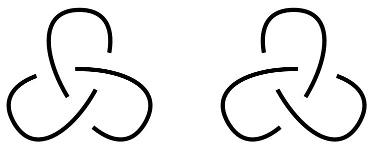
PHYS 570/MA 595: Unit 6 Exercises, Due November 25, 2025, Accepted until December 11

Note: our November 20th class is an asynchronous video lecture.
Exercises will be added to this list but no new problems will be added after Friday November 14. Thank you for your patience while I catch up on TeXing lecture notes; in the meantime, please refer to the board photos on Brightspace.

Tuesday November 4 and Thursday November 6

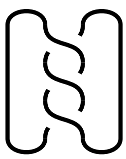
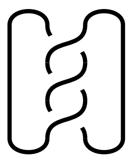
1. The trefoil knots are the smallest nontrivial examples of knots.



left-handed trefoil   right-handed trefoil

The left- and right-handed trefoil knots are mirror knots that are not isotopic. A diagram for a mirror of a knot K can be obtained by reversing each crossing in a diagram for K.

Using the fact that the trefoil knots can be represented as the trace of the braid diagrams [1] of σ_1^3 and σ_1^{-3},

$\mathrm{Tr}(\sigma_1^3) =$  $\mathrm{Tr}(\sigma_1^{-3}) =$ 

Derive a formula for the Reshetikhin-Turaev invariant of the trefoil knot colored by an anyon a using the graphical calculus for a multiplicity-free skeletal UMTC $(L, N_c^{ab}, [F_d^{abc}]_{m,n}, R_c^{ab})$. (6 points)

2. **Evaluate** the invariants of the trefoil knots for the nonabelian Ising anyon σ using the data for an Ising UMTC is given below.

$$L = \{1, \sigma, \psi\}$$

$$\begin{cases} \sigma \otimes \sigma = 1 \oplus \psi \\ \sigma \otimes \psi = \sigma \\ \psi \otimes \psi = 1 \end{cases}$$

[1](Don't worry about conventions for what we're calling σ_1 versus σ_1^{-1} in the 2-strand braid group, we've changed conventions for crossings since we first introduced braid diagrams much earlier in the course.)

$$F^{\sigma\sigma\sigma}_{\sigma} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, [F^{\psi\sigma\psi}_{\sigma}]_{\sigma,\sigma} = -1, [F^{\sigma\psi\sigma}_{\psi}]_{\sigma,\sigma} = -1$$

$$R^{\sigma\sigma}_{1} = e^{-\pi i/8}, R^{\sigma\sigma}_{\psi} = e^{3\pi i/8}, R^{\psi\psi}_{1} = -1, R^{\psi,\sigma}_{\sigma} = R^{\sigma\psi}_{\sigma} = -i$$

Any admissible F-symbols or R-symbols not explicity listed are equal to 1.

Determine whether the Ising anyon detect the handedness of the trefoil knot.
(4 points)

Tuesday November 11

3. There are several equivalent criteria that detect abelianness of anyons in a UMTC. A usual let L be the label set of anyon types.

   An anyon a is abelian

   (I) if $a \otimes a^* = 1$

   (II) if its quantum dimension $d_a = 1$.

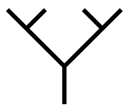
   (III) if there exists an anyon c such that $a \otimes b = c$ for all anyons $b \in L$

   Prove that (I) $\implies$ (II) $\implies$ (III) $\implies$ (I). (3 points)
   Hint: for (II) $\implies$ (III) use that the quantum dimension is equal to the Frobenius-Perron dimension by unitarity.

4. (a) **Show** that a qubit can be stored in the state space $V_1^{\sigma\sigma\sigma\sigma}$ of four Ising anyons σ with trivial total charge. (3 points)

   (b) Compute the 3 quantum gates that come from exchanging adjacent pairs of anyons. (7 points)

   Hint: to simplify the calculation you should work in the fusion tree basis with the following shape:

   (c) Consider the quantum gates from part (b) and their squares up to an overall phase. **Identify** any familiar single-qubit gates. (2 points)

Thursday November 13

5. Consider the anyon chain build on the Ising unitary *fusion* category (forgetting R-symbols) with reference simple object σ.

   (a) **Show** that the total Hilbert space $\mathcal{H}_{total}$ is only nontrivial for anyon chains with an even number of edges. (2 points)

(b) Without computing formulas for how the symmetry generators D^1, D^σ, and D^ψ act on $\mathcal{H}_{total}$, **show** that $D^\sigma D^\sigma = D^1 + D^\psi$. (3 points) *Hint:* how can you use the graphical calculus in a skeletal fusion category to resolve the fusion of two loops into the anyon chain into a fusion of a single loop?