobenius algebra. (7 points)

5. **Read** the definitions below.

- Let $(\mathcal{C}, \otimes, \overset{\mathcal{C}}{\otimes}, 1_{\mathcal{C}}, \alpha^{\mathcal{C}})$ and $(\mathcal{D}, \overset{\mathcal{D}}{\otimes}, 1_{\mathcal{D}}, \alpha^{\mathcal{D}})$ be two monoidal categories and let $(F, J^F)$, $(G, J^G)$ be a monoidal functor from $\mathcal{C}$ to $\mathcal{D}$. A natural transformation $\phi : F \to G$ is monoidal if $\phi_1 : F(1_{\mathcal{C}}) \to G(1_{\mathcal{C}})$ is an isomorphism and

$$\begin{tikzcd} F(X \overset{\mathcal{C}}{\otimes} Y)) \arrow[r, "J^F_{X,Y}"] \arrow[d, "\phi_{X \otimes Y}"'] & F(X) \overset{\mathcal{D}}{\otimes} F(Y) \arrow[d, "\phi_X \otimes \phi_Y"] \\ G(X \overset{\mathcal{C}}{\otimes} Y) \arrow[r, "J^G_{X,Y}"'] & G(X) \overset{\mathcal{D}}{\otimes} G(Y) \end{tikzcd}$$

  Monoidal functors and monoidal natural transformations form a category $\mathbf{Fun}_{\otimes}(\mathcal{C}, \mathcal{D})$.

- Now let $(\mathcal{C}, \otimes, \overset{\mathcal{C}}{\otimes}, 1_{\mathcal{C}}, \alpha^{\mathcal{C}}, \beta^{\mathcal{C}})$ and $(\mathcal{D}, \overset{\mathcal{D}}{\otimes}, 1_{\mathcal{D}}, \alpha^{\mathcal{D}}, \beta^{\mathcal{D}})$ be two braided categories. A monoidal functor $(F, J)$ is braided if the diagram below commutes.

$$\begin{tikzcd}
F(X \overset{\mathcal{C}}{\otimes} Y) \arrow[r, "F(\beta^{\mathcal{C}}_{X,Y})"] \arrow[d, "J_{X,Y}"'] & F(Y \overset{\mathcal{C}}{\otimes} X) \arrow[d, "J_{Y,X}"] \\
F(X) \overset{\mathcal{D}}{\otimes} F(Y) \arrow[r, "\beta^{\mathcal{D}}_{F(X),F(Y)}"'] & F(Y) \overset{\mathcal{D}}{\otimes} F(X)
\end{tikzcd}$$

Note that for a functor being monoidal is a structure, while for a monoidal functor being braided is a property.

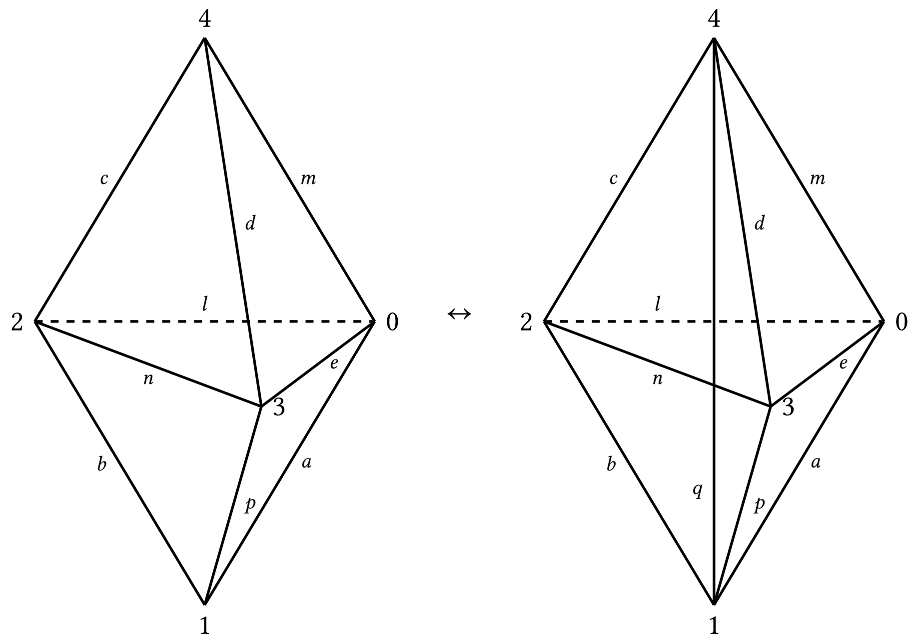
Braided monoidal functors and monoidal natural transformations form a category $\mathbf{Fun}^{br}_{\otimes}(\mathcal{C}, \mathcal{D})$.

**Write** a one or two sentence physical intepretation of isomorphisms in the category of (symmetric) braided monoidal functors $\mathbf{Fun}^{br}_{\otimes}(\mathbf{Bord}_3, \mathbf{Vec})$. (4 points)

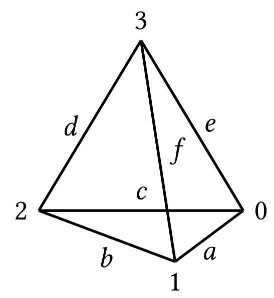
These exercises are working up towards showing you that Atiyah-Segal type $2d$-TQFTs are classified by commutative Frobenius algebras.

**Thursday October 23, Tuesday October 28**

6. Consider the 3D Pachner move that relates 3-manifold triangulations by moving between 2 and 3 positively oriented tetrahedra, with the following vertex ordering and edge labels in a multiplicity-free unitary fusion category with skeletal data $(L, N^{ab}_c, [F^{abc}_d]_{m,n})$.




By comparing the (assume all positively oriented) tetrahedra in the two triangulations with the model tetrahedron (we are ignoring orientations of edges for clarity)




**show** that the TVBW state-sum invariant is invariant under the 2-3 Pachner move if and only if the pentagon equations hold. (8 points)

*Hint:* Look at the tetrahedra inscribed by the vertices (0123), (0243) on the left hand side and the tetrahdedra inscribed by (0124), (0143), and (1243) on the right hand side.