Amer. Math. Monthly **98**(1991)50–53.

Teaching and Testing Mathematics Reading

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When we discuss teaching undergraduate mathematics with our colleagues, we frequently distinguish two sorts of goals: first, we want students to be able to do computations related to the course and, second, we want students to know the theory from the course. In a calculus course, we want students to be able to integrate polynomials, so in class we tell them how, and on the exam, we give them polynomials to integrate. In linear algebra class, we want the students to know the theorems concerning determinants, so we prove the theorems from the book, and on the exam, we ask them to prove that if A is an invertible matrix, then $\det(A^{-1}) = 1/\det(A)$. Our hallway conversations elicit general agreement that most of our students do learn to do the calculations we put on the exam and most don't master the theory to the point that they can write a coherent proof of any but the most trivial theorems.

I want to argue in this note that there is another goal, between these in accessibility, that should be a conscious part of our teaching effort:

We should teach our students to read and understand mathematics.

And to reinforce our effort, we must test their ability to do so. Very few of our students will ever, after leaving our courses, have to give a formal proof of a theorem and few will have to do something so mundane as "Find the definite integral." On the other hand, many of them *will* have to read and understand mathematical writing to apply new ideas to the problems of their jobs. Moreover, learning to read and understand mathematics will make the further step of learning to prove theorems more possible and will have benefits for every course that follows. We need to convince our students that, in the long run, understanding what the theorems mean will be easier and have a bigger payoff than simply memorizing their statements. I do not believe that our current goals should be dropped, they are an important part of every mathematics course. On the other hand, I do not believe that learning to read can be left simply to chance; that's what we've been doing and it hasn't worked!

If you need evidence that we have a problem, let one of your "B" students explain the statement of a theorem from a homework exercise that they have not been able to prove or have them explain the statement and proof of a theorem from a section in the book that you have skipped. My students, at least, do not have the innate ability to read and understand what they have read. When I ask them to read a problem and explain it to me, the majority simply recite the same words back again.

To help them learn to read, we must give them opportunities to practice. In class, it is helpful to discuss the meaning of a new theorem before proving it. It is especially important to discuss hypotheses. Many students have not learned what role the hypothesis of a theorem plays. You might give several examples of objects that could satisfy the hypotheses of a theorem and get your class to tell you which actually do. Asking them to restate the conclusion in a particular instance is frequently a helpful way to deal with the conclusion. Drawing out corollaries that have not yet been stated is another way to deal with the conclusion. For example, in an advanced linear algebra course, I prove the following theorem:

Theorem. *If A is an n by n matrix and μ is a number such that*

$$\|A\| < |\mu|$$

then $\mu I - A$ is invertible.

Comment: This theorem is true for any submultiplicative norm on matrices; the norm used below is the one–norm:

$$\|A\| = \max\{\sum_{i=1}^{n} |a_{ij}| : j = 1, \cdots, n\}$$

After writing the statement of the theorem on the blackboard, we can have the following (idealized) discussion:

Instructor: What is the one–norm of the matrix

$$A = \begin{pmatrix} .7 & -.2 \\ .5 & 1.4 \end{pmatrix}$$

Student: $\|A\| = 1.6$

Instructor: Good! What does this theorem say about the matrix

$$2I - A = \begin{pmatrix} 1.3 & .2 \\ -.5 & .6 \end{pmatrix}$$

Student: The theorem says it's invertible.

Instructor: That's right! The μ in this case is 2 and since $\|A\| = 1.6 < 2 = |\mu|$, the hypothesis of the theorem is satisfied and the theorem says that $2I - A$ is invertible. In fact, the determinant of $2I - A$ is $.88 \neq 0$ and we can find the inverse by direct computation.

Instructor: What does this theorem say about the matrix

$$I - A = \begin{pmatrix} .3 & .2 \\ -.5 & -.4 \end{pmatrix}$$

Student 1: The theorem says it's not invertible.

Instructor: Is that correct?

Student 2: No, the theorem doesn't say anything about this matrix.

Instructor: Student 2 is right! The μ in this case is 1 and since $\|A\| = 1.6 > 1 = |\mu|$, the hypothesis of the theorem is not satisfied and we cannot tell *from this theorem* if the matrix $I - A$ is invertible or not. In fact, the determinant of $I - A$ is $-.02$ so it *is* invertible. Similarly, for $\mu = 1.2$, the hypothesis of the theorem is not satisfied and we cannot tell *from this theorem* if the matrix $1.2I - A$ is invertible or not. In this case, we find that the determinant of $1.2I - A$ is 0, so the matrix $1.2I - A$ isn't invertible.

Instructor: Now, who can remind us of the definition of eigenvalue and tell some things we know about eigenvalues?

Student: The number λ is an eigenvalue of the matrix A if there is a non–zero vector v for which $Av = \lambda v$. We know that λ is an eigenvalue if and only if $\lambda I - A$ is singular.

Instructor: What does this theorem say about the eigenvalues of A?

Student: If the number λ satisfies $|\lambda| > \|A\|$ then it is not an eigenvalue of A.

Typically, I would then formally state the corollary just given by the student and prove the theorem (by adding up the geometric series) and the corollary. Classes need varying amounts of coaxing and help in such a discussion but they require less coaxing after they are accustomed to the style, especially if the instructor refuses to answer the questions for the class!

A second opportunity for practice is in assigning homework over material not covered in class. Most textbooks include much more material than can be covered in a typical course and much of the material that must be skipped is interesting and valuable. Asking the students to read a section of the book and do some related homework is a reasonable way for them to get practice in reading for understanding. Such material needs to be carefully chosen for this purpose to be sufficiently theoretical to be challenging yet down to earth enough that students can be successful. For example, in some real analysis courses, it might be reasonable to have students read a section in the text defining monotone function and culminating with the proof of the theorem that a monotone function on a finite interval has at most countably many discontinuities. Students could then be asked to prove that the sum of two increasing functions is increasing, give an example of two monotone functions whose sum is not monotone, and perhaps prove that a monotone function on the whole line has at most countably many discontinuities.

Finally, if we really expect students to take learning to read seriously, then we must put it on our tests. Whatever other qualities our students have, they have learned to play the student game well. For their part, the students learn to pass our tests and, for our part, we faculty have capitulated to that hated question "Professor, will this be on the test?" Now we only expect them to learn what we are putting on the test! We can use this to our advantage if we test them on their reading ability.

Clearly, on an hour test, we cannot give the students a book chapter to read and describe. To test reading mathematics that involves the ideas of the course, I have frequently included on an exam a new (to the students) theorem, together with its proof, followed by a question that (I hope) can be answered only by those who read and understood the theorem and proof. The theorems need not be significant, although it's nicer if they are. Moreover, since we want it to be possible to answer the question only by understanding the proof, the conclusion may not be stated in the most elegant or cleanest way. Here are two examples.

The first example is from a course for mathematics education majors. The aims of the course are to develop some understanding of theorems, proofs, and mathematical statements in general. The topics include material on sets, number theory, real numbers, and polynomial equations. This question comes from the first test of the semester which follows the material on number theory; the course does not usually include any discussion of Pythagorean triples, solutions of Diophantine equations, or writing integers as sums of squares.

Theorem. *If m and n are positive integers such that*

$$2m = n^2 + 1$$

then m is the sum of the squares of two integers.

Proof. Since $n^2 + 1$ is even, n^2 and therefore n must be odd. That is, there is an integer k so that $n = 2k + 1$. This means that

$$2m = n^2 + 1 = (2k + 1)^2 + 1 = (4k^2 + 4k + 1) + 1 = 4k^2 + 4k + 2$$

It follows that

$$m = 2k^2 + 2k + 1 = k^2 + (k^2 + 2k + 1) = k^2 + (k + 1)^2$$

which expresses m as the sum of the squares of two integers. ■

Problem:

$$2 \cdot 3785 = 7570 = 87^2 + 1$$

Write 3785 as the sum of the squares of two integers.

Of course, if we were really going to state and prove this "theorem" in a book, we would probably spell out how m is written as the sum of two squares, but to do so in this context would obviate the need to understand the proof. What I expect from this question is that students will see from the proof that for $k = (n - 1)/2$, we have $m = k^2 + (k + 1)^2$. To write 3785 as the sum of two squares, I expect them to see that in the notation of the theorem, $n = 87$, so $k = 43$ and by the proof of the theorem $3785 = 43^2 + 44^2$. If students perform this calculation, I infer that they read and understood the proof well enough to do so.

The second example comes from a linear algebra course for engineers. At the time of the test, the students had studied the spectral mapping theorem for polynomials which implies that the positive integer powers of a diagonalizable matrix are similar to the powers of the diagonal matrix, but had not seen the theorem on square roots of positive definite matrices.

Theorem. *If A is a diagonalizable matrix all of whose eigenvalues are non–negative, then there is a matrix B with non–negative eigenvalues such that $B^2 = A$.*

Proof. Since A is diagonalizable, there is an invertible matrix S such that $L = S^{-1}AS$ is diagonal. The diagonal entries $\lambda_1, \lambda_2, \cdots, \lambda_n$ of L (which are the eigenvalues of A) are non–negative by hypothesis. Let $\mu_j = \sqrt{\lambda_j}$ be the non–negative square roots of the eigenvalues and let M be the diagonal matrix with diagonal entries $\mu_1, \mu_2, \cdots, \mu_n$. Clearly, $M^2 = L$. Since similar matrices have the same eigenvalues, the matrix $B = SMS^{-1}$ has eigenvalues $\mu_1, \mu_2, \cdots, \mu_n$, which are non–negative. Moreover,

$$B^2 = (SMS^{-1})^2 = SMS^{-1}SMS^{-1} = SM^2S^{-1} = SLS^{-1} = A$$

■

Problem: The matrix $A = \begin{pmatrix} 10 & -9 \\ 6 & -5 \end{pmatrix}$ has eigenvalues 1 and 4. Find a matrix S as above and use it to find a matrix B with positive eigenvalues such that $B^2 = A$.

In this question, I expect the students to understand from the proof of the theorem that the problem is to be solved by diagonalizing A and solving the corresponding problem for the diagonal matrix. In particular, I expect them to use the given eigenvalues of A to find a basis of eigenvectors, and thereby to construct S, L, M, and B.

I have not done a comparative analysis of my classes to see if trying to teach reading makes a difference in how much students learn, but I am convinced it helps. At the very least, if we communicate to the students that we think it is reasonable that they read mathematics and understand it and that we expect it of them, then they may come to expect it of themselves. We will all be better off if the students view the textbook as a source of information, not just as a list of exercises interspersed with messages for the instructor.

I'd like to thank Dick Hunt, J. J. Price, and Bob Zink for their support and these colleagues and the referee for suggestions concerning this note.