Invariant Subspaces and Complex Analysis

Carl C. Cowen

Math 727 cowen@purdue.edu

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Invariant Subspaces and Complex Analysis

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Much of this work is joint with

Eva Gallardo Gutiérrez

Departamento Análisis Matemático,

Univ. Complutense de Madrid

(who visited Purdue in Fall 2014)

Main messages today:

Most progress comes from looking with a different perspective.

You should develop your intuition.

You should work in areas where you have intuition.

Advisors and collaborators should help you develop your intuition!

My research areas:

Operator Theory

Complex Analysis

Linear Algebra

Doing Operator Theory is:

Doing Linear Algebra

and Calculus (with complex numbers)

in an Infinite Dimensional Euclidean Space.

Linear algebra: Euclidean spaces: $\mathbb{R}^n$, $\mathbb{C}^n$

Problems:

Classify $n \times n$ matrices up to similarity: Jordan Canonical Form

For a given matrix A,

which matrices B satisfy $AB = BA$,

and what subspaces M satisfy $AM \subset M$?

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$\Uparrow$ **Invariant Subspaces!**

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An important example(!):

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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The goal in answering these questions is to understand the *structure* of linear transformations.

The eigenspaces of linear transformations are invariant subspaces and play a key role in describing the structure!

The analysis of differential equations necessitated extension

to infinite dimensional spaces:

Hilbert spaces are infinite dimensional Euclidean spaces: $\mathbb{C}^n$ expands to ℓ^2

$$v = (a_0, a_1, a_2, \cdots) \text{ with } \|v\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \text{ and } \langle v, w \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

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Problems:

Classify operators up to similarity. (unsolved!)

For a given operator A,

which operators B satisfy $AB = BA$, (unsolved!)

and what subspaces M satisfy $AM \subset M$? (unsolved!)

An important example(!):

$$\text{On} \quad \ell^2 = \{v = (a_0, a_1, a_2, \cdots) : \|v\|^2 = \| \sum |a_n|^2 < \infty\}$$

the *unilateral shift operator* is:

$$Sv = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix}$$

An easy example of an operator that is not self-adjoint, normal, or compact, types of operators with much better understood structure than generic ones.

A theorem from linear algebra states that any two

n-dimensional

complex vector spaces

with an inner product

are isometrically isomorphic.

In other words, looking at them with your Euclidean space glasses on,

they look exactly alike!

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For example, $\mathbb{C}^n$ with the Euclidean inner product

is isometrically isomorphic to

the vector space of polynomials of degree $n-1$ or less, with complex coefficients,

and the inner product

$$\langle p, q \rangle = \int_0^1 p(x)\overline{q(x)}\, dx$$

The same is true with infinite dimensional Hilbert spaces!

All Hilbert spaces of the same dimension are isometrically isomorphic,

so ℓ^2 is the same as any other Hilbert space with dimension $\aleph_0$!

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But Hilbert spaces of the same dimension, but different definitions for their description, are mathematically the same,

but elicit *different* mathematical ideas for studying them!

A breakthrough in understanding the *unilateral shift operator* arises from connecting the operator to complex analysis!!

Defining the Hardy space on the unit disk, $\mathbb{D}$, by

$$H^2(D) = \{f \text{ analytic on } \mathbb{D} : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } \|f\|^2 = \sum |a_n|^2 < \infty\}$$

We see $\ell^2 \leftrightarrow H^2$ and $S \leftrightarrow T_z$ where $T_z(f) = zf$

The analytic Toeplitz operators T_ψ, for ψ a bounded analytic function on the unit disk are defined by

$$T_\psi f = \psi f$$

and these operators are continuous with

$$\|T_\psi\| = \|\psi\|_\infty = \sup\{|\psi(z)| : |z| < 1\}$$

For bounded analytic ψ, the matrix for T_ψ is lower triangular and is constant along diagonals:

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & 0 & \cdots \\ a_2 & a_1 & a_0 & 0 & \cdots \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

where $\psi(z) = \sum_{j=0}^{\infty} a_j z^j$.

Definition:

If A is a bounded operator on a space $\mathcal{H}$, the *commutant of* A is the set of operators that commute with A, that is,

$$\{A\}' = \{S \in \mathcal{B}(\mathcal{H}) : AS = SA\}$$

For example, for T_z on H^2,

$$\{T_z\}' = \{T_\psi : \psi \in H^\infty\}$$

The precise terminology:

If A is a bounded linear operator mapping a Banach space $\mathcal{X}$ into itself,

a closed subspace M of $\mathcal{X}$ is an *invariant subspace for* A

if for each v in M, the vector Av is also in M.

The subspaces $M = (0)$ and $M = \mathcal{X}$ are *trivial* invariant subspaces and we are not interested in these.

The *Invariant Subspace Question* is:

- Does every bounded operator on a Banach space have a non-trivial invariant subspace?

We will only consider vector spaces over the complex numbers.

If the dimension of the space $\mathcal{X}$ is finite and at least 2, then any linear transformation has eigenvectors and each eigenvector generates a one dimensional (non-trivial) invariant subspace.

The Jordan Canonical Form Theorem provides the information to construct all of the invariant subspaces of an operator on a finite dimensional space.

Some history:

- Spectral Theorem for self-adjoint operators on Hilbert spaces gives

  invariant subspaces

- Beurling (1949): completely characterized the invariant subspaces of

  operator of multiplication by z on the Hardy Hilbert space, H^2

- von Neumann ('30's, 40's?), Aronszajn & Smith ('54):

  Every compact operator on a Banach space has invariant subspaces.

Some history:

- Spectral Theorem for self-adjoint operators
- Beurling (1949): invariant subspaces of isometric shift
- von Neumann ('30's, 40's?), Aronszajn & Smith ('54): compact operators
- Lomonosov ('73):

If S is an operator that commutes with an operator $T \neq \lambda I$,

and T commutes with a non-zero compact operator

then S has a non-trivial invariant subspace.

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The *(revised) Invariant Subspace Question* is:

- Does every bounded operator on a ~~Banach~~ Hilbert space have a non-trivial invariant subspace?

Rota's Universal Operators:

Defn: Let $\mathcal{X}$ be a Banach space, let U be a bounded operator on $\mathcal{X}$.

We say U is *universal for* $\mathcal{X}$ if for each bounded operator A on $\mathcal{X}$,

there is an invariant subspace M for U and a non-zero number λ

such that λA is similar to $U|_M$.

In other words, a universal operator on $\mathcal{X}$ has a miniature copy of *every* bounded operator on $\mathcal{X}$!!

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Rota proved in 1960 that if $\mathcal{X}$ is a separable, infinite dimensional Hilbert space,

there are universal operators on $\mathcal{X}$!

Theorem (Caradus (1969))

If $\mathcal{H}$ is separable Hilbert space and U is bounded operator on $\mathcal{H}$ such that:

- The null space of U is infinite dimensional.
- The range of U is $\mathcal{H}$.

then U is universal for $\mathcal{H}$.

Theorem (Caradus (1969))

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- The null space of U is infinite dimensional.
- The range of U is $\mathcal{H}$.

then U is universal for $\mathcal{H}$.

So far, every known example of a universal operator on a separable Hilbert space used Caradus' Theorem to prove it is universal and all have been equivalent to an analytic Toeplitz operator.

For φ an analytic map of $\mathbb{D}$ into itself, the *composition operator* C_φ is

$$(C_\varphi h)(z) = h(\varphi(z)) \quad \text{for } h \text{ in } H^2$$

These are all bounded operators on H^2, much is known about them,

and they are a big part of my research.

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For f in H^∞ and φ an analytic map of $\mathbb{D}$ into itself, the *weighted composition operator* $W_{f,\varphi} = T_f C_\varphi$ is

$$(W_{f,\varphi} h)(z) = f(z)h(\varphi(z)) \quad \text{for } h \text{ in } H^2$$

Theorem:(C., Gallardo, 2012)

There are analytic functions, ψ and f, on the disk

and an analytic map, φ, of the disk into itself

so that T_ψ^* is a universal operator and, for $W_{f,\varphi} = T_f C_\varphi$,

the operator $W_{f,\varphi}^*$ is a compact operator commuting with T_ψ^*.

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the operator $W_{f,\varphi}^*$ is a compact operator commuting with T_ψ^*.

More recently, using this result,

we have posed a question from complex analysis

and proved that an affirmative answer to the question

proves the invariant subpace theorem!!

Thank You!

Slides available: `http://www.math.purdue.edu/~cowen`